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Critical Dimension for a Model of Branching in a Random Medium

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Summary. A critical branching random walk in a d-dimensional spatial random medium (environment) is considered. It is said to be "persistent" if there is no loss of particle intensity in the large time limit. A critical dimension d_c is shown to exist such that the system is persistent if $d>d_c$ and fails to be persistent if $d<d_c$.

1. Introduction

Random walks in a spatial random medium have been investigated by a number of authors including Dobrushin, Kesten, Kozlov, Sinai and Spitzer (cf. [13] for references to the recent literature). Examples of branching random walks in a random medium which determines the offspring distribution were introduced in Dawson and Fleischmann [1]. The objective of this paper is to obtain criteria for "persistence" (called "stability" in earlier papers) for a general random medium model starting with an initial Poisson random field. Persistence occurs if the tendency to local extinction of critical branching is offset by the spatial dispersion of the infinitely many particles. It is shown that there is a critical dimension d_c where the behavior switches, that is, if the dimension $d>d_c$ then the system is persistent whereas if $d<d_c$ then it is not. The results are proved for random walks in the domain of attraction of a stable law and for a large class of ergodic random media. The model includes the "classical" branching random walk, that is, a medium which is non-random and constant. But even in the latter case the result is of independent interest. The ideas and methods introduced in the paper [6] of Kallenberg in the context of the classical branching random walk play an important role in the present approach.

2. Preliminaries

Let N denote the set of all counting measures defined on the d-dimensional ($d \geqq 1$) lattice Z^d, that is, non-negative measures such that $\phi(x)=\phi(\{x\})$ is in-

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teger-valued for all $x \in Z^d$. Equipped with the vague topology, N is a Polish space. The Dirac measure δ_y concentrated at $y \in Z^d$ is interpreted as a *particle* situated at y. Hence each ϕ in N can be interpreted as a system of particles. We adopt the convention that if a topological space is denoted by the boldface Roman capital letter A, then the corresponding script letter $\mathscr{A}$ always refers to the σ-field of Borel subsets of A.

Let D denote the set of all probability distributions on $(N, \mathscr{N})$. Furnished with the weak topology, D is a Polish space. The mapping $x \to \delta_x$ associates with each distribution v on Z^d a distribution Q_v on D describing a one particle system with random position distributed by v.

A *clustering mechanism* (often called a "cluster field") is a mapping $x \to \kappa_{(x)}$ from Z^d into D. For a given clustering mechanism κ and $\phi \in M$ we introduce the notation

$$\kappa_{(\phi)} := \mathop{\ast}_{x \in Z^d} (\kappa_{(x)})^{*\phi(x)}$$

where $\kappa_{(x)}^{*k}$ denotes the k-fold convolution of $\kappa_{(x)}$, provided that the infinite convolution exists in D (cf. [10, Sect. 2.1]). This means that each particle at each $x \in Z^d$ is replaced (independently) by a random number of particles distributed via $\kappa_{(x)}$ and that $\kappa_{(\phi)}$ is given by the superposition of these. We interpret $\kappa_{(x)} = \kappa_{(\phi)}$ for $\phi = \delta_x$ as the distribution of a random particle system generated by a particle δ_x and $\kappa_{(\phi)}$ as the distribution of the *progeny* of ϕ if all particles in ϕ are clustered independently according to κ.

The mapping $\phi \to \kappa_{(\phi)}$ is a measurable mapping from $(N, \mathscr{N})$ to $(D, \mathscr{D})$ (cf. [10; 4.1.3]). Thus if ϕ is a random element in N with distribution $P \in D$, then for $B \in \mathscr{N}$,

$$P_\kappa(B) = \int \kappa_{(\phi)}(B)\, P(d\phi)$$

yields the distribution of the progeny of ϕ (provided that the integral makes sense). A distribution $P \in D$ is said to be *cluster invariant* if P_κ exists and coincides with P.

In the following we shall use the translation group $(T_z : z \in Z^d)$ (actually three isomorphic groups denoted by the same symbol) defined by

$$T_z h(x) = h(x+z),\ h \text{ a function on } Z^d,\ x \in Z^d,$$

$$T_z \mu(x) = \mu(x-z),\ \mu \text{ a measure on } Z^d,\ x \in Z^d,$$

$$T_z H(B) = H(\{\phi \in N : T_z \phi \in B\}),\ H \text{ a measure on } N,\ B \in \mathscr{N}.$$

3. Branching Random Walks in a Spatially Varying Medium

From now on let λ be a fixed non-degenerate distribution on Z^d, that is, $\lambda \neq \delta_z$ for any $z \in Z^d$. λ is called the *displacement distribution*. The following hypothesis concerning λ is used in the statement of the main results.

Hypothesis D.1. *Assume that the displacement distribution λ belongs to the domain of attraction of a truly d-dimensional stable distribution defined on R^d with exponent $0 < \beta \leqq 2$. In other words, there exist constants a_n and $c_n > 0$ such that* *the normalized sums*

$$c_n\left(\sum_{i=1}^{n}\xi_i - a_n\right) \tag{3.1}$$

of independent, identically distributed random variables ξ_i with common distribution λ converge in distribution and that the limiting stable distribution is not concentrated on a $(d-1)$-dimensional hyperplane. (Note that the value $\beta=2$ corresponds to the case of a limiting normal distribution.)

Lemma 3.1. (a) *Under Hypothesis D.1 the normalizing sequences $\{c_n : n=1,2,3,\dots\}$ are regularly varying at infinitely with index $(-1/\beta)$, that is,*

$$\lim_{n\to\infty} c_{[tn]}/c_n = t^{-1/\beta}, \quad 0<t<1,$$

where $[x]$ denotes the largest integer less than or equal to x.

(b) *If λ satisfies Hypothesis D.1, then the sequence*

$$\{c_n^{-d}\sup_x \lambda^{*n}(x) : n=1,2,3,\dots\}$$

is bounded away from zero and infinity.

Proof. (a) Refer to Siegmund-Schultze [12: Lemma 5.2].

(b) Refer to [12: Theorem 3.1].

λ is said to belong to the *normal domain of attraction* if the normalizing constants c_n in (3.1) satisfy:

$$c_n \sim \text{const}\, n^{-1/\beta}, \quad \text{as } n\to\infty, \tag{3.2}$$

(cf. Feller [2: 17.6]). (Here and below "const" denotes a positive constant which may vary from formula to formula.)

Let F denote the set of all families $f=\{f_x : x\in Z^d\}$ of critical offspring generating functions f_x,

$$f_x(s)=\sum_{n=0}^{\infty} p_x(n)s^n, \quad 0\leqq s\leqq 1,\ f_x'(1-)=1$$

corresponding to distributions p_x on $Z^+=\{0,1,2,\dots\}$ with first moment $\sum_{n=0}^{\infty} p_x(n)n=1$. We interpret f as a spatially varying medium. The smallest σ-field of subsets of F such that for each $x\in Z^d$ and $s\in[0,1]$ the mapping $f\to f_x(s)$ is measurable, is denoted by $\mathscr{F}$.

