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On the Asymptotic Geometrical Behaviour of a Class of Contact Interaction Processes with a Monotone Infection Rate

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1. Introduction

Let R^d denote the d-dimensional Euclidean space and let Z^d be the d-dimensional square lattice of points ("sites") $x=(x^1,\ldots,x^d)$, where each x^i is an integer ($d\geqq 1$). Points x and y in Z^d are called *neighbours* (we write $x\sim y$) iff $\|x-y\|=1$ where $\|\cdot\|$ denotes the Euclidean norm. Write N_x for the set of all neighbours of $x\in Z^d$. If A is any set, denote by $|A|$ or $\# A$ the number of elements in A (finite or infinite). By I_A we denote the indicator function of A. Let Ξ be the family of all subsets of Z^d. Each $\xi\in\Xi$ can be identified with a map $\xi\colon Z^d\to\{0,1\}$ where $\xi(x)=1$ iff $x\in\xi$, $x\in Z^d$. (Interpretation: Site $x\in Z^d$ is occupied ("infected") in $\xi\in\Xi$ if $\xi(x)=1$, and vacant (not infected) otherwise.) The elements of Ξ are also called *configurations*. Put

$$\Xi_0=\{\xi|\xi\in\Xi,\ 0<|\xi|<\infty\} \quad\text{and}\quad \Xi_1=\{\xi|\xi\in\Xi,\ \xi\neq\emptyset\}.$$

The class of contact interaction processes we would like to study in this paper can now intuitively be described as follows. Consider any $\xi\in\Xi_1$ and any $x\in Z^d$ such that at time t $\xi(x)=0$ and $|N_x\cap\xi|=i\geqq 1$. Then the probability that during a short time interval $(t,t+h)$ x becomes occupied (infected) is given by $c_i h+o(h)$. The constants c_i ($1\leqq i\leqq 2d$) which can be interpreted as *infection rates* are assumed to depend only on i but neither on x nor on ξ. We exclude the possibility for occupied sites to become vacant again. Such a process is a special case of nearest-neighbour interactions which were introduced by Harris in [5] (in Harris' notation we assume $\mu_0=\mu_1=\ldots=\mu_{2d}=0$ and additionally $\lambda_0=0$). We assume throughout $0<c_i<\infty$, $i=1,\ldots,2d$, and put

(1.1) $$c'=\min_{1\leqq i\leqq 2d} c_i,\qquad c''=\max_{1\leqq i\leqq 2d} c_i.$$

The monotonicity property

(M) $$0<c_1\leqq c_2\leqq\ldots\leqq c_{2d}<\infty$$

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plays a central role in what follows (we always indicate when a result is proved under (M)).

Harris showed in [5] by using results of Holley [6] and Liggett [9] that the processes in consideration can be constructed as Hunt processes with state space $(\Xi, \mathscr{E})$ where $\mathscr{E}$ denotes the family of Borelian subsets of Ξ with respect to the product topology of Ξ. Following the notation of Blumenthal/Getoor [2] as closely as possible we designate such Hunt processes more explicitly by $(\Omega, \mathscr{M}, \mathscr{M}_t, \xi_t, \theta_t, P_\xi)$. The expectation taken with respect to P_ξ is denoted by E_ξ. We write P_0 and E_0 instead of $P_{\{0\}}$ and $E_{\{0\}}$, respectively.

Let

$$(1.2)\quad \tau(x)=\tau(\omega, x)=\inf\{t|\xi_t(\omega, x)\neq 0\}, \qquad x\in Z^d,\ \omega\in\Omega,$$

i.e. $\tau(x)$ is the first instant at which x is occupied (we put $\tau(x)=\infty$ if no such instant exists, and the same convention holds for all similar definitions). Clearly $\tau(x)$ is a stopping time. It is useful to extend $\tau(\cdot)$ to all of R^d as follows. Introduce the cubes

$$(1.3)\quad Q_x=\{y|y\in R^d,\ x^i-\tfrac{1}{2}<y^i\leqq x^i+\tfrac{1}{2},\ i=1,\dots,d\}, \qquad x\in Z^d.$$

Then put

$$(1.4)\quad \tau(y)=\tau(x) \qquad \text{if } y\in Q_x,\ x\in Z^d,$$

and

$$(1.5)\quad \tau_\Delta(x)=\Delta\tau\left(\frac{x}{\Delta}\right), \qquad \Delta>0,\ x\in R^d.$$

Using a coupling argument (see Harris [5], p. 978) it can be shown that under (M) $\{\xi_t\}$ has the following "isotonicity" property: If $J\neq\emptyset$ is any finite or countable index set, we have for all configurations $\xi, \eta\in\Xi$ such that $\xi\subset\eta$,

$$(1.6)\quad P_\xi(\tau(x_j)\geqq t_j, j\in J)\geqq P_\eta(\tau(x_j)\geqq t_j, j\in J), \qquad x_j\in R^d,\ t_j>0,\ j\in J,$$

and

$$(1.7)\quad P_\xi(\tau(x_j)\leqq t_j, j\in J)\leqq P_\eta(\tau(x_j)\leqq t_j, j\in J), \qquad x_j\in R^d,\ t_j>0,\ j\in J$$

(these inequalities are used in the case $|J|>1$ only near the end of this paper).

