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On the Convergence of Supercritical General (C-M-J) Branching Processes

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Summary. Convergence in probability of Malthus normed supercritical general branching processes (i.e. Crump-Mode-Jagers branching processes) counted with a general characteristic are established, provided the latter satisfies mild regularity conditions. If the Laplace transform of the reproduction point process evaluated in the Malthusian parameter has a finite '$x \log x$-moment' convergence in probability of the empirical age distribution and more generally of the ratio of two differently counted versions of the process also follow.

Malthus normed processes are also shown to converge a.s., provided the tail of the reproduction point process and the characteristic both satisfy mild regularity conditions. If in addition the '$x \log x$-moment' above is finite a.s. convergence of ratios follow.

Further, a finite expectation of the Laplace-transform of the reproduction point process evaluated in any point smaller than the Malthusian parameter is shown to imply a.s. convergence of ratios even if the '$x \log x$-moment' above equals infinity.

Straight-forward generalizations to the multi-type case are available in Nerman (1979).

1. Introduction

Let us outline the general branching process following Jagers (1975). Assume that a typical individual reproduces at ages according to a random point process ξ on $[0, \infty)$. It is alive during the age interval $[0, \lambda)$ where $0 \leqq \lambda \leqq \infty$ (thus if $\lambda = 0$ the individual is never alive). In general branching processes no particular dependence structure are assumed between ξ and λ.

By L we denote the distribution function of the life length, i.e.

$$L(u) = P[\lambda \leqq u], \tag{1.1}$$

and by $\xi(t)$ the ξ-measure of $[0, t]$, i.e.

$$\xi(t) = \xi([0, t]). \tag{1.2}$$

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$\mu = E[\xi]$ denotes the intensity measure of ξ. We write

$$\mu(t) = E[\xi(t)], \tag{1.3}$$

and call $\mu(t)$ the reproduction function.

Let us make the convention:

In the integral $\int_a^b$, we include a only if a is zero and we exclude b only if b is infinity.

We suppose throughout that:

(i) μ is not (as a measure) concentrated on any lattice $\{0, h, 2h, \ldots\}$, $h > 0$. (All results could be modified to the lattice case.)

(ii) There exists a Malthusian parameter $\alpha \in (0, \infty)$, i.e. a finite positive solution of the equation

$$\int_0^\infty e^{-\alpha t} \mu(dt) = 1. \tag{1.4}$$

(iii) The first moment of $e^{-\alpha t} \mu(dt)$ is finite, i.e.

$$\int_0^\infty u e^{-\alpha u} \mu(du) < \infty. \tag{1.5}$$

We write x for an individual, $x = (i_1, \ldots, i_n)$, if x is the i_n:th child of the i_{n-1}:th child of ... of the i_1:th child of the ancestor, and let 0 denote the ancestor.

$$\mathscr{I} = \{0\} \cup \left(\bigcup_{n=1}^{\infty} \mathscr{I}_n \right), \tag{1.6}$$

where

$$\mathscr{I}_n = \{(i_1, \ldots, i_n);\ i_j \in \{1, 2, \ldots\}\}, \tag{1.7}$$

is to be called the individual space.

The basic probability space is

$$(\Omega, \mathscr{B}, P) = \prod_{x \in \mathscr{I}} (\Omega_x, \mathscr{B}_x, P_x) \tag{1.8}$$

where $(\Omega_x, \mathscr{B}_x, P_x)$ are identical spaces on which we define (ξ_x, λ_x) distributed like (ξ, λ).

We let σ_x stand for the birth time of x:

$$\begin{aligned} &\sigma_0 = 0, \quad \text{and if } x = (x', i) \\ &\sigma_x = \sigma_{x'} + \inf\{u;\ \xi_{x'}(u) \geqq i\}. \end{aligned} \tag{1.9}$$

(Thus, for an individual x never born, by convention, $\sigma_x = \infty$.)

In the sequel we shall associate with each individual x random entities defined on $(\Omega_x, \mathscr{B}_x, P_x)$ (and by extension also on the product space $(\Omega, \mathscr{B}, P)$). Such entities will for short be introduced without indices x just like ξ and λ

were. Similarly restrictions will sometimes be formulated for an anonymous typical individual.

Suppose that besides the reproduction ξ and the life length λ, we assume the existence of a product-measurable, separable, non-negative random process $\phi(t)$, assigning some kind of score to the typical individual at age t (alive or not!). For simplicity we define $\phi(t)$ for all $t \in R$, but require that

$$\phi(t)=0 \quad \text{for } t<0. \tag{1.10}$$

We then define (cf. Jagers (1975), Sect. 6.9)

$$Z_t^\phi = \sum_{x \in \mathscr{I}} \phi_x(t-\sigma_x), \tag{1.11}$$

and say that $\{Z_t^\phi\}_t$ is a general branching process counted with characteristic ϕ. Since Z_t^ϕ is linear in ϕ generalizations to not necessarily nonnegative ϕ:s could easily be made at several places in the sequel. Observe that we require independence of ϕ_x between individuals, but that dependence of $\phi_x, \lambda_x, \xi_x \ldots$ is allowed for fixed x, (for a relaxation of this requirement cf. Sect. 7).

If

$$\phi(t)=\begin{cases} 1 & \text{if } 0 \leqq t < \inf(a,\lambda) \\ 0 & \text{otherwise,} \end{cases} \tag{1.12}$$

then Z_t^ϕ counts the number of individuals alive at time t whose ages are less than a. We denote this particular $Z_t^\phi = Z_t^a$ and name $\{Z_t^a\}_{t,a}$ the age process. Denote $Z_t = Z_t^\infty$. Also T_t = the total number of births up to and including time t has a simple characteristic representation:

$$\phi(t)=1 \quad \text{if } t \geqq 0. \tag{1.13}$$

For other interesting examples cf. Jagers (1975) and Härnqvist (1981).

We shall study the asymptotics of $e^{-\alpha t} Z_t^\phi$, as $t \to \infty$. Special interest will be focused on ages, i.e. on the process $e^{-\alpha t} Z_t^a$, and we give age results explicitly. Moreover, we shall be interested in the empirical age distribution at time t, Z_t^a/Z_t, and generally in ratios $Z_t^{\phi_1}/Z_t^{\phi_2}$, for pairs ϕ_1, ϕ_2 of characteristics. We formulate the results for a fixed age a, but because of the monotonicity of Z_t^a/Z_t as a function of a, it is possible to deduce functional variants of our age distribution results.

Let us make a brief historical sketch. Doney (1972), (1976) showed

Proposition 1.1. *As $t \to \infty$, $e^{-\alpha t} Z_t$ converges in distribution to a random variable W, and with the definition*

$${}_\alpha\xi(t) = \int_0^t e^{-\alpha u}\, \xi(du), \tag{1.14}$$

the following sequence of equivalences holds

$$\begin{aligned} &E[{}_\alpha\xi(\infty) \log^+ {}_\alpha\xi(\infty)] < \infty. \Leftrightarrow \\ &E[W] > 0. \Leftrightarrow \\ &E[e^{-\alpha t} Z_t] \to E[W], \quad \text{as } t \to \infty. \Leftrightarrow \\ &W > 0 \quad \text{a.s. on } \{T_t \to \infty\}. \quad \square \end{aligned} \tag{1.15}$$

For Bellman-Harris processes with finite Malthusian parameter

$$\begin{aligned} &E[{}_\alpha\xi(\infty)\log^+{}_\alpha\xi(\infty)]<\infty. \Leftrightarrow \\ &E[\xi(\infty)\log^+\xi(\infty)]<\infty. \end{aligned} \tag{1.16}$$

However, generally not even finiteness of $\xi(\infty)$ follows from the finiteness of $E[{}_\alpha\xi(\infty)\log^+{}_\alpha\xi(\infty)]$. For Bellman-Harris processes Proposition 1.1 is due to Athreya (1969).

The results for Galton-Watson processes in Kesten and Stigum (1966) suggest the possibility of strengthening the weak convergence in Proposition 1.1 to a.s. convergence. For Bellman-Harris processes this was done by Athreya and Kaplan (1976). Their method starts from studying Z_t^a/Z_t, the age distribution at time t on the set $\{T_t\to\infty\}$. With the help of a strong law of large numbers they show the convergence

$$\frac{Z_t^a}{Z_t}\to\frac{\int_0^a e^{-\alpha u}(1-L(u))\,du}{\int_0^\infty e^{-\alpha u}(1-L(u))\,du},\qquad \text{a.s., as } t\to\infty, \tag{1.17}$$

provided $E[\xi(\infty)\log^+\xi(\infty)]<\infty$, and then use a certain martingale, the so called reproductive value martingale, to connect it with the convergence of $e^{-\alpha t}Z_t^a$. Further, they demonstrate convergence in probability of the age distribution, even if $E[\xi(\infty)\log^+\xi(\infty)]=\infty$, assuming mild conditions on the life span distribution L. Later Athreya and Kaplan (1978) refine their results somewhat and recently (independent of this paper) Kuczek (1980) shows that a.s. convergence for the empirical age-distribution always holds for supercritical Malthusian Bellman-Harris processes.

For the general case Savits (1975) used a quite different method, convergence of the bivariate Laplace transform of $(e^{-\alpha t}Z_t^a, e^{-\alpha t}Z_t)$ to show convergence in probability of Z_t^a/Z_t, provided $E[{}_\alpha\xi(\infty)\log^+{}_\alpha\xi(\infty)]$ and $\mu(\infty)$ are both finite.

K. Rama-Murthy (1978) adopted the Athreya-Kaplan method and modified it to the general case. However, this leads to conditions in form of uniform restrictions on all conditional residual reproductions and life lengths, given the history of an individual up to any age.

In the next section we shall give some elementary results and present our key tool, a martingale, related to the reproductive value martingale. In Sect. 3 this new martingale is combined with a weak law of large numbers, in the spirit of Athreya and Kaplan (1976), to show that $e^{-\alpha t}Z_t^\phi$ converges in probability, as $t\to\infty$, under weak restrictions on ϕ. As a corollary to this, for ϕ_1 and ϕ_2 restricted like ϕ, we also show convergence in probability, on $\{T_t\to\infty\}$, of the ratio $Z_t^{\phi_1}/Z_t^{\phi_2}$, provided $E[{}_\alpha\xi(\infty)\log^+{}_\alpha\xi(\infty)]$ is finite.

Section 4 will contain a strong law of large numbers with a probabilistic proof and for the ease of reference Lévy's generalized Borel-Cantelli theorem both results which we need in Sects. 5 and 6.

In Sect. 5 we show that $e^{-\alpha t} Z_t^\phi$ converges a.s., as $t \to \infty$, under weak restrictions on ϕ, provided there exists a non-increasing integrable function g such that

$$E[\sup_t ({}_\alpha\xi(\infty) - {}_\alpha\xi(t))/g(t))] < \infty. \tag{1.18}$$

We also show a.s. convergence on $\{T_t \to \infty\}$, of the ratio $Z_t^{\phi_1}/Z_t^{\phi_2}$ provided that (1.18) holds and $E[{}_\alpha\xi(\infty) \log^+ {}_\alpha\xi(\infty)]$ is finite.

In Sect. 6 we shall show that if there exists a $\beta < \alpha$, such that $E[{}_\beta\xi(\infty)]$ is finite, then a.s. ratio-convergence holds generally.