For each $f\in F$ we introduce the clustering mechanism

$$\kappa^f_{(x)} = f_x(T_x Q_\lambda) = \sum_{n=0}^{\infty} p_x(n)(T_x Q_\lambda)^{*n}, \quad \text{for } x\in Z^d.$$

Consequently, a particle situated at x generates with probability $p_x(n)$ exactly n particles which are displaced independently and spatially homogeneously according to λ.

Let $\Omega_0 = N^{Z^+}$ and $\Phi = \{\Phi_t : t \in Z^+\}$ denote the canonical process

$$\Phi_t(\omega) = \omega(t) \qquad \text{for } \omega \in \Omega_0.$$

$\mathscr{F}_0$ is defined to be the smallest σ-algebra generated by $\{\Phi_t, t \in Z^+\}$. Let $f \in F$. A probability measure P_f on Ω_0 defines a branching random walk in the spatially varying medium f with Poisson initial field if under P_f:

(a) Φ_t is Markov with transition function $\phi \to \kappa^f_{(\phi)}(d\psi)$, and

(b) Φ_0 is a homogeneous Poisson random field with intensity one.

Let P^t_f be the probability measure on N defined by $P^t_f(A) = P_f(\Phi_t \in A)$. In view of the Poisson initial field, P^t_f is an infinitely divisible measure on N for $t \geqq 0$.

Lemma 3.2. *For each $f \in F$ the probability measures P^t_f converge weakly as $t \to \infty$ to some limit P^∞_f which is cluster invariant and satisfies:*

$$\int_N \Phi(x) P^\infty_f(d\Phi) \leqq 1 \qquad \text{for all } x \in Z^d.$$

Proof. Refer to Liemant [8].

The branching random walk P_f (for a given $f \in F$) is said to be *persistent* (called "stable" in earlier papers) if $\int_N \Phi(x) P^\infty_f(d\Phi) = 1$ for each $x \in Z^d$.

Given a spatially varying cluster mechanism κ^f we need to determine whether or not it yields a persistent branching random walk. To accomplish this we next formulate a generalization of a persistence criterion due to Kallenberg [6] for the case of a spatially homogeneous branching random walk.

For each $f \in F$ let

$$g^f_x(s) = 1 - f'_x(1-s), \qquad x \in Z^d,\ 0 \leqq s \leqq 1,$$

where f'_x denotes the derivative if f_x. The function $g^f_x(\cdot)$ is monotone increasing, convex and satisfies $g^f_x(0) = 0$. Consequently,

$$a g^f_x(s) \leqq g^f_x(as), \qquad 0 \leqq as \leqq s \leqq 1, \tag{3.3}$$

or equivalently,

$$g^f_x(as) \leqq a g^f_x(s), \qquad 0 \leqq s \leqq as \leqq 1. \tag{3.4}$$

Let $\Omega_W = (Z^d)^{Z^+}$ and $\zeta = \{\zeta_t : t \in Z^+\}$ denote the canonical process $\zeta_t(\omega) = \omega(t)$ for $\omega \in \Omega_W$. Let W_z denote the probability law on Ω_W of a random walk governed by the "reflected" displacement distribution $\lambda^-(x) = \lambda(-x)$ starting at $\zeta_0 = z \in Z^d$.

Theorem 3.1. *Let $f \in F$. Then the branching random walk P_f in the spatially varying medium f is persistent if and only if for all $y, z \in Z^d$,*

$$\sum_{t=1}^{\infty} g^f_{\zeta_t}(\lambda^{*t}(y - \zeta_t)) < \infty, \tag{3.5}$$

holds almost surely with respect to W_z.

The proof of Theorem 3.1 is postponed to Sect. 7.

4. The Random Medium

Let $\Omega = F \times \Omega_0$. For $\omega = (f, \omega_0)$, $f \in F$, $\omega_0 \in \Omega_0$, $t \in Z^+$, $f_x(\omega) = f_x$ and $\Phi_t(\omega) = \omega_0(t)$. The branching random walk in the random medium is prescribed the probability measure P_M on Ω defined by

$$P_M(A \times B) = \int_A P_f(B) M(df), \qquad A \in \mathscr{F},\ B \in \mathscr{F}_0.$$

We introduce three hypotheses on the random medium which are referred to in the statement of the results below.

Hypothesis M.1. *Assume that the probability measure M is ergodic with respect to the translation group $\{T_z : z \in Z^d\}$.*

Hypothesis M.2. *Let $f = \{f_x : x \in Z^d\}$ be a family of independent, identically distributed random variables with respect to P_M.*

Hypothesis M.3. *Assume that the branching is non-degenerate, in the sense that*

$$P_M(\{f_0(s) \equiv s\}) < 1.$$

Proposition 4.1. *Under Hypothesis M.1 the set*

$$\{f \in F\colon\ P_f \text{ is persistent}\} \tag{4.1}$$

has P_M-probability either zero or one.

Proof. By Theorem 3.1

$$\begin{aligned} A &= \{f \in F : P_f \text{ is persistent}\} \\ &= \left\{ f \in F : W_z\left[\zeta : \sum_{t=1}^{\infty} g_t^f(\lambda^{*t}(y - \zeta_t)) < \infty\right] = 1 \text{ for all } y,\ z \in Z^d \right\}. \end{aligned}$$

Since $g_z^{T_x f} = g_{z+x}^f$, it follows that the event A is invariant, that is, $A = T_x A$. Therefore by Hypothesis, $P_M(A)$ is either zero or one. □

Thus there exists a dichotomy between those ergodic random media which yield persistent branching random walks and those which do not. In this paper we obtain a characterization of those random media which yield persistence subject to Hypothesis M.2.

The persistence or non-persistence of a branching random walk in an ergodic random medium depends both on the displacement distribution λ and the random medium. We next obtain a necessary consequence of persistence.

Let $°\lambda$ denote the "symmetrized distribution $\lambda^- * \lambda$. It is said to be *transient* if the random walk on Z^d governed by $°\lambda$ is transient.

Proposition 4.2. *Assume that the random medium satisfies the hypotheses M.2 and M.3 and yields persistence. Then the symmetrized displacement distribution $°\lambda$ is transient.*

Proof. For $\omega = (\omega_1, \omega_2) \in \Omega_W \times \Omega_W$ and $t \in Z^+$ let $\zeta(t) = \omega_1(t)$, $\zeta'(t) = \omega_2(t)$ and let $W_{00}^2 = W_0 \times W_0$ (product measure). Under W_{00}^2, ζ and ζ' are independent random walks starting at 0 and governed by the reflected distribution λ^-.

