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Two-Sided Bounds on the Rate of Convergence to a Stable Law

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Summary. We derive necessary and sufficient conditions for several characterizations of the rate of convergence of a sum of independent variables to a stable law. The technique used is to obtain upper and lower bounds on the rate in terms of functions depending in a very simple way on the common underlying distribution. This permits a general approach to the problem of rates of convergence.

1. Introduction

Let $S_n=\sum_1^n X_j$ be a sum of independent and identically distributed random variables whose common distribution lies in the domain of normal attraction of a stable law of exponent α, $0<\alpha<2$. Then there exist constants μ_n such that $(S_n-\mu_n)/n^{1/\alpha}$ converges in distribution to a stable law as $n\to\infty$. There are two principal approaches to estimating the rate of convergence in this limit theorem. The first is to impose a condition on the pseudomoments, or on the difference moments, of the underlying distribution, and the second is to use an order of magnitude condition on the tails of the distribution. See for example the work of Banis [1, 2], Butzer and Hahn [3], Christoph [4, 5], Cramér [6, 7], Egorov [8], Kalinauskaite [15], Paulauskas [17, 18, 19, 20], Satyabaldina [22, 23] and Zolotarev [25]. Some of these results resemble those of Heyde [12], Ibragimov [13] and Lifshits [16] regarding the rate of convergence to a normal law. Our aim in the present paper is to present a different approach to rates of convergence to a stable law, which makes it a matter to derive general characterizations of rates. Our technique seems to be new in the context of stable convergence, although it has been used before in connection with normal convergence; see [10]. We pause here to describe it.

Let Δ_n denote the uniform distance between the distribution function of $(S_n-\mu_n)/n^{1/\alpha}$ and that of the limiting stable law. We shall prove that under mild

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conditions there exist positive constants C_1, C_2 and c, and positive functions f and g depending in a simple way on the underlying distribution, such that $\Delta_n \leqq C_1[f(n)+n^{-c}]$ and $g(n) \leqq C_2(\Delta_n+n^{-c})$. Therefore $f(n)$ and $g(n)$ describe the behaviour of Δ_n up to terms of $O(n^{-c})$. Many questions about the asymptotic behaviour of Δ_n can now be rephrased in terms of f and g, which are much easier to handle than Δ_n itself. To emphasize the generality of this approach we note that some of the classic sufficient conditions for rates of convergence, such as those of Cramér [6, 7], may be derived in this manner.

The theory of non-normal stable laws falls naturally into three classes: $1<\alpha<2$, $\alpha=1$ and $0<\alpha<1$. We shall follow tradition by treating these classes separately, in Sects. 2, 3 and 4, respectively. The proofs of our main results are placed together in Sect. 5.

We close this section with some notation. Let F denote the common distribution function of the summands X_j. A necessary and sufficient condition for F to lie in the domain of normal attraction of a stable law of exponent α is that it admit the representations

$$1-F(x)=c_1 x^{-\alpha}+o(x^{-\alpha}) \quad \text{and} \quad F(-x)=c_2 x^{-\alpha}+o(x^{-\alpha})$$

as $x \to \infty$, for nonnegative constants c_1 and c_2 with $c_1+c_2>0$ [14, Theorem 2.6.7, p. 92]. Define

$$S(x)=1-F(x)+F(-x)-(c_1+c_2)x^{-\alpha}, \qquad x>0$$

(the remainder in the tail sum), and

$$D(x)=1-F(x)-F(-x)-(c_1-c_2)x^{-\alpha}, \qquad x>0$$

(the remainder in the tail difference). We shall always assume that $|S(x)| + |D(x)| = o(x^{-\alpha})$, and if $1<\alpha<2$, that $E(X_1)=0$. Let

$$a_1=(c_1+c_2)\int_0^\infty u^{-\alpha}\sin u\, du$$

for $0<\alpha<2$ (the integral converges only in the Riemann sense if $\alpha \leqq 1$), and

$$a_2=\begin{cases} (c_1-c_2)\int_0^\infty u^{-\alpha}(1-\cos u)\,du & \text{if } 1<\alpha<2 \\ c_1-c_2 & \text{if } \alpha=1 \\ -(c_1-c_2)\int_0^\infty u^{-\alpha}\cos u\, du & \text{if } 0<\alpha<1. \end{cases}$$