The purpose of the present paper is to study the asymptotic behaviour of $\tau_\Delta(x)$ (for small Δ) and that of ξ_t (for large t). The main result proved under (M) says that there exists a norm $N(\cdot)$ (on R^d) independent of $\xi\in\Xi_0$ such that for all $0<\varepsilon<1$ and all $\xi\in\Xi_0$ we have a.s. (P_ξ) for all sufficiently large t

$$(1.8)\quad \{x|N(x)\leqq(1-\varepsilon)t\}\subset\{x|\tau(x)\leqq t\}\subset\{x|N(x)\leqq(1+\varepsilon)t\}$$

(it is understood that these sets are subsets of R^d and not merely of Z^d). This result might be called a "strong law of large configurations". It shows that under (M) the set of all sites $x\in R^d$, which are occupied at time t grows in a very regular geometric manner. An easy consequence of (1.8) is (as $t\to\infty$)

(1.9) $\frac{1}{t^d}|\xi_t| \sim L_d\{x|N(x) \leqq 1\}$ a.s. (P_ξ), $\xi \in \Xi_0$

(L_d denoting the d-dimensional Lebesgue measure).

Richardson studied in [12] a stochastic growth process for which he posed five axioms ([12], p. 517), and he showed that they are sufficient for (1.8) to hold in probability. Examining Richardson's proofs more closely, one sees that his first axiom (A1) may be replaced by the axioms

(A1.1) $E\tau_\Delta(x_1+x_2) \leqq E\tau_\Delta(x_1) + E\tau_\Delta(x_2) + o(1)$, $x_1, x_2 \in R^d$,

and (V denoting variance)

(A1.2) $V\tau_\Delta(2x) \leqq 2V\tau_\Delta(x) + o_x(1)$, $x \in R^d$

(we write τ_Δ instead of Richardson's t_Δ). Here and in the sequel $o(1)$ denotes a real function depending on $\Delta > 0$ in such a way that $\lim_{\Delta \downarrow 0} o(1) = 0$ (if $o(1)$ also depends on $\xi \in \Xi$ and $x \in R^d$ (say), we indicate this by writing $o_{\xi,x}(1)$ instead of $o(1)$). In the process studied in the present paper it is not difficult to verify (A1.1) as well as Richardson's axioms (A2)–(A5). Except for (A1.1) the condition (M) is not needed. Instead of verifying (A1.2), we prove (using (M)) another inequality (see (3.28)). The latter allows the application of a result of Kesten [7] and Hammersley ([4], p. 674), which together with a more careful examination of error terms (in proving axiom (A3) of Richardson) yields

(1.10) $\lim_{n\to\infty} \frac{\tau(2^n m x)}{2^n m} = N(x)$ a.s. (P_0), $x \in R^d$, $m = 1, 2, \ldots$

Finally Richardson's reasoning is still applicable to derive from (1.10) the above strong law of large configurations. This, in turn, implies that for all $\xi \in \Xi_0$,

(1.11) $\lim_{\Delta \downarrow 0} \tau_\Delta(x) = N(x)$ a.s. (P_ξ), $x \in R^d$,

which is stronger than the analogous "in probability" result (for $\xi = \{0\}$) due to Richardson.

Mollison studied in [11] certain classes of spread processes in R^d. He looked at the convex hull H_t of the set of all points inhabited at time t, and gave bounds for H_t in terms of the so-called front velocities (compare also Biggins [1] for branching random walks). Finally we would like to mention that the present study was stimulated by thinking about a more complicated process described in Schürger/Tautu [13], [14] (the latter process, in turn, being a generalization of a process introduced in Williams/Bjerknes [15]).

2. Results Not Assuming (M)

In this section certain extensions of the axioms (A2), (A3) and (A4) of [12] are proved. First we introduce some additional notations and terminology. Call a sequence $x_1, \ldots, x_n$ $(n \geqq 1)$ of mutually distinct sites a *chain* (of length n) if $x_i \sim x_{i+1}$, $i=1, \ldots, n-1$. Comparing the process in consideration with a process in which all infection rates are equal to c'', one can show (see Harris [5], p. 973) that for all $\xi \in \Xi_1$

(2.1) $P_\xi(0<\tau(x_1)<\ldots<\tau(x_n) \leqq t) \leqq F_n(t), \quad t>0,$

where

(2.2) $F_n(t)=\dfrac{(c'')^n}{(n-1)!}\int_0^t u^{n-1} e^{-c''u} du, \quad t \geqq 0, \; n=1,2,\ldots,$

is the n-fold convolution of the exponential distribution function with parameter c'' defined by (1.1).

A chain $x_1, \ldots, x_n$ $(n \geqq 1)$ such that $x_1 = x$, is called a *contact chain* for x with respect to $\omega \in \Omega$ ([5], p. 973) if $\tau(\omega, x_1)=0$ in case $n=1$, and

$$0=\tau(\omega, x_n)<\tau(\omega, x_{n-1})<\ldots<\tau(\omega, x_1)<\infty$$

in case $n \geqq 2$. We say that a contact chain $x_1, \ldots, x_n$ for x with respect to ω *reaches* x by (time) $t \geqq 0$ if

$$0=\tau(\omega, x_n)<\tau(\omega, x_{n-1})<\ldots<\tau(\omega, x_1) \leqq t.$$

Let the event $H(x,t)$, $x \in Z^d$, $t>0$, be defined by

$$H(x,t)=\{\text{for some } n \geqq 1 \text{ there exists a contact chain } x_1, \ldots, x_n \text{ reaching } x \text{ by } t\}.$$

It is easy to show $H(x,t) \in \mathscr{M}_t$, $x \in Z^d$, $t>0$. We have ([5], p. 973)

(2.3) $P_\xi(H(x,t) \mid \tau(x) \leqq t)=1, \quad \xi \in \Xi_1, \; x \in Z^d, \; t>0.$

The following lemma shows that axiom (A2) of [12] is also valid for higher moments.