Finally, Sect. 7 is an addendum concerning a generalization of all results to random characteristics ϕ_x permitted to depend on the whole daughter process of the individual x.

2. Two Basic Results and a Crucial Martingale

It is a fundamental idea in the analysis of branching processes to split the process in a sum of a stochastic number of time translated copies of the original process plus a residual. The following general variant we quote from Jagers (1975):

Proposition 2.1. *It holds that*

$$Z_t^\phi = \phi_0(t) + \sum_{i=1}^{\xi_0(t)} {}_{(i)}Z_{t-\sigma(i)}^\phi, \tag{2.1}$$

where $\{{}_xZ_t^\phi\}_t$ *is the general* ϕ*-counted process of* x*-descendants initiated by the assumption that* x *is born at* 0. *The processes* $\{{}_{(i)}Z_t^\phi\}_t$ $i = 1, 2, \ldots$ *are independent copies of* $\{Z_t^\phi\}_t$, *also independent of* ξ_0 *and* ϕ_0. □

We write

$$m_t^\phi = E[e^{-\alpha t} Z_t^\phi], \tag{2.2}$$

and

$$m_t^a = E[e^{-\alpha t} Z_t^a]. \tag{2.3}$$

Since ϕ is non-negative, we conclude easily that m_t^ϕ satisfies the renewal equation

$$m_t^\phi = e^{-\alpha t} E[\phi(t)] + \int_0^t m_{t-s}^\phi e^{-\alpha s} \mu(ds). \tag{2.4}$$

Let μ_α be the measure on $[0, \infty)$ defined by

$$\mu_\alpha(t) = \int_0^t e^{-\alpha s} \mu(ds). \tag{2.5}$$

Then (cf. Jagers (1975), Theorem 6.9.2).

Proposition 2.2. *Suppose that $E[\phi(t)]$ is continuous a.e. with respect to Lebesgue measure, and that*

$$\sum_{k=0}^{\infty} \sup_{k \leq t \leq k+1} (E[\phi(t)]\, e^{-\alpha t}) < \infty. \tag{2.6}$$

Then

$$m_t^{\phi} \to m_{\infty}^{\phi} = \frac{\int_0^{\infty} e^{-\alpha t} E[\phi(t)]\, dt}{\int_0^{\infty} u \mu_{\alpha}(du)}, \quad \text{as } t \to \infty. \tag{2.7}$$

In particular

$$m_t^{a} \to m_{\infty}^{a} = \frac{\int_0^{a} e^{-\alpha t} (1 - L(t))\, dt}{\int_0^{\infty} u \mu_{\alpha}(du)}, \quad \text{as } t \to \infty. \tag{2.8}$$

Proof. The proposition follows from the renewal theorem (e.g. Jagers (1975), Theorem 5.2.6) if we show that $E[Z_t^{\phi}]$ is bounded on finite intervals.

Consider t in some interval $[0, s]$. Positivity of terms implies that

$$E[Z_t^{\phi}] = \sum_x E[\phi_x(t - \sigma_x)]. \tag{2.9}$$

Since ϕ_x and σ_x are independent for each fixed x, we deduce

$$E[Z_t^{\phi}] \leq (\sup_{t \leq s} E[\phi(t)])\, E[T_s]. \tag{2.10}$$

Due to (2.6) and the finiteness of $E[T_s]$ (Jagers (1975), Theorem 6.3.3), $E[Z_t^{\phi}]$ is bounded, and hence Proposition 2.2 holds. □

Suppose that $x_1, x_2, x_3, \ldots$ are the first, second, third, ... individual born in the process, so that

$$0 = \sigma_{x_1} \leq \sigma_{x_2} \leq \ldots.$$

If several births take place at the same point of time we order the individuals at first hand with respect to generation and at second hand with respect to some arbitrary rule.

Recall ${}_{\alpha}\xi_x(t)$ from (1.14). We define

$$\begin{aligned} &R_0 = 1, \quad \text{and} \\ &R_n = 1 + \sum_{i=1}^{n} e^{-\alpha\sigma_{x_i}} ({}_{\alpha}\xi_{x_i}(\infty) - 1), \quad \text{for } n = 1, 2, \ldots, \end{aligned} \tag{2.11}$$

rewrite

$$R_n = 1 + \sum_{i=1}^{n} \sum_{k=1}^{\xi_{x_i}(\infty)} e^{-\alpha\sigma(x_i, k)} - \sum_{i=1}^{n} e^{-\alpha\sigma_{x_i}},$$

and conclude that R_n is a weighted (weights $e^{-\alpha\sigma_x}$ for x) sum of children of the n first individuals. All children except $x_2, \ldots, x_n$ are included.

Further, we define $\mathscr{A}_n$ as the σ-algebra generated by the biographies of the whole lives of $x_1, \ldots, x_n$. Formally, denote by $\mathscr{B}'_y$ the σ-algebra in the product space $(\Omega, \mathscr{B})$ generated by the projection on $(\Omega_y, \mathscr{B}_y)$. Then $\mathscr{A}_n$ is the smallest σ-algebra, such that

$$\{x_1 = y_1, \ldots, x_n = y_n\} \in \mathscr{A}_n \qquad \text{for all } (y_1, \ldots, y_n)$$

and

$$A \in \mathscr{B}'_y \Rightarrow A \cap \{y \in \{x_1, \ldots, x_n\}\} \in \mathscr{A}_n \qquad \text{for all } y.$$

Lemma 2.3. *$\{R_n\}$ is a non-negative martingale with respect to $\{\mathscr{A}_n\}$.*

Proof. R_n and $\sigma_{x_{n+1}}$ are $\mathscr{A}_n$-measurable. Further ${}_\alpha\xi_{x_{n+1}}(\infty)$ is independent of $\mathscr{A}_n$, and

$$E[{}_\alpha\xi_{x_{n+1}}(\infty)] = \mu_\alpha(\infty) = 1. \tag{2.12}$$

Hence

$$E[R_{n+1} - R_n \mid \mathscr{A}_n] = e^{-\alpha\sigma_{x_{n+1}}} E[{}_\alpha\xi_{x_{n+1}}(\infty) - 1] = 0. \quad \square \tag{2.13}$$

Now, we define

$$\mathscr{I}(t) = \{x = (x', i); \sigma_{x'} \leqq t \quad \text{and} \quad t < \sigma_x < \infty\}. \tag{2.14}$$

This means that $\mathscr{I}(t)$ is composed of the individuals to be born after t, whose mothers are born before or at t. Suppose that

$$Y_t = \sum_{x \in \mathscr{I}(t)} e^{-\alpha\sigma_x}. \tag{2.15}$$

Then, since $\mathscr{I}(t)$ consists of exactly the children of the first T_t individuals to be born after t, it holds that $Y_t = R_{T_t}$.

Finally

Proposition 2.4. *$\{Y_t\}_t$ is a non-negative martingale with respect to $\{\mathscr{A}_{T_t}\}_t$.*

This has the immediate

Corollary 2.5. *There exists a random variable $Y_\infty < \infty$, such that $Y_t \to Y_\infty$ a.s., as $t \to \infty$.*

Proof. For fixed t, observe that T_t is a stopping time with respect to $\{\mathscr{A}_n\}$. A variant of the optional sampling theorem (e.g. Neveu (1975), Theorem II-2-13) shows that $(Y_t\}$ is a supermartingale with respect to $\{\mathscr{A}_{T_t}\}$.

As pointed out $E[T_t]$ is finite. Further

$$E[|R_{n+1} - R_n| \mid \mathscr{A}_n] = e^{-\alpha\sigma_{x_{n+1}}} E[|{}_\alpha\xi(\infty) - 1|] \leqq 2. \tag{2.16}$$

Together, (e.g. Breiman (1968), Proposition 5.33) these facts imply that

$$E[Y_t] = 1. \tag{2.17}$$

Hence $\{Y_t\}$ is not only a supermartingale but a martingale as well. $\square$

3. Convergence in Probability

The main result of this section is

Theorem 3.1. *Suppose that $E[\phi(t)]$, as a function of t, is continuous a.e. with respect to Lebesgue measure,*

$$\sum_{k=0}^{\infty} \sup_{k \leq t \leq k+1} (e^{-\alpha t} E[\phi(t)]) < \infty, \tag{3.1}$$

and

$$E[\sup_{s \leq t} \phi(s)] < \infty, \quad \textit{for all } t < \infty. \tag{3.2}$$

Then

$$e^{-\alpha t} Z_t^{\phi} \to Y_{\infty} m_{\infty}^{\phi} \textit{ in probability, as } t \to \infty. \tag{3.3}$$

We postpone the proof.
There is an immediate

Corollary 3.2. *For each $a \in (0, \infty]$*

$$e^{-\alpha t} Z_t^a \to Y_{\infty} m_{\infty}^a \quad \textit{in probability, as } t \to \infty. \quad \square \tag{3.4}$$

Further,

Corollary 3.3. *Suppose that*

$$E[{}_{\alpha}\xi(\infty) \log^{+} {}_{\alpha}\xi(\infty)] < \infty. \tag{3.5}$$

Then, with ϕ satisfying the conditions of Theorem 3.1, the convergence in (3.3) holds in the sense of L^1-convergence.

Proof of Corollary 3.3. Since $e^{-\alpha t} Z_t \to Y_{\infty} m_{\infty}$ in probability by (3.4), $Y_{\infty} m_{\infty}$ must have the distribution of W in Proposition 1.1. Thus from (1.15)

$$E[Y_{\infty}] = 1. \tag{3.6}$$

Therefore

$$\lim_{t \to \infty} E[e^{-\alpha t} Z_t^{\phi}] = m_{\infty}^{\phi} = E[Y_{\infty} m_{\infty}^{\phi}], \tag{3.7}$$

which yields L^1-convergence (e.g. Bauer (1968), Theorem 20.4 and Corollary 20.5). $\square$

For ratios we obtain

Corollary 3.4. *Suppose that ξ satisfies (3.5)*[1]*, and ϕ_1 and ϕ_2 the conditions of Theorem 3.1. Then, on $\{T_t \to \infty\}$,*

$$\frac{Z_t^{\phi_1}}{Z_t^{\phi_2}} \to \frac{m_{\infty}^{\phi_1}}{m_{\infty}^{\phi_2}} \quad \textit{in probability, as } t \to \infty. \tag{3.8}$$

---

[1] Cf. Theorem 6.3 and Corollary 6.4 for other conditions ensuring the convergences in (3.8) and (3.9)

In particular, on the same set

$$\frac{Z_t^a}{Z_t} \to \frac{\int_0^a e^{-\alpha u}(1-L(u))\,du}{\int_0^\infty e^{-\alpha u}(1-L(u))\,du} \quad \textit{in propability, as } t\to\infty. \tag{3.9}$$

((3.9) was stated for the case that $\mu(\infty)$ is finite by Savits (1975).)

Proof of Corollary 3.4. From Proposition 1.1 and the convergence in probability in (3.4) we know that

$$\{Y_\infty > 0\} = \{T_t \to \infty\} \quad \text{a.s.}, \tag{3.10}$$

and (3.8) follows. □

For the proof of Theorem 3.1 we need two lemmas. Let us define

$$\mathscr{I}(t,c) = \{x \in \mathscr{I}(t);\, t+c < \sigma_x < \infty\}, \qquad \text{for } c>0, \tag{3.11}$$

i.e.

$$\mathscr{I}(t,c) = \{x = (x', i);\, \sigma_{x'} \leqq t,\, t+c < \sigma_x < \infty\}, \tag{3.12}$$

and

$$Y_{t,c} = \sum_{x \in \mathscr{I}(t,c)} e^{-\alpha \sigma_x}. \tag{3.13}$$

$Y_{t,c}$ is the contribution to Y_t from the individuals born later than $t+c$.