Lemma 4.3. *If the symmetrized distribution* $^\circ\lambda$ *is recurrent, then the random subset of* Z^d *defined by*

$$\Xi = \{x \in Z^d\text{: there exists } t \text{ for which } \zeta_t = \zeta'_t = x\}$$

is infinite with W^2_{00}*-probability one.*

Proof. For k a positive integer we introduce the stopping times:

$$\tau_0 = 0, \quad \tau_{n+1} = \min\{t \geqq \tau_n + 2k : \zeta_t = \zeta'_t\}, \quad n = 1, 2, \ldots.$$

By the recurrence of $^\circ\lambda$, $W^2_{00}(\tau_n < \infty) = 1$ for each n. Since λ is nondegenerate we can choose $x \neq 0$ such that $\lambda(-x) > 0$. Then

$$W^2_{00}(|\zeta_{\tau_1}| \geqq 2k) \geqq W^2_{00}(\zeta_{2k} = \zeta'_{2k} = 2kx) \geqq (\lambda(-x))^{4k} > 0,$$

and therefore,

$$W^2_{00}(|\zeta_{\tau_1}| < 2k) < 1. \tag{4.2}$$

Therefore $\{\zeta_{\tau_{n+1}} - \zeta_{\tau_n} : n \in Z^+\}$ is a family of independent, identically distributed random variables which satisfy

$$W^2_{00}(|\zeta_{\tau_{n+1}} - \zeta_{\tau_n}| < 2k) < 1.$$

Therefore

$$W^2_{00}(|\zeta_{\tau_{n+1}} - \zeta_{\tau_n}| < 2k \text{ for all } n \in Z^+) = 0,$$

and consequently

$$W^2_{00}(|\zeta_{\tau_n}| < k \text{ for all } n \in Z^+) = 0.$$

Therefore,

$$W^2_{00}(\{\{x : \zeta_{\tau_n} = x \text{ for some } n\} \text{ is finite}\}) \leqq \sum_{k=1}^{\infty} W^2_{00}(|\zeta_{\tau_n}| < k \text{ for all } n \in Z^+) = 0.$$

This completes the proof of the lemma.

Completion of the Proof of Proposition 4.2. Assume that Φ^f is persistent P_M-almost surely. By Theorem 3.1

$$\sum_{t=1}^{\infty} g^f_{\zeta_t}(\lambda^{*t}(-\zeta_t)) < \infty, \quad W_0\text{-almost surely.}$$

Then taking note of (3.3) this implies that

$$\sum_{t=1}^{\infty} \lambda^{*t}(-\zeta_t)\, g^f_{\zeta_t}(1) < \infty, \quad W_0\text{-almost surely.} \tag{4.3}$$

Since

$$\lambda^{*t}(-\zeta_t) = W^2_{00}(\zeta_t = \zeta'_t | \sigma(\zeta_t\text{: } t \in Z^+)),$$

(4.3) implies that

$$\sum_{t=1}^{\infty} 1_{\{0\}}(\zeta_t - \zeta'_t)\, g^f_{\zeta_t}(1) < \infty, \quad W^2_{00} \times P_M\text{-almost surely,}$$

where $1_{\{0\}}(\cdot)$ denotes the indicator function of the point 0. Hence

$$\sum_{x \in \Xi} g_x^f(1) < \infty, \quad W_{00}^2 \times P_M\text{-almost surely,} \tag{4.4}$$

where Ξ is the random set introduced in Lemma 4.3. If $°\lambda$ is recurrent and Hypothesis M.2 is satisfied, then (4.4) is a sum of infinitely many non-negative, independent and identically distributed random variables. But such a series can be finite only if each term is zero, that is,

$$g_0^f(1) = 1 - f_0'(0) = 0, \quad P_M\text{-almost surely.}$$

But this contradicts Hypothesis M.3 and therefore we must conclude that $°\lambda$ is transient. This completes the proof of Proposition 4.2. □

For a given displacement distribution which is transient the system can turn out to be either persistent or non-persistent depending on the tail behavior of the offspring distributions. To investigate this dependence we introduce the following hypothesis.

Hypothesis M.4. *Let* $0 < \alpha \leqq 1$. *Assume that*

$$E_M g_0^f(s) \textit{ is regularly varying at zero with index } \alpha.$$

(A measurable positive function R defined on a non-empty interval $(0, s_0)$ is said to be *regularly varying* at zero with index $-\infty < \alpha < \infty$ if

$$\lim_{s \downarrow 0} R(cs)/R(s) = c^\alpha, \; 0 < c < 1.)$$

Hypothesis M.4 covers a large class of random critical offspring distributions. For the sake of illustration we include an example which was introduced in [1].

Example 4.4. Let $0 < \alpha \leqq 1$ and set

$$f_K(s) = (1 - K^{-1}) + K^{-1} s^K, \quad 0 \leqq s \leqq 1,$$

where K is a random positive integer not identical to 1. If $\alpha = 1$, let K have a finite expectation. If $0 < \alpha < 1$, the tail probabilities are assumed to satisfy

$$P(K > k) \sim \text{const}\, k^{-\alpha} \quad \text{as } k \to \infty.$$

Then the function $g_0^f(s) = 1 - f_K'(1-s)$ satisfies Hypothesis M.4, more precisely,

$$E_K g_0^f(s) \sim \text{const}\, s^\alpha \quad \text{as } s \to 0. \tag{4.6}$$

Another family of examples is obtained by taking a random mixture of the offspring distribution given by $E_K f_K$ with one which has a finite variance.

Example 4.5. Let $0 < \alpha < 1$. For each $C > 0$,

$$1 - f_C(1-s) = (C\alpha + s^{-\alpha})^{-1/\alpha}, \quad 0 \leqq s \leqq 1,$$

defines a critical offspring generating function f_C for which the corresponding offspring distribution does not have a finite second moment (cf. Zolotarev [14]). If C is chosen to be a positive random variable with finite expectation then f_C satisfies Hypothesis M.4.

5. The Critical Dimension

In this section we assume that the displacement distribution belongs to the domain of attraction of a truly d-dimensional stable distribution of index β (Hypothesis D.1) and that the random medium f is ergodic (Hypothesis M.1) and is such that $g_0^f(s)$ is regularly varying at zero with index α (Hypothesis M.4).

The persistence or non-persistence of the branching random walk can depend on the indices $0<\alpha\leqq 1$, $0<\beta\leqq 2$ and the dimension d. The positive integer d_c is said to be the *critical dimension* if the branching random walk is persistent when $d>d_c$ and is non-persistent when $d<d_c$. (We also show by example that the system can be either persistent or non-persistent when $d=d_c$.) The main objective of this section is to prove that $d_c=[\beta/\alpha]$.