(2.4) **Lemma.** *For all* $C>2c''de$ *and* $m=1,2,\ldots$

(2.5) $E_\xi \tau_\Delta^m(x) \geqq \dfrac{\|x\|^m}{C^m}+o_{\xi,x}(1), \quad \xi \in \Xi_0, \; x \in R^d.$

Proof. Proceed as in the proof of Theorem 3 of [12], p. 524, and use (2.3), (2.1) and (2.2).

Now we show that also axiom (A4) of [12] is valid for higher moments. Put

(2.6) $p(t)=1-e^{-c't}, \quad 0 \leqq t<\infty, \; p=p(1)$

(c' being defined in (1.1)). Comparing our process with one in which all infection rates are equal to c', we easily get

(2.7) **Lemma.** *For all* $x \in Z^d$ *and all* $\xi \in \Xi_1$ *such that* $\xi \cap N_x \neq \emptyset$,

(2.8) $P_\xi(\tau(x) \leqq t) \geqq p(t), \quad 0 \leqq t<\infty.$

Using this result we derive upper bounds for $E_\xi \tau_\Delta^m(x)$, $\xi \in \Xi_1$, $x \in R^d$, $\Delta > 0$, $m = 1, 2, \ldots$ To this end consider any $\xi \in \Xi_1$ and any chain $x_0, x_1, \ldots, x_n$ such that $x_0 \in \xi$ and $x_n \notin \xi$. By Lemma (2.7) we have for $m = 0, 1, 2, \ldots$

(2.9) $$P_\xi(\xi_{m+1}(x_{j+1}) = 1 \mid \xi_m(x_j) = 1) \geqq p, \qquad j = 0, 1, \ldots, n-1.$$

We may now think of an auxiliary particle occupying x_0 at time 0 and displaying the following behaviour given the particle occupies x_j at time m: At time $m+1$ it jumps to x_{j+1} with probability p, or at time $m+1$ it stays at x_j with probability $1-p$ $(0 \leqq j \leqq n-1,\ m \geqq 0)$, where jumps are only possible at times $t = 1, 2, \ldots$ Comparing $\tau(x_n)$ with the first instant at which the particle reaches x_n, and taking into account (2.9) we deduce

(2.10) $$P_\xi(\tau(x_n) \leqq t) \geqq \sum_{0 \leqq j \leqq t-n} \binom{j+n-1}{j} p^n (1-p)^j, \qquad t \geqq 0$$

(void sums being defined as zero). Since negative binomial distributions are convolutions of geometric distributions, (2.10) implies

$$E_\xi \tau^m(x_n) \leqq \frac{m!}{p^m} n^m, \qquad m = 1, 2, \ldots$$

This reasoning (see also [12], p. 525) yields

(2.11) $$E_0 \tau^m(x) \leqq \frac{m!}{p^m} \left(\sum_{i=1}^{d} |x^i| \right)^m, \qquad x \in Z^d,\ m = 1, 2, \ldots,$$

and furthermore

(2.12) **Lemma.** *For $m = 1, 2, \ldots$ and all $\xi \in \Xi_1$ we have*

(2.13) $$E_\xi \tau_\Delta^m(x) \leqq m! \left(\frac{\sqrt{d}}{p} \right)^m \|x\|^m + o_{\xi, x, m}(1), \qquad x \in R^d,$$

and

(2.14) $$E_\xi \tau_\Delta^m(x) \leqq m! \left(\frac{2\sqrt{d}}{p} \right)^m \|x\|^m + o_{\xi, m}(1), \qquad x \in R^d.$$

This shows that for the process in consideration the axiom (A4) of [12] is also valid for higher moments. Another consequence of the above reasoning is given by

(2.15) **Lemma.** *Let $x_0, x_1, \ldots, x_n$ $(n \geqq 1)$ be any chain. Then for all $\xi \in \Xi_1$ such that $x_0 \in \xi$,*

(2.16) $$P_\xi(\tau(x_n) > m) \leqq \sum_{i=0}^{n-1} \binom{m}{i} p^i (1-p)^{m-i}, \qquad m = 0, 1, 2, \ldots$$

It was observed by Richardson (Theorem 4 of [12], p. 524) that his axiom (A3) is a consequence of our next result (2.18). Examining more carefully the function $o(1)$ occurring in (2.18), we arrive at (2.19). This allows the application of the Borel-Cantelli lemma which yields (2.21) being stronger than the axiom (A3) of [12]. Theorem (2.20) roughly says that for all $\xi \in \Xi_1$ we have a.s. (P_ξ) that asymptotically (i.e. for $\Delta \downarrow 0$) $\tau_\Delta(x)$ is uniformly continuous as a function of x.