Lemma 3.5.

$$E[Y_{t,c}] \to \frac{\int_c^\infty (1-\mu_\alpha(s))\,ds}{\int_0^\infty (1-\mu_\alpha(s))\,ds} = k(c), \qquad \text{as } t\to\infty, \tag{3.14}$$

where

$$k(c) \downarrow 0, \qquad \text{as } c \to \infty. \tag{3.15}$$

Proof of Lemma 3.5. If we let

$$\phi(s) = ({}_\alpha\xi(\infty) - {}_\alpha\xi(s+c))\, e^{\alpha s}, \qquad \text{for } s \geqq 0, \tag{3.16}$$

then

$$Y_{t,c} = e^{-\alpha t} Z_t^\phi. \tag{3.17}$$

Hence (3.14) follows from Proposition 2.2. □

Let $N(t,c)$ denote the number of individuals in $\mathscr{I}(t) \setminus \mathscr{I}(t,c)$.

Lemma 3.6. *Suppose that*

$$\mu(c) > 1. \tag{3.18}$$

Then, on $\{T_t \to \infty\}$,

$$\liminf_{t\to\infty} \frac{N(t,c)}{T_t} \geqq \mu(c) - 1 > 0, \quad \text{a.s.} \tag{3.19}$$

Proof of Lemma 3.6. The lemma follows from strong law of large numbers applied to $\{\xi_{x_i}([0,c])\}_i$ and the inequality

$$N(t,c) \geqq \sum_{i=1}^{T_t} \xi_{x_i}([0,c]) - T_t. \tag{3.20}$$

(This argument is due to S. Asmussen.) □

It is time for the

Proof of Theorem 3.1. First we suppose that

$$\phi(t) = 0, \qquad \text{for } t \geqq \text{some } s, \tag{3.21}$$

and make the convention

$$Z_t^\phi = 0, \qquad \text{for } t < 0. \tag{3.22}$$

By (3.21), every individual that contributes to Z_{t+s}^ϕ must be born after t. Therefore

$$Z_{t+s}^\phi = \sum_{x \in \mathscr{I}(t)} {}_x Z_{t+s-\sigma_x}^\phi, \tag{3.23}$$

which we rewrite as

$$\begin{aligned} e^{-\alpha(t+s)} Z_{t+s}^\phi &= \sum_{x\in\mathscr{I}(t)} e^{-\alpha\sigma_x} e^{-\alpha(t+s-\sigma_x)} {}_x Z_{t+s-\sigma_x}^\phi \\ &= \sum_{x \in \mathscr{I}(t) \setminus \mathscr{I}(t,c)} e^{-\alpha\sigma_x} e^{-\alpha(t+s-\sigma_x)} {}_x Z_{t+s-\sigma_x}^\phi \\ &\quad + \sum_{x\in\mathscr{I}(t,c)} e^{-\alpha\sigma_x} e^{-\alpha(t+s-\sigma_x)} {}_x Z_{t+s-\sigma_x}^\phi. \end{aligned} \tag{3.24}$$

We shall show that for any fixed $\varepsilon > 0$

$$P[|e^{-\alpha(t+s)} Z_{t+s}^\phi - Y_\infty m_\infty^\phi| \geqq \varepsilon] \leqq \varepsilon, \qquad \text{when } t \geqq t_0, \tag{3.25}$$

for t_0, s (and c) satisfying (A), (B), (C) and (D):

$$\mu(c) > 1, \qquad \text{and}$$

$$E[Y_{t,c}] \leqq \frac{\varepsilon^2}{16 \sup_t m_t^\phi}, \qquad \text{for } t \geqq t_0, \tag{A}$$

(this is possible because of the fact that m_t^ϕ is finite on finite intervals and, by Proposition 2.2, converges to m_∞^ϕ, and because of Lemma 3.5).

(3.21) is valid, and

$$|m_t^\phi - m_\infty^\phi| \leqq \frac{\varepsilon^2}{16} \quad \text{for } t \geqq s-c \geqq 0, \tag{B}$$

(this is possible according to Proposition 2.2).

$$P\left[|Y_t - Y_\infty| \geqq \frac{\varepsilon}{4m_\infty^\phi}\right] \leqq \frac{\varepsilon}{4}, \quad \text{for } t \geqq t_0, \tag{C}$$

(this is possible by Corollary 2.5).
and

$$P\left[\Big|\sum_{x\in\mathscr{I}(t)\setminus\mathscr{I}(t,c)} e^{-\alpha\sigma_x}(e^{-\alpha(t+s-\sigma_x)}\,{}_xZ^\phi_{t+s-\sigma_x} - m^\phi_{t+s-\sigma_x})\Big| \geqq \frac{\varepsilon}{4}\right] \leqq \frac{\varepsilon}{4},$$
$$\text{for } t \geqq t_0. \tag{D}$$

We show that (D) is possible by proving

Lemma 3.7. *Suppose that* $\mu(c) > 1$. *Then, for* ϕ *satisfying the conditions of Theorem* 3.1 *and* $s \geqq c$, *on* $\{T_t \to \infty\}$,

$$\sum_{x\in\mathscr{I}(t)\setminus\mathscr{I}(t,c)} e^{-\alpha\sigma_x}(e^{-\alpha(t+s-\sigma_x)}\,{}_xZ^\phi_{t+s-\sigma_x} - m^\phi_{t+s-\sigma_x}) \to 0 \tag{3.26}$$

in probability, as $t \to \infty$.

Proof of Lemma 3.7 (inserted). If $N(t,c)$ is positive we may write

$$\begin{aligned}&\sum_{x\in\mathscr{I}(t)\setminus\mathscr{I}(t,c)} e^{-\alpha\sigma_x}(e^{-\alpha(t+s-\sigma_x)}\,{}_xZ^\phi_{t+s-\sigma_x} - m^\phi_{t+s-\sigma_x})\\ &= (N(t,c)\,e^{-\alpha(t+c)})\\ &\quad \frac{\sum_{x\in\mathscr{I}(t)\setminus\mathscr{I}(t,c)} e^{\alpha(t+c-\sigma_x)}(e^{-\alpha(t+s-\sigma_x)}\,{}_xZ^\phi_{t+s-\sigma_x} - m^\phi_{t+s-\sigma_x})}{N(t,c)}.\end{aligned} \tag{3.27}$$

Since $\sigma_x \leqq t+c$ for $x \in \mathscr{I}(t)\setminus\mathscr{I}(t,c)$,

$$N(t,c)\,e^{-\alpha(t+c)} \leqq Y_t \to Y_\infty < \infty \quad \text{a.s., as } t \to \infty. \tag{3.28}$$

Let

$$A(t) = \frac{\sum_{x\in\mathscr{I}(t)\setminus\mathscr{I}(t,c)} e^{\alpha(t+c-\sigma_x)}(e^{-\alpha(t+s-\sigma_x)}\,{}_xZ^\phi_{t+s-\sigma_x} - m^\phi_{t+s-\sigma_x})}{N(t,c)}. \tag{3.29}$$

By Lemma 3.6, (3.26) follows if we for any $\varepsilon_1 > 0$, can show that

$$P[|A(t)| > \varepsilon_1,\, N(s,c) \to \infty] \to 0, \quad \text{as } t \to \infty. \tag{3.30}$$

We shall use the following law of large numbers to show (3.30). For an elementary proof cf. Athreya and Kaplan (1976).

Proposition 3.8. *Consider a family of sequences* $\{X_{tk}\}$ $k=1,\ldots,n_t, t>0$, *that are independent for fixed* t, *and such that* $n_t\to\infty$, *as* $t\to\infty$. *Suppose that,* $\overset{s}{\leqq}$ *denoting stochastic order,*

$$0\leqq X_{tk}\overset{s}{\leqq}X, \qquad \text{with } E[X]<\infty. \tag{3.31}$$

Then, with $m_{tk}=E[X_{tk}]$,

$$\frac{\sum_{k=1}^{n_t}(X_{tk}-m_{tk})}{n_t}\to 0 \qquad \text{in probability, as } t\to\infty. \quad \square \tag{3.32}$$

Recall that $\mathscr{A}_{T_t}$ is the σ-algebra generated by the whole lives of individuals born before or at time t. $N(t,c)$ is $\mathscr{A}_{T_t}$-measurable and, on $\{x\in\mathscr{I}(t)\setminus\mathscr{I}(t,c)\}$, so is σ_x. Further, conditioned on $\mathscr{A}_{T_t}$, the processes $\{{}_xZ_u^\phi\}$ are independent for $x\in\mathscr{I}(t)\setminus\mathscr{I}(t,c)$ and their distributions are unaffected by the conditioning.

Moreover, for each $u\leqq s$

$$Z_u^\phi=\sum_x\phi_x(u-\sigma_x)\leqq\sum_{\substack{x;\\ \sigma_x\leqq s}}\sup_{u\leqq s}\phi_x(u). \tag{3.33}$$

Since σ_x and ϕ_x are independent for x fixed we deduce

$$E[\sum_{\substack{x;\\ \sigma_x\leqq s}}\sup_{u\leqq s}\phi_x(u)]=E[T_s]\cdot E[\sup_{u\leqq s}\phi_x(u)], \tag{3.34}$$

which is finite due to (3.2) and the finiteness of $E(T_s]$. Hence, on $\{N(t,c)\to\infty\}$,

$$P[|A(t)|>\varepsilon_1|\mathscr{A}_{T_t}]\to 0 \qquad \text{a.s., as } t\to\infty, \tag{3.35}$$

by virtue of Proposition 3.8. Dominated convergence yields (3.30). $\square$

We return to the proof of Theorem 3.1. Recall the definition of $Y_t = \sum_{x\in\mathscr{I}(t)} e^{-\alpha\sigma_x}$. By repeated use of Boole's and Markov's inequalities (3.25) follows:

$$\begin{aligned}
&P[|e^{-\alpha(t+s)}Z_{t+s}^\phi-Y_\infty m_\infty^\phi|\geqq\varepsilon]\\
&\quad=P[|\sum_{x\in\mathscr{I}(t)\setminus\mathscr{I}(t,c)}e^{-\alpha\sigma_x}(e^{-\alpha(t+s-\sigma_x)}\,{}_xZ_{t+s-\sigma_x}^\phi)\\
&\quad+\sum_{x\in\mathscr{I}(t,c)}e^{-\alpha\sigma_x}(e^{-\alpha(t+s-\sigma_x)}\,{}_xZ_{t+s-\sigma_x}^\phi)-Y_\infty m_\infty^\phi|\geqq\varepsilon]\\
&\quad\leqq P\left[|\sum_{x\in\mathscr{I}(t)\setminus\mathscr{I}(t,c)}e^{-\alpha\sigma_x}(e^{-\alpha(t+s-\sigma_x)}\,{}_xZ_{t+s-\sigma_x}^\phi-m_{t+s-\sigma_x}^\phi)|\geqq\frac{\varepsilon}{4}\right]\\
&\quad+P\left[|\sum_{x\in\mathscr{I}(t)\setminus\mathscr{I}(t,c)}e^{-\alpha\sigma_x}m_{t+s-\sigma_x}^\phi\right.\\
&\quad\left.+\sum_{x\in\mathscr{I}(t,c)}e^{-\alpha\sigma_x}(e^{-\alpha(t+s-\sigma_x)}\,{}_xZ_{t+s-\sigma_x}^\phi)-Y_t m_\infty^\phi|\geqq\frac{\varepsilon}{2}\right]\\
&\quad+P\left[|Y_t-Y_\infty|m_\infty^\phi\geqq\frac{\varepsilon}{4}\right]\leqq\frac{\varepsilon}{4}+P\left[\sum_{x\in\mathscr{I}(t)\setminus\mathscr{I}(t,c)}e^{-\alpha\sigma_x}|m_{t+s-\sigma_x}^\phi-m_\infty^\phi|\geqq\frac{\varepsilon}{4}\right]\\
&\quad+P\left[\sum_{x\in\mathscr{I}(t,c)}e^{-\alpha\sigma_x}|e^{-\alpha(t+s-\sigma_x)}\,{}_xZ_{t+s-\sigma_x}^\phi-m_\infty^\phi|\geqq\frac{\varepsilon}{4}\right]+\frac{\varepsilon}{4}.
\end{aligned} \tag{3.36}$$