Theorem 5.1. *Assume that λ satisfies Hypothesis D.1 and that the random medium satisfies Hypotheses M.1 and M.4. Then*

(a) *P_f is persistent for P_M-almost every f provided that $d>\beta/\alpha$;*

(b) *if in addition the random medium satisfies Hypothesis M.2, then the persistence of P_f for P_M-almost every f implies that $d\geqq\beta/\alpha$.*

Proof. (a) Assume that $d>\beta/\alpha$ and choose θ such that $d/\beta>\theta>1/\alpha$. Then by Lemma 3.1.b we have

$$M_t=\sup_{x\in Z^d}\lambda^{*t}(x)\leqq \text{const}\, c_t^d \quad \text{for } t\geqq 1.$$

By Lemma 3.1.a it can be shown that (cf. [2: Lemma 8.8.2]),

$$c_t^d\leqq \text{const}\, t^{-\theta}, \text{ and therefore,}$$

$$M_t\leqq \text{const}\, t^{-\theta}. \tag{5.1}$$

Choose ρ such that $\theta\alpha>\rho>1$. Then for fixed ζ and y and all sufficiently large t we obtain

$$\begin{aligned} E_M g_{\zeta_t}^f(\lambda^{*t}(y-\zeta_t)) &= E_M g_0^f(\lambda^{*t}(y-\zeta_t)) \\ &\leqq E_M g_0^f(M_t) \quad \text{(since } g \text{ is monotone increasing)} \\ &\leqq E_M(g_0^f(\text{const}\, t^{-\theta})) \quad \text{by (5.1),} \\ &\leqq \text{const}\, t^{-\rho} \quad \text{(by Hypothesis M.4).} \end{aligned}$$

Since $\sum_{t=1}^{\infty} t^{-\rho}<\infty$, this implies that for each ζ and y

$$\sum_{t=1}^{\infty} g_{\zeta_t}^f(\lambda^{*t}(y-\zeta_t))<\infty \quad P_M\text{-almost surely.}$$

Thus, for all y and z, $W_z \times P_M$-almost surely, the finiteness condition (3.5) is satisfied. Therefore Proposition 3.4 implies that the branching random walk P_f is persistent for P_M-almost every f and the proof of (a) is complete.

(b) Assume that the random medium satisfies Hypothesis M.2 and that P_f is persistent for P_M-almost every f. Then by Proposition 4.2 the symmetrized displacement distribution $°\lambda$ is transient. By Theorem 3.1

$$\sum_{t=1}^{\infty} g^f_{\zeta_t}(\lambda^{*t}(-\zeta_t)) < \infty, \quad W_0 \times P_M\text{-almost surely.} \tag{5.2}$$

For each $t \in Z^+$, let

$$A_t = \{\zeta \in \Omega_W \colon \zeta_{t'} \neq \zeta_t \text{ for all } t' > t\}.$$

Then (5.2) yields

$$\sum_{t=1}^{\infty} 1_{A_t}(\zeta) g^f_{\zeta_t}(\lambda^{*t}(-\zeta_t)) < \infty, \quad W_0 \times P_M\text{-almost surely.} \tag{5.3}$$

Consider a fixed ζ. Then the sum which appears in (5.3), for values of t for which $1_{A_t}(\zeta) = 1$, the $g^f_{\zeta_t}$ are independent, identically distributed, non-negative, and bounded random variables. We can then conclude (cf. Kolmogorov's Three Series Theorem) that

$$\sum_{t=1}^{\infty} 1_{A_t}(\zeta)(E_M g^f_{\zeta_t})(\lambda^{*t}(-\zeta_t)) < \infty, \quad W_0\text{-almost surely.}$$

Then by Hypothesis M.4, we can conclude that for any $\alpha' > \alpha$,

$$\sum_{t=1}^{\infty} 1_{A_t}(\zeta)(\lambda^{*t}(-\zeta_t))^{\alpha'} < \infty, \quad W_0\text{-almost surely.} \tag{5.4}$$

Let $\eta_t = \lambda^{*t}(-\zeta_t)$. Then $\eta_t \leqq M_t$ where M_t is as defined in the proof of (a). Moreover,

$$E_{W_0} \eta_t = °\lambda^{*t}(0) \geqq \text{const}\, M_{2t} \qquad \text{for all } t \in Z^+.$$

where E_{W_0} denotes expectation with respect to W_0.

The last inequality follows from a lemma of Esseen and Enger (cf. [6: p. 20, formula (9)]). On the other hand by Lemma 3.1.b, $M_{2t} \geqq \text{const}\, M_t$ for $t \geqq 1$. Therefore there is a $0 < \delta < 1/2$ such that

$$E_{W_0} \eta_t \geqq 2\delta M_t \qquad \text{for } t \geqq 1.$$

Hence

$$(1-\delta) M_t W_0(M_t - \eta_t \geqq (1-\delta) M_t) \leqq E_{W_0}(M_t - \eta_t) \leqq (1-2\delta) M_t,$$

and therefore

$$W_0(\eta_t > \delta M_t) \geqq 1 - (1-2\delta)(1-\delta)^{-1} \geqq \delta, \qquad \text{for all } t \geqq 1.$$

Therefore for each positive integer t_0,

$$\sum_{t=1}^{t_0} \delta(\delta M_t)^{\alpha'} \leqq \sum_{t=1}^{t_0} W_0(\eta_t > \delta M_t)(\delta M_t)^{\alpha'} \leqq \sum_{t=1}^{t_0} E_{W_0}[(\eta_t)^{\alpha'}].$$

Note that $E_{W_0} 1_{A_t}(\zeta) = W_0(A_0) = r > 0$ by the transience of $^\circ\lambda$. Since $1_{A_t}(\zeta)$ and η_t are independent,

$$E_{W_0}\left(\sum_{t=1}^{t_0} 1_{A_t}(\zeta)(\eta_t)^{\alpha'}\right) = r \sum_{t=1}^{t_0} E_{W_0}((\eta_t)^{\alpha'}) \geqq r\,\delta^{1+\alpha'} \sum_{t=1}^{t_0} M_t^{\alpha'}. \tag{5.5}$$

Let k be a positive integer and note that

$$\sum_{t=1}^{t_0} 1_{A_t}(\zeta)(\eta_t)^{\alpha'} \leqq \sum_{t=1}^{t_0} M_t^{\alpha'}.$$

Therefore,

$$E_{W_0}\left(\sum_{t=1}^{t_0} 1_{A_t}(\zeta)(\eta_t)^{\alpha'}\right) \leqq k + W_0\left(\sum_{t=1}^{t_0} 1_{A_t}(\zeta)(\eta_t)^{\alpha'} > k\right) \sum_{t=1}^{t_0} (M_t)^{\alpha'}. \tag{5.6}$$

Then (5.5) and (5.6) imply that

$$W_0\left(\sum_{t=1}^{t_0} 1_{A_t}(\zeta)(\eta_t)^{\alpha'} > k\right) \geqq r\,\delta^{1+\alpha'} - k\left[\sum_{t=1}^{t_0} (M_t)^{\alpha'}\right]^{-1}. \tag{5.7}$$

Assume for the moment that $\sum_{t=1}^{\infty} (M_t)^{\alpha'} = \infty$. Letting $t_0 \to \infty$ and then $k \to \infty$ in (5.7) we obtain

$$W_0\left(\sum_{t=1}^{\infty} 1_{A_t}(\zeta)(\eta_t)^{\alpha'} = \infty\right) \geqq r\,\delta^{1+\alpha'} > 0$$

which contradicts (5.4). Thus we must conclude that

$$\sum_{t=1}^{\infty} (M_t)^{\alpha'} < \infty. \tag{5.8}$$

Let $\theta > d/\beta$. Then by (5.8) and Lemma 3.1 we obtain

$$\sum_{t=1}^{\infty} t^{-\theta\alpha'} < \infty.$$

In other words, $\theta\alpha' > 1$ for every $\theta > d/\beta$ and $\alpha' > \alpha$. But this implies that $d\alpha/\beta \geqq 1$, that is, $d \geqq \beta/\alpha$, and the proof of (b) is complete. □

We next demonstrate that under additional assumptions the persistence of the branching random walk is equivalent to the condition $d > \beta/\alpha$.