(2.17) **Lemma.** *There exists a constant $s>0$ such that for all $x, y \in R^d$ and $\xi \in \Xi_1$*

$$(2.18)\quad P_\xi\left(|\tau_\Delta(x)-\tau_\Delta(y)| \leqq \frac{\|x-y\|+\sqrt{\Delta}}{s}\right) \geqq 1-\Delta^d \cdot o(1).$$

For all $x, y \in R^d$ and $\xi \in \Xi_1$ we may choose

$$(2.19)\quad o(1)=\frac{1}{1-p}\left(\frac{1}{e\Delta^{\sqrt{\Delta}}}\right)^{2d/\sqrt{\Delta}}, \qquad 0<\Delta \leqq \frac{1}{d}.$$

Proof. Choose $s=\dfrac{\delta p}{(1-p)\sqrt{d}}$ where $0<\delta \leqq \dfrac{d(1-p)}{d+1}$ is any number for which

$$(1-p)^{(1-p)/(p\delta)} \leqq \frac{\delta}{2}\exp(-3(1+\sqrt{d})).$$

Then apply the strong Markov property, Lemma (2.15) and use estimations similar to those in the proof of Theorem 5 of [12], p. 525 $\left(\text{consider the cases } \|x-y\|<\dfrac{\Delta}{\sqrt{d}} \text{ and } \|x-y\| \geqq \dfrac{\Delta}{\sqrt{d}}\right)$.

Now we can prove

(2.20) **Theorem.** *There exists a constant $r>0$ such that for all $\xi \in \Xi_1$, $z \in R^d$ and $\varepsilon>0$*

$$(2.21)\quad \limsup_{\Delta \downarrow 0} \sup_{x:\, \|x-z\| \leqq r\varepsilon} |\tau_\Delta(x)-\tau_\Delta(z)| \leqq \varepsilon \quad a.s.\ (P_\xi).$$

Proof. Fix $\xi \in \Xi_1$, $z \in R^d$ and $\varepsilon>0$. Put $r=\dfrac{s}{2}$ where s is the constant occurring in Lemma (2.17). It follows from Lemma (2.17) (compare the reasoning in the proof of Theorem 4 of [12], p. 524) that

$$P_\xi\left(\sup_{\frac{1}{n+1} \leqq \Delta \leqq \frac{1}{n}} \sup_{x:\, \|x-z\| \leqq r\varepsilon} |\tau_\Delta(x)-\tau_\Delta(z)| > \frac{(n+1)\varepsilon}{n}\right) = O\left(n^{2d}\left(\frac{1}{e}\right)^{2d\sqrt{n}}\right).$$

Hence the Borel-Cantelli lemma implies (2.21).

3. Results about the Asymptotic Behaviour Valid under (M)

First we prove an inequality (see (3.3) below) containing the axiom (A1.1) (see Introduction) as a special case. Define the translations $t_x\colon \Xi \to \Xi$, $x \in Z^d$, by

$$(3.1)\quad t_x(\xi)(y)=\xi(y-x), \qquad \xi \in \Xi,\ x, y \in Z^d.$$

(3.2) **Lemma.** *Under* (M) *for* $m=1,2,\ldots$ *and* $x,y\in R^d$

(3.3) $$(E_0\,\tau^m(x+y))^{1/m}\leqq(E_0\,\tau^m(x))^{1/m}+(E_0\,\tau^m(y))^{1/m}+\frac{3dm}{2p}.$$

Proof. First we show that for $m=1,2,\ldots$

(3.4) $$E_0(|\tau(x+y)-\tau(x)|^m)\leqq E_0\,\tau^m(y),\qquad x,y\in Z^d.$$

To prove (3.4) fix $x,y\in Z^d$. Then for $m=1,2,\ldots$ we get using the strong Markov property, (2.13), the right continuity of the sample paths as well as (1.6),

$$\begin{aligned}&E_0((\tau(x+y)-\tau(x))^m I_{\{\tau(x+y)>\tau(x)\}})\\&\quad=\int_0^\infty E_0(I_{\{\tau(x+y)>\tau(x)\}}P_{t-x\xi_{\tau(x)}}(\tau^m(y)>t))\,dt\leqq P_0(\tau(x+y)>\tau(x))\,E_0\,\tau^m(y).\end{aligned}$$

Similarly

$$E_0((\tau(x)-\tau(x+y))^m I_{\{\tau(x)>\tau(x+y)\}})\leqq P_0(\tau(x)>\tau(x+y))\,E_0\,\tau^m(-y).$$

Since by reasons of symmetry $E_0\,\tau^m(-y)=E_0\,\tau^m(y)$, addition of the obtained inequalities yields (3.4). Now fix $x,y\in R^d$ and let $x\in Q_{\tilde{x}}$, $y\in Q_{\tilde{y}}$, $x+y\in Q_{\tilde{z}}$. Application of Minkowski's inequality together with (3.4) gives for $m=1,2,\ldots$

$$\begin{aligned}(E_0\,\tau^m(x+y))^{1/m}&=(E_0\,\tau^m(\tilde{z}))^{1/m}\\&\leqq(E_0|\tau(\tilde{z})-\tau(\tilde{z}-\tilde{x})|^m)^{1/m}+(E_0\,\tau^m(\tilde{y}))^{1/m}+(E_0|\tau(\tilde{z}-\tilde{x})-\tau(\tilde{y})|^m)^{1/m}\\&\leqq(E_0\,\tau^m(x))^{1/m}+(E_0\,\tau^m(y))^{1/m}+(E_0\,\tau^m(\tilde{z}-\tilde{x}-\tilde{y}))^{1/m}.\end{aligned}$$

By (2.11)

$$(E_0\,\tau^m(\tilde{z}-\tilde{x}-\tilde{y}))^{1/m}\leqq\frac{m}{p}\sum_{i=1}^d|\tilde{z}^i-\tilde{x}^i-\tilde{y}^i|\leqq\frac{3dm}{2p}$$

which proves (3.3).