Here the first $\varepsilon/4$ comes from assumption (D) and the second from (C). The second term is less than

$$P\left[Y_t \frac{\varepsilon^2}{16} \geqq \frac{\varepsilon}{4}\right] \leqq \frac{\varepsilon}{4} \tag{3.37}$$

by (B) and Proposition 2.4, implying that $E[Y_t]=1$.

$$E[\sum_{x \in \mathscr{I}(t,c)} e^{-\alpha\sigma_x} |e^{-\alpha(t+s-\sigma_x)} {}_x Z^{\phi}_{t+s-\sigma_x} - m^{\phi}_{\infty}|] \leqq E[Y_{t,c}] \sup m^{\phi}_t \leqq \frac{\varepsilon^2}{16} \tag{3.38}$$

due to (A). This and Markov's inequality applied to the third term completes the proof under the restriction (3.21), which will now be removed.

Truncate a general ϕ to

$$\phi'(t) = \begin{cases} \phi(t) & \text{for } t < c' \\ 0 & \text{otherwise.} \end{cases} \tag{3.39}$$

Of course

$$Z^{\phi}_t - Z^{\phi'}_t \geqq 0, \tag{3.40}$$

and

$$e^{-\alpha t} Z^{\phi'}_t \to Y_{\infty} m^{\phi'}_{\infty} \quad \text{in probability, as } t \to \infty. \tag{3.41}$$

Moreover, by definition,

$$m^{\phi'}_{\infty} \to m^{\phi}_{\infty}, \qquad \text{as } c' \to \infty. \tag{3.42}$$

Further,

$$|e^{-\alpha t} Z^{\phi}_t - Y_{\infty} m^{\phi}_{\infty}| \leqq |e^{-\alpha t}(Z^{\phi}_t - Z^{\phi'}_t)| + |e^{-\alpha t} Z^{\phi'}_t - Y_{\infty} m^{\phi'}_{\infty}| + |m^{\phi'}_{\infty} - m^{\phi}_{\infty}|\, Y_{\infty}. \tag{3.43}$$

The probability that the left hand term should be greater than $\varepsilon > 0$, can thus be shown to be small: choose a large c', then a large t, and apply Boole's and Markov's inequalities to the right hand terms. □

4. A Strong Law of Large Numbers

To get a.s. convergence results we shall work with convergence on certain lattices in both of Sects. 5 and 6. The following strong law of large numbers is needed.

Proposition 4.1. *Let n_i $i=1,2,\dots$ be a sequence of positive integers, and let X_{ij}, $j=1,\dots,n_i$ be independent for fixed i, $i=1,2,\dots$. Suppose that*

$$|X_{ij}| \overset{s}{\leqq} Y, \quad \textit{with } E[Y] < \infty, \tag{4.1}$$

and

$$\liminf_{i \to \infty} \frac{n_{i+1}}{n_1 + \dots + n_i} > 0. \tag{4.2}$$

then

$$S_i = \frac{\sum_{j=1}^{n_i} (X_{ij} - E[X_{ij}])}{n_i} \to 0 \quad \textit{a.s., as } i \to \infty \tag{4.3}$$

or, seemingly stronger, for any $\varepsilon > 0$

$$\sum_{i=1}^{\infty} P[|S_i| > \varepsilon] < \infty. \tag{4.4}$$

For the proof of this proposition (for a more direct approach cf. Asmussen and Kurtz (1980):

Lemma 4.2. *Assume that* X_i, $i = 1, 2, \ldots$, *are independent,*

$$|X_i| \overset{s}{\leqq} Y \quad \textit{and} \quad E[Y] < \infty. \tag{4.5}$$

Then

$$\frac{\sum_{i=1}^{n} (X_i - E[X_i])}{n} \to 0 \quad \textit{a.s., as } n \to \infty. \tag{4.6}$$

Proof. Construct U_i, $i = 1, 2, \ldots$, independent and independent of all X_i, each U_i uniformly distributed on $[0,1]$. Let $F_i(x) = P(|X_i| \leqq x]$, $F(x) = P[Y \leqq x]$, $F_i\{x\} = F_i(x) - F_i(x-)$ and $F^{-1}(x) = \inf\{t; F(t) \geqq x\}$. Define

$$Y_i = F^{-1}(F_i(|X_i|) - U_i F_i\{|X_i|\}) \qquad i = 1, 2, \ldots. \tag{4.7}$$

Certainly Y_i will be independent and have Y's distribution, and by construction

$$|X_i| \leqq Y_i. \tag{4.8}$$

Let $\chi(A)$ denote the indicator function of the event A. Choose c so large that

$$E[Y\chi(Y > c)] \leqq \frac{\varepsilon}{2}. \tag{4.9}$$

Then

$$\begin{aligned}
\limsup_{n\to\infty} \left| \frac{\sum_{i=1}^{n} (X_i - E[X_i])}{n} \right| &\leqq \limsup_{n\to\infty} \left| \frac{\sum_{i=1}^{n} (X_i \chi(Y_i \leqq c) - E[X_i \chi(Y_i \leqq c)])}{n} \right| \\
&\quad + \limsup_{n\to\infty} \left| \frac{\sum_{i=1}^{n} X_i \chi(Y_i > c)}{n} \right| + \limsup_{n\to\infty} \left| \frac{\sum_{i=1}^{n} E[X_i \chi(Y_i > c)]}{n} \right| \\
&\leqq 0 + \limsup_{n\to\infty} \left| \frac{\sum_{i=1}^{n} Y_i \chi(Y_i > c)}{n} \right| + \frac{\varepsilon}{2} \leqq 0 + \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.
\end{aligned} \tag{4.10}$$

The law of large numbers for independent random variables (e.g. Breiman (1968), Theorem 3.27) yields the zero. This, (4.8) and (4.9) justify the second inequality. The last one is implied by the classical law of large numbers for independent and identically distributed random variables (Theorem 3.30 the same reference). □

Proof of Proposition 4.1. We may assume that $E[X_{ij}]=0$. To prove (4.4), it is no restriction to assume independence between all the X_{ij}. Then Lemma 4.2 shows that

$$\frac{\sum_{i=1}^{k}\sum_{j=1}^{n_i} X_{ij}}{\sum_{i=1}^{k} n_i} \to 0 \quad \text{a.s., } as\ k\to\infty. \tag{4.11}$$

But

$$\frac{\sum_{j=1}^{n_k} X_{kj}}{n_k} = \frac{\sum_{i=1}^{k}\sum_{j=1}^{n_i} X_{ij}}{\sum_{i=1}^{k} n_i}\,\frac{\sum_{i=1}^{k} n_i}{n_k} - \frac{\sum_{i=1}^{k-1}\sum_{j=1}^{n_i} X_{ij}}{\sum_{i=1}^{k-1} n_i}\,\frac{\sum_{i=1}^{k-1} n_i}{n_k}, \tag{4.12}$$

and thus (4.11) and (4.2) show (4.3). The inequality (4.4) follows from the independence of S_k, $k=1,2,\ldots$ by virtue of the converse Borel-Cantelli theorem. □

Finally, we shall have use for the following direct consequence of Lévy's generalized Borel-Cantelli theorem, see Meyer (1972) Theorem 21.

Proposition 4.3. *Assume that* $\{\mathscr{F}_n\}$ *is an increasing sequence of* σ*-algebras and that* $\{A_n\}$ *is any sequence of events. Then, with*

$$E=\left\{\sum_{n=1}^{\infty} P[A_n|\mathscr{F}_n]<\infty\right\}, \tag{4.13}$$

$P[\{A_n\}$ *infinitely often* $|E]=0$. □

5. Almost Sure Convergence

Our technique in proving Theorem 3.1 was to split $e^{-\alpha(t+s)}Z_{t+s}^{\phi}$ (ϕ satisfying $\phi(t)=0$, for $t\geqq s$) into ($\mathscr{I}(t,c)=\{x\in\mathscr{I}(t);\ \sigma_x>t+c\}$)

$$\sum_{x\in\mathscr{I}(t)\setminus\mathscr{I}(t,c)} e^{-\alpha(t+s)}\,{}_xZ_{t+s-\sigma_x}^{\phi}+\sum_{x\in\mathscr{I}(t,c)} e^{-\alpha(t+s)}\,{}_xZ_{t+s-\sigma_x}^{\phi}. \tag{5.1}$$

Then, we used a weak law of large numbers to show that the first sum is close to $(Y_t-Y_{t,c})m_\infty^{\phi}$, where $Y_{t,c}=\sum_{x\in\mathscr{I}(t,c)} e^{-\alpha\sigma_x}$, for s,c and t large. By renewal theory we found that the expectation of $Y_{t,c}$ is small for large c and t. Thus Markov's inequality yielded that $Y_{t,c}$ should be close to zero. Hence, the first sum in (5.1) is close to $Y_\infty m_\infty^\phi$ for large s, c and t. On the other hand, a conditional expectation of the second sum in (5.1) is smaller than $Y_{t,c} \sup_t m_t^\phi$, which is approximately zero for s, c and t large.

To strengthen the convergence of $e^{-\alpha t} Z_t^\phi$ to convergence a.s. with a method along the same lines we need a strong law of large numbers applicable to the first sum in (5.1) and something like

$$\lim_{c\to\infty} \limsup_{t\to\infty} Y_{t,c} = 0 \quad \text{a.s.} \tag{5.2}$$

To obtain the latter we require

Condition 5.1. *There exists on* $[0, \infty)$ *a non-increasing, bounded, positive integrable function* g, *such that*

$$E\left[\sup_t \frac{{}_\alpha\xi(\infty) - {}_\alpha\xi(t)}{g(t)}\right] < \infty. \tag{5.3}$$

(The boundedness of g is superfluous: if ξ satisfies the rest of the condition we can always choose g bounded.)