Theorem 5.2. *Assume that in addition to the hypotheses of Theorem* 5.1 *the following two conditions are satisfied:*

(i) λ *belongs to a normal domain of attraction, and*

(ii) $E g_0^f(s) \sim \text{const}\, s^\alpha$, *as* $s \to 0$.

Under these assumptions the branching random walk is persistent if and only if $d > \beta/\alpha$.

Proof. Assume that the branching random walk is persistent. Then proceeding as in the proof of Theorem 5.1 we obtain

$$\sum_{t=1}^{\infty} 1_{A_t}(\zeta)(\lambda^{*t}(-\zeta_t))^{\alpha} < \infty, \quad W_0\text{-almost surely}, \tag{5.9}$$

and

$$\sum_{t=1}^{\infty} (M_t)^{\alpha} < \infty. \tag{5.10}$$

Since λ is in a normal domain of attraction,

$$c_t^d \sim \text{const}\, t^{-d/\beta} \quad \text{as } t \to \infty.$$

Therefore by Lemma 3.1.b,

$$M_t \geqq \text{const}\, t^{-d/\beta} \quad \text{for } t \geqq 1. \tag{5.11}$$

Consequently,

$$\sum_{t=1}^{\infty} t^{-\alpha d/\beta} < \infty,$$

and therefore $d > \beta/\alpha$. □

6. The Classical Medium

In this section we specialize the results of the previous section to the classical medium, that is, a spatially homogeneous and non-random medium.

Let h be a critical offspring generating function and assume that

$$P_M(f_x = h \text{ for all } x) = 1. \tag{6.1}$$

We also impose the following hypothesis.

Hypothesis C.1. *Assume that the critical offspring generating function h has a representation of the form:*

$$h(s) = s + R_h(1-s), \quad 0 \leqq s \leqq 1, \tag{6.2}$$

where R_h is a regularly varying function at zero with index $\gamma := 1 + \alpha$.

Consequently, under Hypothesis C.1, the critical offspring distribution determined by h belongs to the domain of attraction of a stable distribution (defined on the real line) with exponent γ. In the case $\gamma = 2$, all critical offspring distributions with positive finite variance are covered.

Theorem 6.1. *Assume that the displacement distribution λ satisfies Hypothesis D.1 and that a classical medium satisfies Hypothesis C.1. Then the branching random walk is persistent if $d > \beta/\alpha$ whereas its persistence implies that $d \geqq \beta/\alpha$.*

Lemma 6.1. *Let R be a regularly varying function at zero with index γ. Moreover assume that R has a monotone derivative R'. Then*

$$\lim_{r \downarrow 0} s R'(s)/R(s) = \gamma.$$

Proof. See Lamperti [7].

Proof of Theorem 6.1. By (6.2) we have

$$h(1-s)-(1-s)=R_h(s),$$

and

$$g^h(s)=R_h'(s)=1-h'(1-s), \qquad 0 \leqq s \leqq 1,$$

is monotone. Therefore by Lemma 6.1,

$$g^h(s) \sim (1+\alpha) R_h(s)/s \qquad \text{as } s \to 0. \tag{6.3}$$

Consequently, g^h is regularly varying with index α. Therefore Hypothesis M.4 is satisfied and Theorem 6.1 follows from Theorem 5.1. □

Theorem 6.2. *Assume that in addition to the hypotheses of Theorem* 6.1, *the displacement distribution λ belongs to a normal domain of attraction and that the function R_h satisfies:*

$$R_h(s) \sim \text{const}\, s^{1+\alpha}, \qquad \textit{as } s \to 0. \tag{6.4}$$

Then the persistence of the branching random walk implies that $d > \beta/\alpha$.

Proof. The asymptotic relations (6.3) and (6.4) imply that

$$g^h(s) \sim \text{const}\, s^\alpha, \qquad s \to 0, \text{ that is,}$$

condition (ii) of Theorem 5.2. The Theorem then follows from Theorem 5.2. □

We next give an example to illustrate the fact that it is possible for a branching random walk to be persistent in the case $d=\beta/\alpha$ when the hypotheses of Theorem 6.2 are not satisfied even in the case of a classical medium.

Example 6.3. Consider a classical medium in which the offspring distribution has positive finite variance, hence $\alpha=1$. By a well-known persistence criterion (cf. [10: Theorem 12.6.4]) P_f is persistent if ${}^\circ\lambda$ is transient. Let $d=1$. Then in order to exhibit a classical medium in which the branching random walk is persistent and for which $\beta=1$, it suffices to obtain a symmetric distribution λ which is transient and which also belongs to the domain of attraction of the Cauchy distribution. The following construction of such a distribution is related to the work of Siegmund-Schultze [11].

For some $\varepsilon>0$ the function

$$k_0(x)=\exp(-|x| \log^2 |x|), \qquad |x|<\varepsilon$$

is convex on $(0,\varepsilon)$. Choosing ε sufficiently small, there is a continuation of k_0 to a symmetric function k_1 on R^1 with

$$k_1(0)=1, \qquad k_1(x)=0 \quad \text{for all } x \geqq \pi,$$

and such that k_1 is convex on $(0,\infty)$. Then by Polya's criterion (cf. [2: Example 15.3.b]), k_1 is the characteristic function of a distribution on R^1. Then the periodic continuation, k, of $k_1|_{[-\pi,\pi]}$ to R^1, is also the characteristic function of a symmetric distribution (cf. [2: 19.5 Theorem 2]) λ on Z^1. Then

$$1-k(x)\sim x\log^2 x \quad \text{as } x\downarrow 0,$$

and therefore

$$\int_0^{\frac{1}{2}}(1-k(x))^{-1}\,dx<\infty.$$

But the latter implies that λ is transient ([2: 18.7, (6.7)]). On the other hand,

$$n[1-k(x(n\log^2 n)^{-1})]\to x \quad \text{as } n\to\infty \text{ for } x>0.$$

Therefore,

$$[k(x(n\log^2 n)^{-1})]^n\to e^{-|x|} \quad \text{for all } x\in R^1.$$

Consequently λ belongs to the domain of attraction of the standard Cauchy distribution on R^1. Hence we have constructed λ with the desired properties.

7. Proof of Theorem 3.1.

7.1. Preliminaries

Let ν be a symmetric probability distribution on Z^d. Then

$$(TQ_\nu)_{(x)}=T_x Q_\nu, \quad x\in Z^d$$

defines a clustering mechanism TQ_ν which describes independent displacements according to ν.