Our next result (a consequence of Lemma (3.2)) extends Lemma 4 of [12] (p. 518) and Theorem 1 of [12] (p. 521).

(3.5) **Theorem.** *Under* (M) *for all* $\xi\in\Xi_0$, $x\in R^d$ *and* $m=1,2,\ldots$ *the limit*

(3.6) $$\lim_{\Delta\downarrow 0}(E_\xi\,\tau_\Delta^m(x))^{1/m}=N^{(m)}(x)$$

exists and is independent of $\xi\in\Xi_0$. *We also have for* $m=1,2,\ldots$

(3.7) $$\begin{aligned}N^{(m)}(x)&=\inf_{\Delta>0}\left((E_0\,\tau_\Delta^m(x))^{1/m}+\frac{3dm\Delta}{2p}\right)\\&=\inf_{n\geqq 1}\frac{1}{n}\left((E_0\,\tau^m(n\,x))^{1/m}+\frac{3dm}{2p}\right),\qquad x\in R^d.\end{aligned}$$

For $m=1,2,\ldots$ $N^{(m)}(\cdot)$ *is a norm on* R^d *(hence equivalent to the Euclidean norm), for which*

(3.8) $$\frac{\|x\|}{2c''de} \leqq N^{(m)}(x) \leqq \frac{m\sqrt{d}}{p}\|x\|, \qquad x\in R^d.$$

It will turn out later that $N^{(m)}(\cdot)$ is independent of $m\geqq 1$ (see Theorem (3.31) below).

Proof of Theorem (3.5). Fix $x\in R^d$ and $m\geqq 1$. It follows from (3.3) and (2.13) that the function

$$t\mapsto (E_0\,\tau^m(t\,x))^{1/m} + \frac{3dm}{2p}, \qquad t>0,$$

is subadditive. Hence (3.6) (for $\xi=\{0\}$) and (3.7) follow from a classical theorem on subadditive functions (cf. Hammersley [3]). The norm properties and (3.8) (still in the case $\xi=\{0\}$) are immediate from (3.3), (2.13) and Lemma (2.4). The fact that the limit in (3.6) exists for all $\xi\in\Xi_0$ and is independent of ξ follows from (3.10) and Lemma (3.11) given below.

Let

(3.9) $$\tau_\xi = \inf\{t \mid \xi\subset\xi_t\}, \qquad \xi\in\Xi_0.$$

Clearly τ_ξ is a stopping time, and the reasoning leading to (2.10) shows that (without using (M)) we have for $m=1,2,\ldots$

(3.10) $$\sup_{\eta\in\Xi_1:\,x\in\eta} E_\eta\,\tau_\xi^m < \infty, \qquad \xi\in\Xi_0,\ x\in Z^d.$$

(3.11) **Lemma.** *Under* (M) *we have for all* $\xi\in\Xi_0$, $\eta\in\Xi_1$ *and* $x\in Z^d$

(3.12) $$E_\eta\,\tau^m(x) \leqq \sum_{i=0}^{m}\binom{m}{i}(E_\xi\,\tau^i(x))(E_\eta\,\tau_\xi^{m-i}), \qquad m=1,2,\ldots$$

Using (3.12) together with (3.10) one can easily finish the proof of Theorem (3.5). Furthermore Lemma (3.11) will enable us later to apply an almost sure convergence result due to Kesten (see [7]), which was generalized by Hammersley (in [4], p. 674).

Proof of Lemma (3.11). Fix $\xi\in\Xi_0$, $\eta\in\Xi_1$, $x\in Z^d$ and $m\geqq 1$. Clearly

(3.13) $$E_\eta\,\tau^m(x) \leqq E_\eta\,\tau_\xi^m + \int_0^\infty P_\eta(\tau(x)>t^{1/m},\ \tau_\xi\leqq t^{1/m})\,dt.$$

For the rest of the proof we assume $\xi\not\subset\eta$ (otherwise (3.12) is an immediate consequence of (1.6)). Put (for the moment)

$$G(t)=\begin{cases}P_\eta(\tau_\xi\leqq t), & t\geqq 0\\ 0, & t<0\end{cases} \quad\text{and}\quad \tilde G(t)=\begin{cases}P_\xi(\tau(x)\leqq t), & t\geqq 0\\ 0, & t<0.\end{cases}$$

It follows

(3.14) $G(0)=0.$

Using the strong Markov property. (3.14), the right continuity of sample paths, (1.6) and (3.10), we find for $t>0$ and $n=1,2,\ldots$

$$(3.15)\quad P_\eta(\tau(x)>t,\ \tau_\xi\leqq t)\leqq\sum_{i=1}^{n}\left(G\left(\frac{it}{n}\right)-G\left(\frac{(i-1)t}{n}\right)\right)\left(1-\tilde G\left(t-\frac{it}{n}\right)\right).$$

Letting $n\to\infty$ in (3.15) we get

$$\int_0^\infty P_\eta(\tau(x)>t^{1/m},\ \tau_\xi\leqq t^{1/m})\,dt\leqq\int_0^\infty\int_{[0,t^{1/m}]}(1-\tilde G(t^{1/m}-u))\,G(du)\,dt$$

$$=m\int_0^\infty\int_0^\infty(1-\tilde G(v))\,(u+v)^{m-1}\,dv\,G(du).$$

Integration by parts and (3.13) together yield (3.12).