Remark. Condition 5.1 is satisfied if there exists a non-increasing integrable positive function g such that

$$\int_0^\infty \frac{1}{g(t)} e^{-\alpha t} \mu(dt) < \infty, \tag{5.4}$$

since then

$$\begin{aligned} \frac{{}_\alpha\xi(\infty) - {}_\alpha\xi(t)}{g(t)} &= \int_t^\infty \frac{1}{g(t)} e^{-\alpha s} \xi(ds) \\ &\leqq \int_t^\infty \frac{1}{g(s)} e^{-\alpha s} \xi(ds) \leqq \int_0^\infty \frac{1}{g(s)} e^{-\alpha s} \xi(ds), \end{aligned} \tag{5.5}$$

and accordingly

$$E\left[\sup_t \frac{{}_\alpha\xi(\infty) - {}_\alpha\xi(t)}{g(t)}\right] \leqq \int_0^\infty \frac{1}{g(s)} e^{-\alpha s} \mu(ds) < \infty. \tag{5.6}$$

For Bellman-Harris processes with finite Malthusian parameter $\mu(\infty)$ is always finite. In this case $g(t) = e^{-\alpha t}$ will do. We conclude that for Bellman-Harris processes Condition 5.1 is superfluous.

Generally, Condition 5.1 is seen to be weaker than

$$\int_0^\infty t^2 e^{-\alpha t} \mu(dt) < \infty, \tag{5.7}$$

and e.g.

$$\int_0^\infty t(\log^+ t)^{1+\varepsilon} e^{-\alpha t} \mu(dt) < \infty, \qquad \text{for some } \varepsilon > 0. \tag{5.8}$$

Besides the condition on ξ we shall need restrictions on ϕ. The following one is a dual of that on ξ. It will be used to conclude a.s. convergence for general ϕ:s from convergence for ϕ:s vanishing outside a bounded interval.

Condition 5.2. *There exists on* $[0,\infty)$ *an integrable, bounded, non-increasing positive function h, such that*

$$U=\sup_t \left(\frac{e^{-\alpha t}\phi(t)}{h(t)}\right) \tag{5.9}$$

has finite expectation.

To bridge the gap from convergence, as $t\to\infty$ on certain lattices, to general convergence we need more restrain on ϕ. We make two definitions:

$$\phi^{\varepsilon}(t)=\begin{cases}\sup\limits_{|s-t|\leqq\varepsilon}\phi(s), & \text{for } t\geqq 0\\ 0 & \text{otherwise,}\end{cases} \tag{5.10}$$

and

$$\phi_{\varepsilon}(t)=\inf_{|s-t|\leqq\varepsilon}\phi(s). \tag{5.11}$$

Lemma 5.3. *If ϕ satisfies Condition* 5.2 *and has paths in the Skorohod D-space (not necessarily right continuous), then* $E[\phi(t)]$, $E[\phi^{\varepsilon}(t)]$ *and* $E[\phi_{\varepsilon}(t)]$ *are a.e. continuous and for almost all t.*

$$E[\phi^{\varepsilon}(t)]\downarrow E[\phi(t)], \quad \textit{and} \quad E[\phi_{\varepsilon}(t)]\uparrow E[\phi(t)], \quad \textit{as} \quad \varepsilon\downarrow 0. \tag{5.12}$$

Proof. Dominated convergence, justified by Condition 5.2 proves that $E[\phi(t)]$ is D-valued and hence a.e. continuous.

Clearly $\phi^{\varepsilon}(t)$ and $\phi_{\varepsilon}(t)$ are D-valued and Condition 5.2 can again be used to dominate and hence $E[\phi^{\varepsilon}(t)]$ and $E[\phi_{\varepsilon}(t)]$ are also a.e. continuous. For continuity points t of ϕ

$$\phi^{\varepsilon}(t)\downarrow\phi(t) \quad \text{and} \quad \phi_{\varepsilon}(t)\uparrow\phi(t), \quad \text{as} \quad \varepsilon\downarrow 0.$$

and since all but countably many t are a.s. continuity points of ϕ the last assertion follows. □

We are ready to formulate the main theorem of this section:

Theorem 5.4. *Suppose that ξ satisfies Condition* 5.1, *and that ϕ is D-valued and satisfies Condition* 5.2. *Then*

$$e^{-\alpha t}Z_t^{\phi}\to Y_{\infty}m_{\infty}^{\phi} \quad \text{a.s., } \textit{as } t\to\infty. \tag{5.13}$$

We postpone the proof.

Some immediate consequences:

„**Corollary 5.5.** *Suppose that ξ satisfies Condition* 5.1. *Then*

$$e^{-\alpha t}Z_t^{a}\to\frac{\int_0^a(1-L(u))\,e^{-\alpha u}\,du}{\int_0^\infty u e^{-\alpha u}\,\mu(du)}\,Y_{\infty} \quad \textit{a.s., as } t\to\infty. \quad \square \tag{5.14}$$

Corollary 5.6. *Suppose that ξ, ϕ_1 and ϕ_2 satisfy the conditions of Theorem 5.4, and that*[2]

$$E[{}_\alpha\xi(\infty)\log^+ {}_\alpha\xi(\infty)] < \infty. \tag{5.15}$$

Then, on $\{T_t \to \infty\}$,

$$\frac{Z_t^{\phi_1}}{Z_t^{\phi_2}} \to \frac{m_\infty^{\phi_1}}{m_\infty^{\phi_2}} \quad \text{a.s., as } t \to \infty. \tag{5.16}$$

In particular, on the same set,

$$\frac{Z_t^a}{Z_t} \to \frac{\int_0^a (1-L(u))\, e^{-\alpha u}\, du}{\int_0^\infty (1-L(u))\, e^{-\alpha u}\, du} \quad \text{a.s., as } t \to \infty. \quad \square \tag{5.17}$$

We shall need an asymptotic bound for T_t, the total number of births up to and including time t:

Lemma 5.7. *There is a constant $K < \infty$, such that*

$$\limsup_{t\to\infty} e^{-\alpha t} T_t \leqq K Y_\infty \quad \text{a.s.} \tag{5.18}$$

Proof. From Lemma 3.6 we have that if $\mu(c) > 1$ then, on $\{T_t \to \infty\}$,

$$\liminf_{t\to\infty} \frac{N(t,c)}{T_t} \geqq \mu(c) - 1 \quad \text{a.s.} \tag{5.19}$$

but $Y_t \geqq e^{-\alpha(t+c)} N(t,c)$ shows

$$\liminf_{t\to\infty} \frac{Y_t}{e^{-\alpha t} T_t} \geqq (\mu(c)-1)\, e^{-\alpha c} \tag{5.20}$$

and the lemma follows. $\square$

Define for any $c > 0$

$$\phi'(t) = \begin{cases} \phi(t) & \text{for } t > c \\ 0 & \text{otherwise.} \end{cases} \tag{5.22}$$

Then

Lemma 5.8. *Assume that ϕ satisfies Condition 5.2. Then there exists a $K < \infty$, such that, for all $c > 0$,*

$$\limsup_{t\to\infty} e^{-\alpha t} Z_t^{\phi'} \leqq K \left(\int_{c-K}^\infty h(t)\, dt \right) Y_\infty \quad \text{a.s.} \tag{5.23}$$

The lemma has the useful

---

[2] Cf. Theorem 6.3 and Corollary 6.4 for other conditions implying (5.16) and (5.17)

Corollary 5.9. *Suppose that ξ satisfies Condition* 5.1. *Then there exists a $K<\infty$, such that, for all $c>0$,*

$$\limsup_{t\to\infty} Y_{t,c} \leqq K\left(\int_{c-K}^{\infty} g(t)\,dt\right) Y_\infty \qquad a.s. \tag{5.24}$$

Proof of the Corollary. With

$$\phi(t)=e^{\alpha t}({}_\alpha\xi(\infty)-{}_\alpha\xi(t)), \tag{5.25}$$

$e^{-\alpha(t+c)}Z_{t+c}^{\phi'}$ is nothing but $Y_{t,c}$. Condition 5.1 on ξ and Condition 5.2 on this particular ϕ are equivalent. □

Proof of Lemma 5.8. First we fix s such that $\mu(s)>1$, and observe that,

$$T_{t+s} \geqq \sum_{i=1}^{T_t} \xi_{x_i}\{[0,s]\}. \tag{5.26}$$

Hence the strong law of large numbers implies that, on $\{T_t\to\infty\}$,

$$\liminf_{t\to\infty} \frac{T_{t+s}}{T_t} \geqq \mu(s)>1 \qquad \text{a.s.} \tag{5.27}$$

and that (argue as in the proof of Proposition 4.1), on $\{T_t\to\infty\}$,

$$\frac{\sum_{i=T_{ks}+1}^{T_{(k+1)s}} U_{x_i}}{T_{(k+1)s}-T_{ks}} \to E[U] \qquad \text{a.s., } as\ k\to\infty. \tag{5.28}$$

Hence, for any $\varepsilon>0$, on $\{T_t\to\infty\}$:

$$\begin{aligned}
\limsup_{t\to\infty} e^{-\alpha(t+c)}Z_{t+c}^{\phi'} &= \limsup_{t\to\infty} e^{-\alpha(t+c)} \sum_{i=1}^{T_t} \phi_{x_i}(t+c-\sigma_{x_i})\\
&\leqq \limsup_{t\to\infty} \sum_{k=0}^{[t/s]} \sum_{i=T_{ks}+1}^{T_{(k+1)s}} e^{-\alpha(t+c)}\phi_{x_i}(t+c-\sigma_{x_i})\\
&\leqq \limsup_{t\to\infty} \sum_{k=0}^{[t/s]} \sum_{i=T_{ks}+1}^{T_{(k+1)s}} e^{-\alpha\sigma_{x_i}} h(t+c-\sigma_{x_i})\,U_{x_i}\\
&\leqq \limsup_{t\to\infty} \sum_{k=0}^{[t/s]} \sum_{i=T_{ks}+1}^{T_{(k+1)s}} e^{-\alpha ks} h(t+c-(k+1)s)\,U_{x_i}\\
&\leqq \limsup_{t\to\infty} \sum_{k=0}^{[t/s]} e^{-\alpha ks} h(t+c-(k+1)s)\,E[U]\,T_{(k+1)s}\\
&\leqq e^{\alpha s}K(Y_\infty+\varepsilon) \limsup_{t\to\infty} \sum_{k=0}^{[t/s]} h(t+c-(k+1)s)
\end{aligned} \tag{5.29}$$

by virtue of Lemma 5.7. But, since h is non-increasing

$$\limsup_{t\to\infty} \sum_{k=0}^{[t/s]} h(t+c-(k+1)s) \leqq (1/s)\int_{c-2s}^{\infty} h(u)\,du. \tag{5.30}$$

The arbitrariness of $\varepsilon>0$ shows the lemma. □

The proof of Theorem 5.4 is built upon the following lemma or rather on its Corollary 5.11.

Define to each $c>0$ and $t_0 \geqq 0$ the lattice $\{t_0, t_1, t_2, \ldots\}$ by

$$t_k = kc + t_0, \qquad k=0,1,2,\ldots, \tag{5.31}$$

and $\{t_{k,n}\}_{k,n}$ by

$$t_{k,n} = \frac{kc}{n}, \quad k=0,1,\ldots, \quad n=1,2,\ldots. \tag{5.32}$$

Lemma 5.10. *Suppose that ξ satisfies Condition 5.1, ϕ satisfies Condition 5.2 and that $E[\phi(t)]$ is continuous a.e. with respect to Lebesgue measure. Further, assume that $\mu(c)>1$. Then*

$$e^{-\alpha t_k} Z_{t_k}^{\phi} \to Y_\infty m_\infty^{\phi} \qquad \text{a.s., as } k \to \infty. \tag{5.33}$$

We postpone the proof of this lemma.