Fix a varying medium $f\in F$. For each $t=0,1,2,\dots,\infty$, let Φ_t^ν denote the random counting measure obtained from Φ_t if all the particles in Φ_t are displaced independently according to ν. In other words, the distribution of Φ_t^ν is given by P_f^t clustered with respect to TQ_ν.

By Lemma 3.1, P_f^t converges in distribution to P_f^∞ as $t\to\infty$. Therefore by a standard continuity theorem for clustering (cf. [10: Prop. 4.7.3]) we conclude that

$$\Phi_t^\nu\to\Phi_\infty^\nu \text{ in distribution as } t\to\infty. \tag{7.1}$$

where the distribution of Φ_∞^ν is obtained by P_f^∞ clustered with respect to TQ_ν. Note that the expectations

$$E_f\Phi_t(x)=E\Phi_t^\nu(x)=1 \quad \text{for } t=0,1,2,\dots, \text{ and } x\in Z^d,$$

and

$$E_f\Phi_\infty(x)=1 \text{ for all } x\in Z^d \text{ if and only if } E\Phi_\infty^\nu(x)=1 \text{ for all } x\in Z^d.$$

Therefore the branching random walk is persistent if and only if

$$E\Phi_t^\nu(x)\to E\Phi_\infty^\nu(x) \quad \text{as } t\to\infty, \text{ for all } x\in Z^d. \tag{7.2}$$

7.2. A Reformulation of Persistence

Since Φ_t^ν is an infinitely divisible random counting measure, it has a cluster representation and this will be used to reformulate the persistence criterion. In this subsection we review some relevant facts concerning the cluster representation, canonical measure and Palm distributions. A systematic exposition of these topics can be found in [5] or [10].

If Φ is an arbitrary infinitely divisible random counting measure on Z^d, then it has the *cluster representation* (cf. [5: Theorem 6.1 and Lemma 6.5]):

$$\Phi = \int \mu \Xi_L(d\mu)$$

where Ξ_L is a Poisson random measure on $N \backslash \{0\}$ with intensity measure L where L is a uniquely determined measure on $N \backslash \{0\}$ satisfying $\int (1 - e^{-\mu(B)}) L(d\mu) < \infty$, for every finite set $B \subset Z^d$. L is called the canonical measure of Φ.

Consider the family of probability distributions $\{L_x\}$ on $(N, \mathcal{N})$ defined by

$$L_x(M) = \frac{\int_M \mu(\{x\}) L(d\mu)}{\int_N \mu(\{x\}) L(d\mu)}, \qquad x \in Z^d, \ M \in \mathcal{N}.$$

provided that $0 < \int_N \mu(\{x\}) L(d\mu) < \infty$.

L_x is referred to as the *Palm distribution* at x of the canonical measure of Φ. For convenience we introduce random counting measures $(\tilde{\Phi})_x$ distributed according to L_x.

The random counting measures Φ_t^ν are infinitely divisible and satisfy (7.1). Therefore, according to [5: Lemma 10.8] the convergence statement (7.2) hence persistence is equivalent to

$$(\tilde{\Phi}_t^\nu)_z \to (\tilde{\Phi}_\infty^\nu)_z \text{ in distribution as } t \to \infty, \text{ for all } z \in Z^d.$$

Now assume for the moment the weaker condition, namely

$$(\tilde{\Phi}_t^\nu)_z \text{ converge in distribution as } t \to \infty$$

for all $z \in Z^d$ in the sense of weak convergence in D. Combined with the fact

$$\lim_{t \to \infty} E\Phi_t^\nu(x) = 1 \qquad \text{for all } x \in Z^d$$

we conclude (cf. [5: Exercise 10.7])

$$E\Phi_\infty^\nu(x) = 1 \qquad \text{for all } x \in Z^d,$$

that is, the persistence of P_f. Thus we have derived the following lemma.

Lemma 7.2.1. *The branching random walk in a spatially varying environment P_f is persistent if and only if*

$$(\tilde{\Phi}_t^\nu)_z \textit{ converge in distribution as } t \to \infty, \textit{ for all } z \in Z^d.$$

7.3. Kallenberg's Backward Tree Method

In this subsection we apply Kallenberg's (cf. [6]) backward tree method to identify the distribution of $(\tilde{\Phi}_t^\nu)_z$.

We begin with an informal description of the ideas. Consider an augmented branching model in which all family relations are recorded and which therefore yields a family tree. Consider a particle in Φ_t^ν "chosen at random" and assume that this particle, called *ego*, is located at $z\in Z^d$. The ego $\psi_0=\delta_z$ has *brothers* described by a counting measure ψ_1, that is, particles which have the same *father* as ego. Moreover it has *cousins* described by a counting measure ψ_2, etc. It turns out that $(\tilde{\Phi}_t^\nu)_z$ can be identified with $(\psi_0+\psi_1+\psi_2+\ldots+\psi_t)$.

We now formalize the construction of the sequence of counting measures $\psi_0, \psi_1, \psi_2, \ldots$. Let $\zeta=\{\zeta_t: t\in Z^+\}$ be a random walk governed by λ^- but starting at ζ_0 which is distributed according to $T_z\nu$. We denote by W_z^ν the probability measure on Ω_W associated with ζ. Note that

$$W_z^\nu=\sum_{y\in Z^d}\nu(y)\,W_{z+y} \tag{7.3}$$

where W_x denotes the probability measure associated with the random walk ζ with $\zeta_0=x$.

For $x\in Z^d$, let δ_{δ_x} denote the Dirac measure concentrated at the particle δ_x. We introduce a notation for repeated clustering according to κ^f as follows:

$$\kappa_{(x)}^{f[0]}=\delta_{\delta_x},$$

$$\kappa_{(x)}^{f[t+1]}=(\kappa_{(x)}^{f[t]})_{\kappa^f} \quad \text{for } t=0,1,2,\ldots \text{ and } x\in Z^d.$$

The Palm distribution of $\kappa_{(x)}^f$ at $y\in Z^d$ is given by

$$(\kappa_{(x)}^f)_y=\delta_{\delta_y}*f_x'(T_xQ_\lambda).$$

Now let ξ_j denote the location of ego's ancestor $j\geqq 1$ generations back. Then $f_{\xi_j}'(T_{\xi_j}Q_\lambda)$ is the distribution of that ancestor's children except the one situated at ξ_{j-1}. Then after $(j-1)$ fold-clustering according to κ^f as well as independent ν-displacements, we obtain ego's relatives of order j, denoted by ψ_j. For a given sequence $\{\xi_j\}$, the $\psi_0, \psi_1, \psi_2, \psi_3, \ldots$ form a sequence of independent random counting measures. If we assume that the ξ_j are distributed according to the measure W_z^ν, then the joint distribution of $\{\xi_j, \psi_j\}$ is given by a probability measure V_z defined on $(Z^d\times N)^{Z^+}$ which is given by:

$$V_z(d[\xi,\psi])=W_z^\nu(d\xi)\,\delta_{\delta_z}(d\psi_0)\prod_{j=1}^{\infty}((f_{\xi_j}'(T_{\xi_j}Q_\lambda))_{\kappa^{f[j-1]}})_{TQ_\nu}(d\psi_j).$$

Lemma 7.3.1. *For all $z\in Z^d$ and $t\in Z^+$, the distribution of $(\tilde{\Phi}_t^\nu)_z$ is given by*

$$V_z(\psi_0+\psi_1+\ldots+\psi_t\in\cdot).$$

Proof. Refer to Liemant [9: Satz 8.2].