In the sequel we put

(3.16) $N(x)=N^{(1)}(x),\qquad x\in R^d,$

$N^{(1)}(\cdot)$ being given by (3.6) ($N(\cdot)$ is the norm figuring in (1.8)).

(3.17) **Lemma.** *Under* (M) *for all* $x,y\in R^d$

$$(3.18)\quad |N(x)-N(y)|\leqq\frac{\sqrt{d}}{p}\,\|x-y\|.$$

Proof. Immediate from (3.16), (3.8) and the norm properties of $N(\cdot)$.

Examining the error term $o_x(1)$ in (A 1.2) (see Introduction) more carefully Kesten arrived at an interesting almost sure convergence result in which no assumptions concerning stationarity were made (see [7]). Kesten's result was later generalized by Hammersley ([4], p. 674). Instead of trying to apply these results directly to sequences of the form $\{\tau(nx)|n=1,2,\ldots\}$, $0\neq x\in Z^d$, we observe that the following theorem (which suffices for our purposes) is implicitly contained in Hammersley's proof of his generalization.

(3.19) **Theorem.** *Let* $X_1,X_2,\ldots$ *be a sequence of nonnegative random variables on some probability space, having finite second moments. Assume that*

$$(3.20)\quad \lim_{n\to\infty}\frac{1}{n}EX_n=\alpha\qquad(-\infty\leqq\alpha<\infty)$$

and

$$(3.21)\quad \lim_{n\to\infty}\frac{1}{n^2}EX_n^2=\beta\qquad(0\leqq\beta<\infty)$$

exist. Furthermore let for $k=1,2,\ldots$

(3.22) $V(X_{2k})+E^2(X_{2k})\leqq 2V(X_k)+4E^2(X_k)$

(V denoting variance). Then we have

(3.23) $\beta=\alpha^2$,

(3.24) $\lim_{n\to\infty}\frac{1}{2^n m}X_{2^n m}=\alpha$ *a.s.*, $m=1,2,\ldots,$

and

(3.25) $\lim_{n\to\infty}\frac{1}{n}X_n=\alpha$ *in quadratic mean.*

(Observe that α is finite by (3.21) and (3.23).)

(3.26) *Remark.* If additionally $X_1, X_2, \ldots$ is almost surely a monotone sequence, instead of (3.24) we even have

(3.27) $\lim_{n\to\infty}\frac{1}{n}X_n=\alpha$ a.s.

(see Remark 2 of [4], p. 675). Unfortunately for all $x\in Z^d$ $(x\neq 0)$ $\{\tau(nx)|n=1,2,\ldots\}$ is not a.s. (P_0) monotone. We can overcome this difficulty by first proving our strong law of large configurations for indices of the form $2^n m$ (m being fixed) and then constructing certain random diameters (see (3.40) and (3.44)) which are monotone.

Now we show that Theorem (3.19) is applicable to the sequences $\{\tau(nx)|n=1,2,\ldots\}$, $0\neq x\in Z^d$. To this end fix $x\in Z^d$ $(x\neq 0)$ and k, $l\geqq 1$. Putting $\eta=\{0\}$ and $\xi=\{lx\}$ we get under (M), using (3.12),

(3.28) $E_0\tau^2((k+l)x)\leqq\sum_{i=0}^{2}\binom{2}{i}(E_0\tau^i(kx))(E_0\tau^{2-i}(lx))$

which implies (3.22) if we define $X_n=\tau(nx)$, $n=1,2,\ldots$ It follows from Theorem (3.5) and (2.11) that (under (M)) Theorem (3.19) is applicable to the sequence $X_n=\tau(nx)$, $n=1,2,\ldots$ Hence (3.24) and (3.25) imply

(3.29) $\lim_{n\to\infty}\frac{1}{2^n m}\tau(2^n m x)=N(x)$ a.s. (P_0), $x\in Z^d$, $m=1,2,\ldots$

and

(3.30) $\lim_{n\to\infty}E_0\left(\left|\frac{1}{n}\tau(nx)-N(x)\right|^2\right)=0,$ $x\in Z^d$.

Using (3.30), (3.4), (2.11), (2.13), (3.18) and (1.6) it is not difficult to prove the next result generalizing Lemma 8 of [12] (p. 519).

(3.31) **Theorem.** *Under* (M) *for all* $\xi\in\Xi_0$ *and* $m=1,2,\ldots$ *we have*

(3.32) $\lim_{\Delta\downarrow 0}\tau_\Delta(x)=N(x)$ *in* $L^m(P_\xi)$, $x\in R^d$, *and*

(3.33) $N^{(m)}(\cdot)=N(\cdot)$.