Corollary 5.11. *Assume that ϕ, ξ, and c satisfy the conditions in Lemma 5.10. Then, for each fixed n,*

$$e^{-\alpha t_{k,n}} Z_{t_{k,n}}^{\phi} \to Y_\infty m_\infty^{\phi} \qquad \text{a.s., as } k \to \infty. \tag{5.34}$$

Proof of the Corollary. Let A_r be the set where (5.33) holds for $t_0 = rc$. On the set $A = \bigcap_{r \in \{\text{rationals in } [0,1]\}} A_r$, obviously (5.34) holds for each fixed n. Finally by Lemma 5.10 $P(A_r)=1$, and hence $P(A)=1$ too. □

Proof of Theorem 5.4 from Corollary 5.11. Fix a c, such that $\mu(c)>1$. According to Lemma 5.3, $E[\phi^{c/n}(t)]$ and $E[\phi_{c/n}(t)]$ are both continuous a.e. Further, it follows from the definition of ϕ^{ε} and ϕ_{ε} that, for $t \in [t_{k,n}, t_{k+1,n}]$,

$$Z_{t_{k,n}}^{\phi_{c/n}} \leqq Z_t^{\phi} \leqq Z_{t_{k+1,n}}^{\phi^{c/n}}. \tag{5.35}$$

Since $\phi_{c/n}$ and $\phi^{c/n}$ satisfy Condition 5.2 we can apply Corollary 5.11 to $\{Z_t^{\phi_{c/n}}\}$ and $\{Z_t^{\phi^{c/n}}\}$. Together with (5.35) this yields,

$$\begin{aligned} e^{-\alpha \frac{c}{n}} Y_\infty m_\infty^{\phi_{c/n}} &\leqq \liminf_{t\to\infty} e^{-\alpha t} Z_t^{\phi} \\ \leqq \limsup_{t\to\infty} e^{-\alpha t} Z_t^{\phi} &\leqq e^{\alpha \frac{c}{n}} Y_\infty m_\infty^{\phi^{c/n}} \qquad \text{a.s.} \end{aligned} \tag{5.36}$$

But, by definition,

$$m_\infty^{\phi} = \frac{\int_0^\infty E[\phi(t)]\, e^{-\alpha t}\, dt}{\int_0^\infty u e^{-\alpha u} \mu(du)} \tag{5.37}$$

Hence, if

$$m_\infty^{\phi^{c/n}} < \infty, \tag{5.38}$$

dominated convergence and the second part of Lemma 5.3 imply that $m_\infty^{\phi^{c/n}}$ and $m_\infty^{\phi_{c/n}}$ both converge towards m_∞^ϕ, as $n \to \infty$. However, (5.38) is a consequence of the fact that $\phi^{c/n}$ satisfies Condition 5.2. Intersection of the sets where (5.36) holds for $n = 1, 2, \ldots$ yields Theorem 5.4. □

It remains to prove Lemma 5.10. For that purpose we need a sublemma. Recall that $N(t, c)$ is the total number of elements of $\mathscr{I}(t) \setminus \mathscr{I}(t, c)$.

Lemma 5.12. *If* $\mu(c) > 1$, *then*

$$P\left[\liminf_{k \to \infty} \frac{N(t_{k+1}, c)}{\sum_{j=1}^{k} N(t_j, c)} > 0 \,|\, T_t \to \infty\right] = 1. \tag{5.39}$$

Proof of Lemma 5.12. We observe that

$$N(t_k, c) \leqq T_{t_{k+1}}. \tag{5.40}$$

Thus, on $\{T_t \to \infty\}$

$$\liminf_{t \to \infty} \frac{N(t_k, c)}{N(t_{k-1}, c)} \geqq \liminf_{t \to \infty} \frac{N(t_k, c)}{T_{t_k}} \geqq \mu(c) - 1 \tag{5.41}$$

by (5.19) which yields the lemma. □

Proof of Lemma 5.10. First, we suppose that

$$\phi(t) = 0 \qquad \text{for } t \geqq n_0 c. \tag{5.42}$$

Then, for $n \geqq n_0$,

$$\begin{aligned} &|e^{-\alpha t_{k+n}} Z_{t_{k+n}}^\phi - m_\infty^\phi Y_\infty| \\ &\leqq \Big| \sum_{x \in \mathscr{I}(t_k) \setminus \mathscr{I}(t_k, nc)} e^{-\alpha \sigma_x} (e^{-\alpha(t_{k+n} - \sigma_x)} {}_x Z_{t_{k+n} - \sigma_x}^\phi - m_{t_{k+n} - \sigma_x}^\phi) \Big| \\ &\quad + \Big| \Big(\sum_{x \in \mathscr{I}(t_k) \setminus \mathscr{I}(t_k, nc)} e^{-\alpha \sigma_x} m_{t_{k+n} - \sigma_x}^\phi \Big) - m_\infty^\phi Y_\infty \Big| = S_1(t_k) + S_2(t_k) \quad \text{say.} \end{aligned} \tag{5.43}$$

Let us rewrite $S_1(t_k)$ as

$$\begin{aligned} S_1(t_k) &= |e^{-\alpha t_k} N(t_k, nc)| \\ &\quad \cdot \left| \frac{\sum_{x \in \mathscr{I}(t_k) \setminus \mathscr{I}(t_k, nc)} e^{-\alpha(\sigma_x - t_k)} (e^{-\alpha(t_{k+n} - \sigma_x)} {}_x Z_{t_{k+n} - \sigma_x}^\phi - m_{t_{k+n} - \sigma_x}^\phi)}{N(t_k, nc)} \right| \\ &= S_{11}(t_k) S_{12}(t_k), \quad \text{say.} \end{aligned} \tag{5.44}$$

Since

$$e^{-\alpha t_k} N(t_k, nc) \leqq e^{\alpha nc} Y_{t_k} \to e^{\alpha nc} Y_\infty < \infty \qquad \text{a.s., as } k \to \infty, \tag{5.45}$$

it is enough to show that, on $\{T_t \to \infty\}$,

$$S_{12}(t_k) \to 0 \qquad \text{a.s., as } k \to \infty, \tag{5.46}$$

in order to conclude that, on $\{T_t \to \infty\}$,

$$S_1(t_k) \to 0 \quad \text{a.s., as } k \to \infty. \tag{5.47}$$

To show (5.46) we shall use Propositions 4.1 and 4.3. $\mathscr{I}(t_k) \setminus \mathscr{I}(t_k, nc)$, $N(t_k, nc)$ and σ_x on $\{x \in \mathscr{I}(t_k) \setminus \mathscr{I}(t_k, nc)\}$ are all measurable with respect to $\mathscr{A}_{T_{t_k}}$. Further, conditioned on $\mathscr{A}_{T_{t_k}}$, the processes $\{{}_xZ_s^\phi\}$, $x \in \mathscr{I}(t_k) \setminus \mathscr{I}(t_k, nc)$, are mutually independent and their distributions are unaffected by the conditioning.

Observe that

$$e^{-\alpha(\sigma_x - t_k)} e^{-\alpha(t_{k+n} - \sigma_x)} = e^{-\alpha cn} < 1, \tag{5.48}$$

and that

$$\sup_{s \leqq cn} Z_s^\phi \leqq \sum_{\substack{x \\ \sigma_x \leqq cn}} \sup_{s \leqq cn} \phi_x(s). \tag{5.49}$$

From the finiteness of $E[T_t]$, the independence of σ_x and ϕ_x, and Condition 5.2,

$$E[\sum_{\substack{x \\ \sigma_x \leqq cn}} \sup \phi_x(s)] = E[T_{cn}]\, E[\sup_{s \leqq cn} \phi_x(s)] < \infty. \tag{5.50}$$

Further, from Lemma 5.12, on $\{T_t \to \infty\}$,

$$\liminf_{k \to \infty} \frac{N(t_k, c)}{\sum_{j=1}^{k} N(t_j, c)} > 0 \quad \text{a.s.} \tag{5.51}$$

Proposition 4.1 yields that, on $\{T_t \to \infty\}$,

$$\sum_{k=1}^{\infty} P[S_{12}(t_k) > \varepsilon \,|\, \mathscr{A}_{T_{t_k}}] < \infty \quad \text{a.s.}, \tag{5.52}$$

which by virtue of Proposition 4.3 yields (5.46), and hence also (5.47).

To settle (5.33) for our special ϕ it suffices to show that to any $\varepsilon > 0$, we can find $n \geqq n_0$, such that

$$\limsup_{k \to \infty} S_2(t_k) \leqq Y_\infty \varepsilon \quad \text{a.s.}. \tag{5.53}$$

Let r be so large that

$$\limsup_{t \to \infty} Y_{t,cr} \leqq \frac{\varepsilon}{2 \sup_t m_t^\phi} Y_\infty \quad \text{a.s.}, \tag{5.54}$$

which is possible by Corollary 5.9.

Assume that $n>n_0$, and r is so large that

$$|m_t^\phi - m_\infty^\phi| \leqq \frac{\varepsilon}{2} \quad \text{if } t>(n-r)c. \tag{5.55}$$

(Certainly Proposition 2.2 is applicable.)
Then, by the definition of Y_t,

$$\begin{aligned}\limsup_{k\to\infty} S_2(t_k) \leqq & \limsup_{k\to\infty} \Big| \sum_{x\in\mathscr{I}(t_k)\smallsetminus\mathscr{I}(t_k,rc)} e^{-\alpha\sigma_x}(m^\phi_{t_k+n-\sigma_x} - m^\phi_\infty)\Big| \\ & + \limsup_{k\to\infty} \Big| \sum_{x\in\mathscr{I}(t_k,rc)\smallsetminus\mathscr{I}(t_k,nc)} e^{-\alpha\sigma_x}(m^\phi_{t_k+n-\sigma_x} - m^\phi_\infty) \\ & - \sum_{x\in\mathscr{I}(t_k,nc)} e^{-\alpha\sigma_x} m^\phi_\infty\Big| + \limsup_{k\to\infty} |m^\phi_\infty Y_{t_k} - m^\phi_\infty Y_\infty| \\ & \leqq \frac{\varepsilon}{2} \limsup_{k\to\infty} Y_{t_k} + (\sup_t m_t^\phi) \limsup_{k\to\infty} Y_{t_k, rc} \leqq \varepsilon Y_\infty \quad \text{a.s.,}\end{aligned} \tag{5.56}$$

by (5.54) and (5.55).
Finally, use Lemma 5.8 to remove (5.42). □

6. a.s. Convergence of Ratios and the Empirical Age Distribution

Consider the ratio Z_t^ϕ/Z_t^ψ for two characteristics ϕ and ψ. If

$$E[_\alpha\xi(\infty)\log^+ {}_\alpha\xi(\infty)] < \infty, \tag{6.1}$$

and ϕ and ψ satisfy the conditions of Theorem 3.1 or ϕ, ψ and ξ the conditions of Theorem 5.4 then Corollary 3.4 and Corollary 5.6 state appropriate ratio-convergences (on $\{T_t\to\infty\}$).
On the other hand, if

$$E[_\alpha\xi(\infty)\log^+ {}_\alpha\xi(\infty)] = \infty \tag{6.2}$$

then Theorem 3.1 and Proposition 1.1 show that $Y_\infty=0$ a.s.. Accordingly ratio-convergences do not follow that easily.
The purpose of this section is to give sufficient conditions on the reproduction ξ, and the characteristics ϕ and ψ, for a.s. convergence (on $\{T_t\to\infty\}$) of Z_t^ϕ/Z_t^ψ, even if (6.2) holds.
We shall work with the following assumption on ξ:

Condition 6.1. *There exists a $\beta<\alpha$ such that $\mu_\beta(\infty)=E[_\beta\xi(\infty)]<\infty$.* (For Bellman-Harris processes with finite Malthusian parameter $\mu(\infty)$ is always finite. Therefore they satisfy the condition with $\beta=0$.)
We shall restrain ϕ and ψ more than in Sect. 5:

Condition 6.2. *There exists a $\beta<\alpha$, such that*

$$V = \sup_t (e^{-\beta t}\phi(t)) \tag{6.3}$$

has finite expectation.