Lemma 7.3.2. *P_f is persistent if and only if for all $z\in Z^d$, $V_z(\psi_0+\psi_1+\ldots+\psi_t\in\cdot)$ converges as $t\to\infty$.*

Proof. This follows from Lemmas 7.2.1 and 7.3.1.

7.4. A Further Reformulation of Persistence

Lemma 7.4.1. *P_f is persistent if and only if for all y and $z \in Z^d$,*

$$\sum_{t=1}^{\infty} g^f_{\zeta_t}(B_t(\zeta_t; \{\mu \in N: \mu(y)>0\})) < \infty, \quad W^\nu_z\text{-almost every } \zeta, \tag{7.4}$$

where

$$B_t(x; M) = ((T_x Q_\lambda)_{\kappa^{f[t-1]}})_{TQ_\nu}(M) \quad \text{for } x \in Z^d,\ M \in \mathcal{N}.$$

Proof. The partial sums $\psi_1 + \ldots + \psi_t$ are monotone increasing in t. Thus the convergence assertion in Lemma 7.3.2 is equivalent to

$$V_z\left(\sum_{t=0}^{\infty} \psi_t(y) < \infty\right) = 1 \quad \text{for all } y, z \in Z^d. \tag{7.5}$$

This follows since almost sure convergence implies weak convergence whereas unboundedness contradicts weak relative compactness. But (7.5) is equivalent to

$$V_z\left(\sum_{t=0}^{\infty} \psi_t(y) < \infty \,|\, \mathcal{F}_\xi\right) = 1, \quad V_z\text{-almost surely},$$

where $\mathcal{F}_\xi = \sigma\{\xi_t: t \geqq 0\}$, or equivalently,

$$V_z(\psi_t(y) > 0 \text{ infinitely often } |\mathcal{F}_\xi) = 0, \quad V_z\text{-almost surely}.$$

Conditioned on the σ-field $\mathcal{F}_\xi$, the events $\{\psi_t(y) > 0\}$ form a sequence of independent events. Therefore by the Borel-Cantelli lemma (7.5) is equivalent to

$$\sum_{t=0}^{\infty} V_z(\psi_t(y) > 0 \,|\, \mathcal{F}_\xi), \quad V_z\text{-almost surely}.$$

But from the definition of V_z, there is a regular conditional distribution which for a given sequence ξ gives: (using again $\{\mu(y) > 0\}$ as a shorthand for $\{\mu \in N: \mu(y) > 0\}$)

$$\begin{aligned} V_z(\psi_t(y) > 0 \,|\, \{\xi\}) &= (f^t_{\xi_t}(T_{\xi_t} Q_\lambda))_{\kappa^{f[t-1]}})_{TQ_\nu}(\{\mu(y) > 0\}) \\ &= 1 - f^t_{\xi_t}(1 - ((T_{\xi_t} Q_\lambda)_{\kappa^{f[t-1]}})_{TQ_\nu}(\{\mu(y) > 0\})) \\ &= g^f_{\xi_t}(B_t(\xi_t; \mu(y) > 0\})). \end{aligned}$$

This completes the proof of Lemma 7.4.1. □

7.5. Proof of Sufficiency

For a random counting measure, ψ, we have $P(\psi(y) > 0) \leqq E\psi(y)$. Therefore,

$$\begin{aligned} ((T_x Q_\lambda)_{\kappa^{f[t-1]}})_{TQ_\nu}(\{\mu(y) > 0\}) &\leqq T_x \lambda * \lambda^{*(t-1)} * \nu(y) \\ &= \lambda^{*t} * \nu(y - x), \end{aligned} \tag{7.6}$$

where we have used the fact that $\kappa^f_{(x)}$ corresponds to a critical offspring distribution. If we set $\nu = \delta_0$, then by the monotonicity of g^f_x,

$$g_x^f(B_t(x;\{\mu(y)>0\}))\leqq g_x^f(\lambda^{*t}(y-x)). \tag{7.7}$$

If condition (3.5) is satisfied, then (7.7) and Lemma 7.4.1 immediately imply that P_f is persistent.

7.6. Proof of Necessity

We now assume that P_f is persistent and that ν is a symmetric probability measure on Z^d with $\nu(x)>0$ for all $x\in Z^d$.

For $x\in Z^d$ let $B_t(x)\in D$ be defined as in the statement of Lemma 7.4.1. If $\{\zeta_t: t\in Z^+\}$ is a sequence in Z^d, then for each $t\in Z^+$ let χ_t^ζ be a random counting measure whose probability distribution is given by $B_t(\zeta_t)$. Then we have (cf. (7.6)),

$$E\chi_t^\zeta(y)=\int\mu(\{y\})B_t(\zeta_t;d\mu)=\lambda^{*t}*\nu(y-\zeta_t).$$

But by assumption on ν, $\lambda^{*t}*\nu(y-\zeta_t)>0$. Therefore the Palm distribution at y of $B_t(\zeta_t)$, denoted by $(B_t(\zeta_t))_y$ can be computed by:

$$(B_t(\zeta_t))_y=V_y\left(\sum_{i=0}^{t-1}\psi_i\in\cdot\,\middle|\,\{\xi_t=\zeta_t\}\right) \quad \text{(cf. [9: Satz 8.2]).} \tag{7.8}$$

Note that in (7.8) $\{\xi_t=\zeta_t\}$ denotes the event

$$\{\xi\in\Omega_W: \xi_t=\zeta_t\}\in\mathscr{F}_\xi.$$

Actually, this event is in $\mathscr{F}_\xi^{t+}=\sigma\{\xi_s: s\geqq t\}$. We also define

$$\mathscr{F}_\xi^\infty=\bigcap_{t=1}^\infty\mathscr{F}_\xi^{t+}.$$

We note that by the Hewitt-Savage zero-one law every event in $\mathscr{F}_\xi^\infty$ has W_z^ν and therefore V_z probability either zero or one. Using this fact together with (7.5) we can conclude that persistence implies that for each $y\in Z^d$,

$$V_y\left(\sum_{i=0}^\infty\psi_i(y)<\infty\,|\,\mathscr{F}_\xi^\infty\right)=1, \quad V_y\text{-almost surely.} \tag{7.9}$$

Lemma 7.6.1. *Let* y, $z\in Z^d$. *Then*

$$\lim_{k\to\infty}\liminf_{t\to\infty}(B_t(\zeta_t))_y(\{\mu: \mu(y)\leqq k\})=1, \quad \textit{for } W_z^\nu \textit{ almost every } \zeta. \tag{7.10}$$