The following theorem (being the almost sure analogon of Lemma 9 of [12], p. 520) shows that the almost sure results obtained so far on all half-lines (see (3.29)) can be nicely combined to give uniform convergence on compact subsets of R^d. More precisely we have

(3.34) **Theorem.** *Under* (M) *for all* $k, m=1,2,\ldots$

(3.35) $$\lim_{n\to\infty} \sup_{x:\, \|x\| \leqq k} \left|\frac{\tau(2^n m x)}{2^n m} - N(x)\right| = 0 \quad a.s.\ (P_\xi), \qquad \xi \in \Xi_0.$$

Proof. Using (3.29), Theorem (2.20) and (3.18) one first shows that

(3.36) $$\lim_{n\to\infty} \frac{\tau(2^n m x)}{2^n m} = N(x) \quad \text{a.s. } (P_0), \qquad x \in R^d,\ m=1,2,\ldots$$

Let us now prove that (3.36) remains valid if P_0 is replaced by any P_ξ, $\xi \in \Xi_0$. First observe that (3.36) together with (1.7) implies that for $m=1,2,\ldots$

(3.37) $$\lim_{l\to\infty} P_\xi(\tau(2^n m x) \leqq (N(x)+\varepsilon)\, 2^n m,\ n \geqq l) = 1, \qquad \varepsilon>0,\ x \in R^d, \text{ if } 0 \in \xi \in \Xi_0.$$

Now fix $x \in R^d$ $(x \neq 0)$ and $m \geqq 1$. Let $\varepsilon>0$. Using the strong Markov property and (1.7) we get for all $\xi \in \Xi_0$ $(0 \notin \xi)$, $j \geqq 4\left(1+\frac{N(x)}{\varepsilon}\right)$ and $l=1,2,\ldots$

$$\begin{aligned} &P_\xi(\tau(2^n m x) \leqq (N(x)+\varepsilon)\, 2^n m,\ n \geqq l) \\ &\quad \geqq \sum_{i=1}^{j} P_\xi\left(\frac{(i-1)(N(x)+\varepsilon)\, 2^l m}{j} < \tau_{\{0\}} \leqq \frac{i(N(x)+\varepsilon)\, 2^l m}{j}\right) \\ &\quad \cdot P_0\left(\tau(2^n m x) \leqq (N(x)+\varepsilon)\left(2^n - \frac{i 2^l}{j}\right) m,\ n \geqq l\right) \\ &\quad \geqq P_\xi\left(\tau_{\{0\}} \leqq \frac{2^l m \varepsilon}{4}\right) P_0\left(\tau(2^n m x) \leqq \left(N(x)+\frac{\varepsilon}{2}\right) 2^n m,\ n \geqq l\right). \end{aligned}$$

Similar calculations (using (1.6)) show that for $0<\varepsilon<\frac{N(x)}{3}$ and $l=1,2,\ldots$

$$\begin{aligned} &P_0(\tau(2^n m x) > (N(x)-\varepsilon)\, 2^n m,\ n \geqq l) \\ &\quad \leqq 2 P_0\left(\tau_\xi > \frac{2^l m \varepsilon}{2}\right) + P_\xi(\tau(2^n m x) \geqq (N(x)-2\varepsilon)\, 2^n m,\ n \geqq l) \quad \text{if } \{0\} \neq \xi \in \Xi_0. \end{aligned}$$

Letting $l \to \infty$ in the above inequalities and using (3.10), (3.36) and (3.37) we get for all $\xi \in \Xi_0$

$$\lim_{n\to\infty} \frac{\tau(2^n m x)}{2^n m} = N(x) \quad \text{a.s. } (P_\xi), \qquad x \in R^d,\ m=1,2,\ldots$$

This together with Theorem (2.20) and (3.18) yields (3.35).

Now we can prove an almost sure version of Richardson's main theorem (Theorem 2 of [12], p. 521).

(3.38) **Lemma.** *Under* (M) *for all* $\xi\in\Xi_0$, $0<\varepsilon<1$ *and* $m=1,2,\ldots$ *we have a.s.* (P_ξ) *for all sufficiently large n*

(3.39) $\{x|N(x)\leqq(1-\varepsilon)2^n m\}\subset\{x|\tau(x)\leqq 2^n m\}\subset\{x|N(x)\leqq(1+\varepsilon)2^n m\}.$

Proof. Use Theorem (3.34), (3.8), (3.33) as well as (2.3) and apply Richardson's reasoning (in the proof of Lemma 10 in [12], p. 520).

In order to prove the strong law of large configurations in full generality, let for $t>0$

(3.40) $D_t(\omega)=\sup\{r|\forall x\in R^d\colon\ N(x)\leqq r \text{ implies } \tau(\omega,x)\leqq t\},\quad \omega\in\Omega.$

Clearly

(3.41) $D_t(\omega)\uparrow$ as $t\uparrow\infty$, $\omega\in\Omega$.

By (3.39) for all $\xi\in\Xi_0$, $0<\varepsilon<1$ and $m=1,2,\ldots$ we have

$$D_{2^n m}\geqq(1-\varepsilon)2^n m \quad \text{a.s. } (P_\xi)$$

for all sufficiently large n implying

(3.42) $$\liminf_{n\to\infty}\frac{1}{2^n m}D_{2^n m}\geqq 1 \quad \text{a.s. } (P_\xi),\qquad \xi\in\Xi_0,\ m=1,2,\ldots$$

But (3.41) and (3.42) together imply

(3.43) $$\liminf_{n\to\infty}\frac{1}{n}D_n\geqq 1 \quad \text{a.s. } (P_\xi),\qquad \xi\in\Xi_0.$$

Similarly let for $t>0$

(3.44) $\tilde D_t(\omega)=\inf\{r|\forall x\in R^d\colon\ \tau(\omega,x)\leqq t \text{ implies } N(x)\leqq r\},\quad \omega\in\Omega.$

Clearly

(3.45) $\tilde D_t(\omega)\uparrow$ as $t\uparrow\infty$, $\omega\in\Omega$.