Just like in Sect. 5 the two conditions are dual.

Let us formulate

Theorem 6.3. *Suppose that ξ satisfies Condition* 6.1[3] *and that ϕ and ψ satisfy Condition* 6.2 *and have D-paths. Then, on $\{T_t \to \infty\}$,*

$$\frac{Z_t^\phi}{Z_t^\psi} \to \frac{m_\infty^\phi}{m_\infty^\psi} \quad a.s., \text{ as } t \to \infty. \tag{6.4}$$

Corollary 6.4. *Suppose that ξ satisfies Condition* 6.1.[3] *Then, on $\{T_t \to \infty\}$,*

$$\frac{Z_t^a}{Z_t} \to \frac{\int_0^a (1-L(u))\, e^{-\alpha u}\, du}{\int_0^\infty (1-L(u))\, e^{-\alpha u}\, du} \quad a.s., \text{ as } t \to \infty. \quad \square \tag{6.5}$$

Remark. For Bellman-Harris processes, Condition 6.1 is always satisfied. And thus the fact that the empirical age distribution of a supercritical Malthusian Bellman-Harris process always converges a.s. towards the stable age distribution follows from Corollary 6.4. As pointed out in the introduction this result has been established also by Kuczek [1980].

The proof of Theorem 6.3 will be built on the next lemma. From Sect. 5 we recall some conventions: to fixed $c > 0$ and $t_0 \geqq 0$,

$$t_k = kc + t_0, \qquad k = 0, 1, 2, \ldots \tag{6.6}$$

and

$$t_{k,n} = \frac{kc}{n}, \qquad k = 0, 1, 2, \ldots, \qquad n = 1, 2, \ldots. \tag{6.7}$$

Lemma 6.5. *Assume that ξ satisfies Condition* 6.1 *and ϕ and ψ satisfy Condition* 6.2, *and that $E[\phi(t)]$ and $E[\psi(t)]$ are continuous a.e. Then, on $\{T_t \to \infty\}$,*

$$\frac{Z_{t_{k+j,n}}^\phi}{Z_{t_{k,n}}^\psi} \to \frac{m_\infty^\phi}{m_\infty^\psi}\, e^{\alpha\left(\frac{jc}{n}\right)} \quad a.s., \text{ as } k \to \infty. \tag{6.8}$$

We postpone the proof of Lemma 6.5.

Proof of Theorem 6.3 from Lemma 6.5. Lemma 5.3 implies that $E[\phi^{c/n}(t)]$, $E[\phi_{c/n}(t)]$, $E[\psi^{c/n}(t)]$, and $E[\psi_{c/n}(t)]$ are continuous a.e. Moreover $\phi^{c/n}$, $\phi_{c/n}$, $\psi^{c/n}$, and $\psi_{c/n}$ satisfy Condition 6.2.

Now, if $t \in [t_{k,n}, t_{k+1,n}]$ then

$$\frac{Z_{t_{k,n}}^{\phi_{c/n}}}{Z_{t_{k+1,n}}^{\psi^{c/n}}} \leqq \frac{Z_t^\phi}{Z_t^\psi} \leqq \frac{Z_{t_{k+1,n}}^{\phi^{c/n}}}{Z_{t_{k,n}}^{\psi_{c/n}}}. \tag{6.9}$$

[3] For other conditions implying ratio stabilization consult Corollary 3.5 and Corollary 5.6

Hence Lemma 6.5 implies that, on $\{T_t \to \infty\}$,

$$e^{-\alpha\frac{c}{n}} \frac{m_\infty^{\phi_{c/n}}}{m_\infty^{\psi_{c/n}}} \leqq \liminf_{t\to\infty} \frac{Z_t^\phi}{Z_t^\psi} \leqq \limsup_{t\to\infty} \frac{Z_t^\phi}{Z_t^\psi} \leqq e^{\alpha\frac{c}{n}} \frac{m_\infty^{\phi_{c/n}}}{m_\infty^{\psi_{c/n}}} \quad \text{a.s.} \tag{6.10}$$

The proof is completed as the proof of Theorem 5.4. □

For the proof of Lemma 6.5 we need some supporting results.

Lemma 6.6. *Suppose that $\mu(c)>1$. Then, on $\{T_t \to \infty\}$, for each fixed $s \in (0, \infty)$*

$$\frac{Y_{t_k+s}}{Y_{t_k}} \to 1 \qquad \textit{a.s., as } k \to \infty. \tag{6.11}$$

With the

Corollary 6.7. *Suppose that $\mu(c)>1$. Then, on $\{T_t \to \infty\}$,*

$$\frac{Y_{t_{k+j,n}}}{Y_{t_{k,n}}} \to 1 \qquad \textit{a.s., as } k \to \infty, \tag{6.12}$$

for any fixed j and n.

Proof of the Corollary. Choose $s = \frac{jc}{n}$ and intersect the sets where (6.11) holds for $t_0 = 0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}$. □

Proof of Lemma 6.6. We prove the lemma for $s \leqq c$. The claim of the lemma then follows if we use this on $s_1 = c$ and on $s_2 = s - \left[\frac{s}{c}\right] c$.

We make the convention

$$Y_t = 1 \qquad \text{for } t < 0. \tag{6.13}$$

Define $\{{}_xY_t\}$ similar to $\{{}_xZ_t\}$. Then, for $s \leqq c$, we can write

$$Y_{t+s} = \sum_{x \in \mathscr{I}(t) \setminus \mathscr{I}(t,c)} e^{-\alpha\sigma_x} {}_xY_{t+s-\sigma_x} + Y_{t,c}. \tag{6.14}$$

From this we obtain ($N(t,c)$ as in Sect. 5)

$$\begin{aligned} \frac{Y_{t+s}}{Y_t} &= 1 + \left\{\frac{N(t,c)\,e^{-\alpha t}}{Y_t}\right\} \left\{\frac{\sum_{x \in \mathscr{I}(t) \setminus \mathscr{I}(t,c)} e^{-\alpha(\sigma_x - t)} ({}_xY_{t+s-\sigma_x} - 1)}{N(t,c)}\right\} \\ &= 1 + S_1(t)\, S_2(t), \quad \text{say.} \end{aligned} \tag{6.15}$$

Since

$$N(t,c)\,e^{-\alpha t} \leqq (Y_t - Y_{t,c})\, e^{\alpha c} \leqq e^{\alpha c}\, Y_t, \tag{6.16}$$

it is enough to show that, on $\{T_t \to \infty\}$,

$$S_2(t_k) \to 0 \quad \text{a.s., as } k \to \infty. \tag{6.17}$$

To do so, we shall use Propositions 4.1 and 4.3 like in the proof of Lemma 5.10.

$\mathscr{I}(t_k) \setminus \mathscr{I}(t_k, c)$, $N(t_k, c)$ and σ_x on $\{x \in \mathscr{I}(t_k) \setminus \mathscr{I}(t_k, c)\}$ are $\mathscr{A}_{T_{t_k}}$-measureable. Further, conditioned on $\mathscr{A}_{T_{t_k}}$, $\{{}_xY_t\}$, $x \in \mathscr{I}(t_k) \setminus \mathscr{I}(t_k, c)$, are mutually independent and their distributions are unaffected by the conditioning. Observe that, for $x \in \mathscr{I}(t_k)$,

$$e^{-\alpha(\sigma_x - t_k)} \leqq 1, \tag{6.18}$$

and that, for $t \leqq s$,

$$Y_t \leqq \begin{cases} \sum_{x \in \mathscr{I}(t) \setminus \mathscr{I}(t, s-t)} e^{-\alpha\sigma_x} + Y_{t, s-t} \leqq T_s + Y_s, & t \geqq 0 \\ 1 & t < 0. \end{cases} \tag{6.19}$$

Finally, from Lemma 5.12

$$P\left[\liminf_{k \to \infty} \frac{N(t_{k+1}, c)}{\sum_{j=1}^{k} N(t_j, c)} > 0 \mid T_t \to \infty\right] = 1. \tag{6.20}$$

Proposition 4.1 yields that, for any $\varepsilon > 0$, on $\{T_t \to \infty\}$,

$$\sum_{k=1}^{\infty} P[S_2(t_k) > \varepsilon \mid \mathscr{A}_{T_{t_k}}] < \infty \quad \text{a.s.,} \tag{6.21}$$

which by virtue of Proposition 4.3 yields (6.17). This completes the proof of the lemma. □

Lemma 6.8. *There exists a $K < \infty$, such that, on $\{T_t \to \infty\}$,*

$$\limsup_{t \to \infty} \frac{e^{-\alpha t} T_t}{Y_t} \leqq K \quad a.s. \tag{6.22}$$

Proof. The lemma follows from the fact that

$$\frac{T_t}{Y_t} \leqq \frac{T_t}{Y_t - Y_{t,c}} \leqq \frac{T_t}{e^{-\alpha c} N(t, c)\, e^{-\alpha t}} \tag{6.23}$$

and Lemma 3.6. □

Before the proof of Lemma 6.5 we need analogues to Lemma 5.8 and Corollary 5.9. Recall from section 5 that

$$\phi^s(t) = \begin{cases} \phi(t) & \text{for } t > s \\ 0 & \text{otherwise.} \end{cases} \tag{6.24}$$

Lemma 6.9. *Suppose that ϕ satisfies Condition* 6.2. *Then, if $\mu(c)>1$, on $\{T_t\to\infty\}$,*

$$\lim_{s\to\infty}\limsup_{k\to\infty}\frac{e^{-\alpha t_k}Z^{\phi'}_{t_k}}{Y_{t_k}}=0 \quad \text{a.s..} \tag{6.25}$$

and, for each n, on the same set,

$$\lim_{s\to\infty}\limsup_{k\to\infty}\frac{e^{-\alpha t_{k,n}}Z^{\phi'}_{t_{k,n}}}{Y_{t_{k,n}}}=0. \tag{6.26}$$

Further (with the help of Corollary 6.7 proved like Corollary 5.9).