Proof. Let $A_k=\{\mu: \mu(y)\leqq k\}$. By (7.8),

$$\begin{aligned}(B_t(\zeta_t))_y(A_k)&=V_y\left(\sum_{i=0}^{t-1}\psi_i(y)\leqq k\,|\,\{\xi_t=\zeta_t\}\right)=V_y\left(\sum_{i=0}^{t-1}\psi_i(y)\leqq k\,|\,\{\xi_s=\zeta_s, s\geqq t\}\right)\\&=V_y\left(\sum_{i=0}^{t-1}\psi_i(y)\leqq k\,|\,\mathscr{F}_\xi^{t+}\right)(\zeta)\\&\qquad\text{(using the regular conditional probability)}\\&\geqq V_y\left(\sum_{i=0}^{\infty}\psi_i(y)\leqq k\,|\,\mathscr{F}_\xi^{t+}\right)(\zeta).\end{aligned} \tag{7.11}$$

Since the σ-fields $\mathscr{F}_\xi^{t+}\downarrow\mathscr{F}_\xi^\infty$,

$$V_y\left(\sum_{i=0}^{\infty}\psi_i(y)\leqq k\,|\,\mathscr{F}_\xi^{t+}\right)(\zeta)\to V_y\left(\sum_{i=0}^{\infty}\psi_i(y)\leqq k\,|\,\mathscr{F}_\xi^{\infty}\right)(\zeta) \tag{7.12}$$

for V_y-almost every ζ, by the reverse martingale convergence theorem.

But by the positivity assumption on v it follows that the measures W_z^v and W_y^v are mutually absolutely continuous. Therefore,

$$\lim_{t\to\infty} V_y\left(\sum_{i=0}^{\infty}\psi_i(y)\leqq k\,|\,\mathscr{F}_\xi^{t+}\right)(\zeta)= V_y\left(\sum_{i=0}^{\infty}\psi_i(y)\leqq k\,|\,\mathscr{F}_\xi^{\infty}\right)(\zeta) \tag{7.13}$$

for W_z^v-almost every ζ. Therefore, by (7.9),

$$\begin{aligned}&\lim_{k\to\infty}\lim_{t\to\infty} V_y\left(\sum_{i=0}^{\infty}\psi_i(y)\leqq k\,|\,\mathscr{F}_\xi^{t+}\right)(\zeta)\\ &\quad=\lim_{k\to\infty} V_y\left(\sum_{i=0}^{\infty}\psi_i(y)< k\,|\,\mathscr{F}_\xi^{\infty}\right)(\zeta)= V_y\left(\sum_{i=0}^{\infty}\psi_i(y)<\infty\,|\,\mathscr{F}_\xi^{\infty}\right)(\zeta)=1,\end{aligned}$$

for W_z^v-almost every ζ. Consequently by (7.11) we conclude

$$\lim_{k\to\infty}\liminf_{t\to\infty}(B_t(\zeta_t))_y(A_k)=1,\quad \text{for } W_z^v\text{-almost every } \zeta. \quad \square$$

Lemma 7.6.2. *For W_z^v-almost every ζ, there exists $k_0(\zeta)$ and $t_0(\zeta)$ such that for all $t\geqq t_0(\zeta)$,*

$$E\chi_t^\zeta(y)\leqq 2k_0(\zeta)\,B_t(\zeta_t;\{\mu\colon \mu(y)>0\})<1.$$

Proof. By Lemma 7.6.1, for given y and W_z^v-almost every ζ, there exists a $k_0(\zeta)$ and $t_0(\zeta)$ such that

$$((B_t(\zeta_t))_y(\{\mu\colon \mu(y)\leqq k_0(\zeta)\})\geqq 1/2 \qquad \text{for } t\geqq t_0(\zeta). \tag{7.14}$$

Recalling that $B_t(\zeta_t)$ is the distribution of $\chi_t^\zeta(y)$ and the definition of the Palm measures $(B_t(\zeta_t))_y$, we obtain

$$(B_t(\zeta_t))_y(\{\mu(y)=i\})=i\,B_t(\zeta_t;\{\mu(y)=i\})/(E\chi_t^\zeta(y)).$$

Then for $t\geqq t_0(\zeta)$,

$$\begin{aligned}k_0(\zeta)\,B_t(\zeta_t;\{\mu(y)>0\})&\geqq\sum_{i=1}^{k_0(\zeta)} i\,B_t(\zeta_t;\{\mu(y)=i\})\\ &=\sum_{i=1}^{k_0(\zeta)}(B_t(\zeta_t))_y(\{\mu(y)=i\})\cdot(E\chi_t^\zeta(y))\\ &=(B_t(\zeta_t))_y(\{\mu(y)\leqq k_0(\zeta)\})\cdot(E\chi_t^\zeta(y))\\ &\geqq 1/2\,E\chi_t^\zeta(y) \quad \text{by (7.14)}.\end{aligned}$$

Since

$$E\chi_t^\zeta(y)=\lambda^{*t}*v(y-\zeta_t), \tag{7.15}$$

we have

$$B_t(\zeta_t;\{\mu(y)>0\})\leqq E\chi_t^\zeta(y)=\lambda^{*t}*\nu(y-\zeta_t)\leqq\sup_x\lambda^{*t}(x)$$

and the last term converges to zero as $t\to\infty$. Therefore, increasing t_0 sufficiently we can insure that $2k_0B_t(\zeta_t;\{\mu(y)>0\})<1$. This completes the proof of the lemma. □

Using Lemma 7.6.2, (7.15) and (3.4) we conclude that for $t>t_0(\zeta)$,

$$g_{\zeta_t}^f(\lambda^{*t}*\nu(y-\zeta_t))\leqq g_{\zeta_t}^f(2k_0(\zeta)B_t(\zeta_t;\{\mu(y)>0\}))\leqq 2k_0(\zeta)g_{\zeta_t}^f(B_t(\zeta_t;\{\mu(y)>0\})). \tag{7.16}$$

By Lemma 7.4.1 persistence implies that

$$\sum_{t=1}^\infty g_{\zeta_t}^f(B_t(\zeta_t;\{\mu(y)>0\})<\infty,\quad \text{for } W_z^\nu\text{-almost every } \zeta.$$

This together with (7.16) yields

$$\sum_{t=1}^\infty g_{\zeta_t}^f(\lambda^{*t}*\nu(y-\zeta_t))<\infty,\quad \text{for } W_z^\nu\text{-almost every } \zeta.$$

Finally, using (3.3),

$$g_{\zeta_t}^f(\lambda^{*t}*\nu(y-\zeta_t))\geqq g_{\zeta_t}^f(\lambda^{*t}(y-\zeta_t)\,\nu(0))\geqq \nu(0)\,g_{\zeta_t}^f(\lambda^{*t}(y-\zeta_t)).$$

Hence we have shown that persistence implies that for W_z^ν-almost every ζ,

$$\sum_{t=1}^\infty g_{\zeta_t}^f(\lambda^{*t}(y-\zeta_t))<\infty. \tag{7.17}$$

Finally note that in view of (7.3) and the positivity assumption on ν, (7.17) is also true for W_z-almost every ζ. This completes the proof of Theorem 3.1. □

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