By (3.39) and (3.45)

(3.46) $$\limsup_{n\to\infty}\frac{1}{n}\tilde D_n\leqq 1 \quad \text{a.s. } (P_\xi),\qquad \xi\in\Xi_0.$$

Since obviously $D_t(\omega)\leqq\tilde D_t(\omega)$, $t>0$, $\omega\in\Omega$, it follows from (3.43), (3.46), (3.41) and (3.45)

(3.47) $$\lim_{t\to\infty}\frac{1}{t}D_t=\lim_{t\to\infty}\frac{1}{t}\tilde D_t=1 \quad \text{a.s. } (P_\xi),\qquad \xi\in\Xi_0.$$

But (3.47) is equivalent to

(3.48) **Theorem.** *Under* (M) *for all* $\xi \in \Xi_0$ *and* $0<\varepsilon<1$ *we have a.s.* (P_ξ) *for all sufficiently large* t

(3.49) $$\{x|N(x) \leqq (1-\varepsilon)\, t\} \subset \{x|\tau(x) \leqq t\} \subset \{x|N(x) \leqq (1+\varepsilon)\, t\}.$$

This is the desired strong law of large configurations in full generality (compare the corresponding "in probability" result of Richardson (Theorem 2 of [12])).

Let us formulate three simple consequences of Theorem (3.48). Denote by $\mathrm{co}(\xi)$, $\xi \in \Xi$, the convex hull (in R^d) of ξ. For $A \subset R^d$ and $t>0$ let $t \cdot A = \{t\,x | x \in A\}$. Then we derive from Theorem (3.48) the following weaker result:

(3.50) **Theorem.** *Under* (M) *for all* $\xi \in \Xi_0$ *and* $0<\varepsilon<1$ *we have a.s.* (P_ξ) *for all sufficiently large* t

(3.51) $$\{x|N(x) \leqq 1-\varepsilon\} \subset \frac{1}{t} \cdot \mathrm{co}(\xi_t) \subset \{x|N(x) \leqq 1+\varepsilon\}.$$

(3.52) **Theorem.** *Under* (M) *for all* $\xi \in \Xi_0$ *we have*

(3.53) $$\lim_{\Delta \downarrow 0} \tau_\Delta(x) = N(x) \quad \text{a.s. } (P_\xi), \qquad x \in R^d.$$

Finally we have (L_d denoting the d-dimensional Lebesgue measure)

(3.54) **Theorem.** *Under* (M) *for all* $\xi \in \Xi_0$ *we have, as* $t \to \infty$,

(3.55) $$\frac{1}{t^d}|\xi_t| \sim L_d\{x|N(x) \leqq 1\} \quad \text{a.s. } (P_\xi).$$

Proof. Fix $\xi \in \Xi_0$ and $0<\varepsilon<1$. By Theorem (3.48) we have a.s. (P_ξ), as $t \to \infty$,

$$\begin{aligned} |\xi_t| &= \#\{x|x \in Z^d,\ \tau(x) \leqq t\} \\ &\leqq \#\{x|x \in Z^d,\ N(x) \leqq (1+\varepsilon)\, t\} \sim L_d\{x|N(x) \leqq (1+\varepsilon)\, t\} \\ &= (1+\varepsilon)^d\, t^d\, L_d\{x|N(x) \leqq 1\}. \end{aligned}$$

Since $0<\varepsilon<1$ was arbitrary, we get

$$\limsup_{t \to \infty} \frac{1}{t^d}|\xi_t| \leqq L_d\{x|N(x) \leqq 1\} \quad \text{a.s. } (P_\xi).$$

The desired converse inequality is proved similarly.

4. Open Problems

An interesting question concerns the shape of the convex set $\{x|N(x) \leqq 1\}$. Is it under (M) a circle (say) or does its shape depend on the infection rates $c_1, \ldots, c_{2d}$?

We wonder what can be said about the asymptotic geometrical behaviour of ξ_t in case (M) is violated.

Let (for the sake of simplicity) $d=2$ and define the *crinkliness* (proposed in Mollison [10]) of $\xi\in\Xi_0$ by

$$(4.1)\quad C(\xi)=\frac{1}{4\sqrt{|\xi|}}\,\#\{(x,y)|x\in\xi,\ y\sim x,\ y\notin\xi\},\quad \xi\in\Xi_0$$

(observe that square arrays have crinkliness 1 and that all other configurations in Ξ_0 have crinkliness >1). The following conjecture was formulated in Schürger/Tautu [14] (for a different process though).

(4.2) **Conjecture.** *Let $d=2$. Under* (M) *there exists an absolute constant $c\geq 1$ depending only on the infection rates c_i such that for all $\xi\in\Xi_0$*

$$(4.3)\quad \lim_{t\to\infty} C(\xi_t)=c \quad a.s.\ (P_\xi).$$

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Note Added in Proof. In a paper entitled "On the asymptotic geometrical behaviour of percolation processes" (to appear), the author has shown recently that a strong law of large configurations also holds for percolation processes on Z^d, satisfying a weak moment condition.