Corollary 6.10. *Suppose that ξ satisfies Condition* 6.1. *Then for each n, on* $\{T_1\to\infty\}$,

$$\lim_{c\to\infty}\limsup_{k\to\infty}\frac{Y_{t_{k,n},c}}{Y_{t_{k,n}}}=0. \quad \square \tag{6.27}$$

Proof of Lemma 6.9. Certainly it is enough to prove (6.25) for $s=rc$ with r integer. Observe that (just like (5.28)), on $\{T_t\to\infty\}$,

$$\frac{\sum\limits_{i=T_{t_k}+1}^{T_{t_{k+1}}} V_{x_i}}{T_{t_{k+1}}-T_{t_k}}\to E[V] \quad \text{a.s., as } k\to\infty. \tag{6.28}$$

Hence, analogous to (5.29), on $\{T_t\to\infty\}$,

$$\begin{aligned}
&\limsup_{k\to\infty}\frac{e^{-\alpha t_{k+r}}Z^{\phi'}_{t_{k+r}}}{Y_{t_{k+r}}}\\
&=\limsup_{k\to\infty}\frac{\sum\limits_{i=1}^{T_{t_k}}\phi_{x_i}(t_{k+r}-\sigma_{x_i})\,e^{-\alpha t_{k+r}}}{Y_{t_{k+r}}}\\
&\leqq\limsup_{k\to\infty}\frac{\sum\limits_{i=1}^{T_{t_0}}V_{x_i}e^{-(\alpha-\beta)t_{k+r}}+\sum\limits_{j=0}^{k-1}\sum\limits_{i=T_{t_j}+1}^{T_{t_{j+1}}}V_{x_i}e^{-(\alpha-\beta)(t_{k+r}-t_{j+1})}e^{-\alpha t_j}}{Y_{t_{k+r}}}\\
&\leqq 0+e^{\alpha c}E[V]\limsup_{k\to\infty}\sum_{j=0}^{k-1}\left(\frac{e^{-\alpha t_{j+1}}T_{t_{j+1}}}{Y_{t_{j+1}}}\right)\left(\frac{Y_{t_{j+1}}e^{-(\alpha-\beta)(k+r-j-1)c}}{Y_{t_{k+r}}}\right) \quad \text{a.s..}
\end{aligned} \tag{6.29}$$

The zero comes from Lemma 6.6. The first factors in the last sum are uniformly dominated (a.s.) due to Lemma 6.8. Reversing the order of summation and using Lemma 6.6 again it is easy to dominate the second factors geometrically and hence to deduce by dominated convergence (K from Lemma 6.8), on $\{T_t\to\infty\}$,

$$\limsup_{k\to\infty}\frac{e^{-\alpha t_{k+r}}Z^{\phi'}_{t_{k+r}}}{Y_{t_{k+r}}}\leqq e^{\alpha c}E[V]\,K\sum_{j=r}^{\infty}e^{-(\alpha-\beta)jc} \quad \text{a.s.} \quad \square \tag{6.30}$$

Proof of Lemma 6.5. The lemma follows by Corollary 6.7 if we show that, on $\{T_t \to \infty\}$,

$$\frac{e^{-\alpha t_{k,n}} Z^{\phi}_{t_{k,n}}}{Y_{t_{k,n}}} \to m^{\phi}_{\infty} \quad \text{a.s., as } k \to \infty. \tag{6.31}$$

Suppose first that

$$\phi(t)=0 \quad \text{for } t \geqq \frac{j_0 c}{n}. \tag{6.32}$$

By the same corollary the limit (6.31) then follows from, on $\{T_t \to \infty\}$,

$$\lim_{j\to\infty} \limsup_{k\to\infty} \left| \frac{e^{-\alpha t_{k+j+r,n}} Z^{\phi}_{t_{k+j+r,n}}}{Y_{t_{k,n}}} - m^{\phi}_{\infty} \right| = 0 \quad \text{a.s.} \tag{6.33}$$

To prove this consider $j \geqq j_0$ and $r > 0$. Then, on $\{T_t \to \infty\}$,

$$\begin{aligned}
&\limsup_{k\to\infty} \left| \frac{e^{-\alpha t_{k+j+r,n}} Z^{\phi}_{t_{k+j+r,n}}}{Y_{t_{k,n}}} - m^{\phi}_{\infty} \right| \\
&\quad \leqq \limsup_{k\to\infty} \left| \frac{e^{-\alpha t_{k,n}} N(t_{k,n}, (j+r)c/n)}{Y_{t_{k,n}}} \right| \\
&\quad \limsup_{k\to\infty} \Big| \sum_{x \in \mathscr{I}(t_{k,n}) \setminus \mathscr{I}(t_{k,n}, (j+r)c/n)} e^{-\alpha(\sigma_x - t_{k,n})} \\
&\quad\quad (e^{-\alpha(t_{k+j+r,n}-\sigma_x)}\, {}_x Z^{\phi}_{t_{k+j+r,n}-\sigma_x} - m^{\phi}_{t_{k+j+r,n}-\sigma_x}) / N(t_{k,n}, (j+r)c/n) \Big| \\
&\quad + \limsup_{k\to\infty} \left| \frac{\sum_{x \in \mathscr{I}(t_{k,n}) \setminus \mathscr{I}(t_{k,n}, jc/n)} e^{-\alpha\sigma_x} (m^{\phi}_{t_{k+j+r,n}-\sigma_x} - m^{\phi}_{\infty})}{Y_{t_{k,n}}} \right| \\
&\quad + \{\sup_t m^{\phi}_t\} \limsup_{k\to\infty} \frac{Y_{t_{k,n}, jc/n}}{Y_{t_{k,n}}} \quad \text{a.s.}
\end{aligned} \tag{6.34}$$

Now, on $\{T_t \to \infty\}$,

$$\frac{e^{-\alpha t_{k,n}} N(t_{k,n}, (j+r)c/n)}{Y_{t_{k,n}}} \leqq e^{\alpha(j+r)c/n}, \tag{6.35}$$

and the second factor of the first term in (6.34) can be shown to be zero a.s. In fact, with $k \to \infty$ only on $\{k_0, k_0+n, k_0+2n, \ldots\}$, this follows just like the convergence of $S_{12}(t_k)$ in the proof of Lemma 5.10. But $k_0 = 1, 2, \ldots, n$ can be chosen arbitrarily. The second term can be made arbitrarily small by choice of r large, according to Proposition 2.2 and the definition of Y_t. By virtue of Corollary 6.10 the last term tends to zero, as $j \to \infty$.

Finally, use Lemma 6.9 to remove (6.32). □

7. Addendum

Consider a branching process counted with a random characteristic:

$$Z^{\phi}_t = \sum_{x \in \mathscr{I}} \phi_x(t - \sigma_x). \tag{7.1}$$

Of course this sum makes sence for not necessarily independent ϕ-processes. The question arises whether our limit theorems are valid for some larger class of characteristics. We shall show that indeed they are for characteristics which may depend not only on x's own life but also on its whole daughter process. Such generalized characteristics play a fundamental role in a recent paper by Jagers (1981), where different aspects of sampling in a supercritical branching process at a late time point are treated.

To any $x \in \mathscr{I}$, let

$$\mathscr{I}_x = \{(x, y);\ y \in \mathscr{I}\} \tag{7.2}$$

where $(x, 0) = x$, i.e. $\mathscr{I}_x$ consists of x and its potential progeny. Let $\{\phi(t)\} = \{\phi(t, \omega)\}_{t \in R}$ denote a real-valued, non-negative stochastic process vanishing for t negative, which may depend on all coordinates in Ω (recall: $(\Omega, \mathscr{B}, P) = \prod_{x \in \mathscr{I}} (\Omega_x, \mathscr{B}_x P_x)$). Denoting by π_x the shift operator that maps the (x, y)-coordinate on the y-coordinate (the x-coordinate on the 0-coordinate) we define

$$\phi_x(t, \omega) = \phi(t, \pi_x(\omega)), \tag{7.3}$$

and

$$z_t^\phi = \sum_{x \in \mathscr{I}} \phi_x(t - \sigma_x) \tag{7.4}$$

as usual.

Since (cf. Proposition 2.1)

$$z_t^\phi = \phi_0(t) + \sum_{i=1}^{\xi_0(t)} {}_{(i)}z^\phi_{t-\sigma(i)}, \tag{7.5}$$

where, and this is the key to our results, the ${}_{(j)}z_t^\phi$-processes are independent copies of $\{z_t^\phi\}$ also independent of ξ_0 (but not necessarily of ϕ_0), renewal theory applies unchanged, and both (2.4) and Proposition 2.2 follow exactly like before. Also Theorem 3.1 and its proof are valid without any changes.

For age-truncated ϕ (i.e. $\phi(a)$ vanishing for all ages a larger than some constant), and reproductions ξ satisfying the conditions of Theorem 5.4 a.s. convergence of $e^{-\alpha t} z_t^\phi$ follows as before. However, since the ϕ_x's need not be independent, the proof of Lemma 5.8 is no longer valid. But, define

$$\psi_x(a) = \begin{cases} \sup\limits_t \dfrac{e^{-\alpha t}\phi_x(t)}{h(t)} = U_x & \text{for } 0 \leqq a \leqq 1 \\ 0 & \text{otherwise.} \end{cases} \tag{7.6}$$

Then, since ψ_x is a characteristic of the truncated type,

$$e^{-\alpha t} z_t^\psi \to Y_\infty m_\infty^\psi \quad \text{a.s., as } t \to \infty. \tag{7.7}$$

where

$$m_\infty^\psi = \int_0^1 E[U_x]\, e^{-\alpha t}\, dt < E[U_x]. \tag{7.8}$$

With $\phi'(a)$ denoting $\phi(a)$ for a larger than c and zero otherwise we have, since h does not increase, that

$$\begin{aligned}
e^{-\alpha t} z_t^{\phi'} &= e^{-\alpha t} \sum_{\sigma_x \leqq t-c} \phi_x(t-\sigma_x) \leqq e^{-\alpha t} \sum_{\sigma_x \leqq t-c} h(t-\sigma_x)\, e^{\alpha(t-\sigma_x)} U_x \\
&\leqq e^{-\alpha t} \sum_{k=1}^{[t-c]+1} \sum_{k-1 \leqq \sigma_x \leqq k} h(t-\sigma_x)\, e^{\alpha(t-\sigma_x)} U_x \\
&\leqq e^{-\alpha t} \sum_{k=1}^{[t-c]+1} h(t-k)\, e^{\alpha(t-k+1)} \sum_{k-1 \leqq \sigma_x \leqq k} U_x \\
&= \sum_{k=1}^{[t-c]+1} h(t-k) e^{\alpha} (e^{-\alpha k} z_k^{\psi}).
\end{aligned} \tag{7.9}$$

We can reverse the order of summation, and use the a.s. convergence of $e^{-\alpha k} z_k^{\psi}$, and the monotonicity of h, to dominate and conclude that

$$\limsup_{t \to \infty} e^{-\alpha t} z_t^{\phi'} \leqq e^{\alpha} \left(\int_{c-2}^{\infty} h(s)\, ds \right) Y_{\infty} \cdot E[U_x] \quad \text{a.s.} \tag{7.10}$$

Thus, the lemma is still valid, and the original proof of Theorem 5.4 works. Also the proof of Lemma 6.9 is invalid, but for age-truncated characteristics Theorem 6.3 follows as before, and with an argument analogous to (7.9) we can prove Lemma 6.9 and hence the theorem holds.

Let us give some examples of processes counted with generalized characteristics.

Example 1. If

$$\phi(t) = \begin{cases} 1 & \text{if } z_t > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{7.11}$$

Then z_t^{ϕ} counts the number of individuals who have descendents alive at t.

Example 2. Let $\phi(t)$ denote the number of pairs of cousins in the second generation both alive at time t. Then z_t^{ϕ} counts the total number of cousin relations between pairs of individuals alive at time t.

Example 3. If

$$\phi(t) = \begin{cases} 1 & \text{if the ancestor is alive, and has exactly } k-1 \text{ children alive at } t \\ 0 & \text{otherwise} \end{cases} \tag{7.12}$$

then z_t^{ϕ} counts the number of "families" consisting of k individuals.

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