In the case $\alpha=1$ assume $\int_1^\infty |D(x)|\,dx<\infty$, and for any $a>0$, define

$$\mu=\int_0^a [1-F(x)-F(-x)]\,dx+\int_a^\infty D(x)\,dx-(c_1-c_2)(\gamma+\log a),$$

where γ denotes Euler's constant. It is easily checked that μ does not depend on a. Set $\mu_n=0$ if $0<\alpha<1$ or $1<\alpha<2$; $n(\mu+a_2\log n)$ if $\alpha=1$. Then $(S_n-\mu_n)/n^{1/\alpha}$ has a limiting stable law whose distribution G has characteristic function ψ given by

$$-\log\psi(t)=\begin{cases}|t|^\alpha[a_1+ia_2\operatorname{sgn}(t)] & \text{if } 0<\alpha<1 \text{ or } 1<\alpha<2\\ |t|[a_1+ia_2\operatorname{sgn}(t)\log|t|] & \text{if } \alpha=1.\end{cases}$$

This may be seen from the results of [9, p. 580]; it will also be proved during the course of our investigations. Let F_n denote the distribution function of $(S_n-\mu_n)/n^{1/\alpha}$, and set

$$\Delta_n=\sup_x|F_n(x)-G(x)|.$$

The symbol C, with or without subscripts, will denote a positive generic constant. It will in general depend on the underlying distribution F, but not on n.

2. The Case $1<\alpha<2$

We shall assume throughout that $E(X_1)=0$. Our first result provides an upper bound without any additional restrictions on F.

Theorem 1. *If F is in the domain of normal attraction of a stable law of exponent α, $1<\alpha<2$, then*

$$\Delta_n\leqq Cn\left\{n^{-2/\alpha}\int_0^{n^{1/\alpha}}x|S(x)|\,dx+n^{-3/\alpha}\int_0^{n^{1/\alpha}}x^2|D(x)|\,dx\right.$$
$$\left.+n^{-1/\alpha}\int_{n^{1/\alpha}}^\infty[|S(x)|+|D(x)|]\,dx+n^{-2/\alpha}\right\}.$$

It follows immediately that if $|S(x)|+|D(x)|=O(x^{-\beta})$, where $\alpha<\beta<2$, then $\Delta_n=O(n^{1-\beta/\alpha})$ (Cramér [6, 7]). By using integral approximations to series it is also easily proved from Theorem 1 that if $\alpha\leqq\beta<2$ and

$$\int_1^\infty x^{\beta-1}[|S(x)|+|D(x)|]\,dx<\infty, \tag{1}$$

then

$$\sum_1^\infty n^{\beta/\alpha-2}\Delta_n<\infty.$$

Condition (1) is equivalent to a restriction on the difference moment of F. Since

$$|1-F(x)-[1-G(x)]|=|1-F(x)-c_1x^{-\alpha}|+O(x^{-2\alpha})$$

and

$$|F(-x)-G(-x)|=|F(-x)-c_2x^{-\alpha}|+O(x^{-2\alpha})$$

then $|S(x)+D(x)|=2|F(x)-G(x)|+O(x^{-2\alpha})$ and $|S(x)-D(x)|=2|F(-x)-c_2x^{-\alpha}|+O(x^{-2\alpha})$. But $|S|+|D|\leqq|S-D|+|S+D|\leqq 2(|S|+|D|)$, and so (1) holds if and only if

$$\int_{-\infty}^{\infty}|x|^{\beta-1}|F(x)-G(x)|\,dx<\infty.$$

This is generally a little weaker than the more common pseudomoment condition,

$$\int_{-\infty}^{\infty}|x|^{\beta}|d[F(x)-G(x)]|<\infty;$$

see Zolotarev [25].

Our next result gives an improved upper bound on Δ_n, and a lower bound, under slightly more restrictive conditions.

Theorem 2. *If S is ultimately monotone then*

$$\Delta_n\leqq Cn\left\{n^{-2/\alpha}\int_0^{n^{1/\alpha}}x|S(x)|\,dx+\int_{n^{1/\alpha}}^{\infty}x^{-1}|S(x)|\,dx\right.$$
$$\left.+n^{-3/\alpha}\int_0^{n^{1/\alpha}}x^2|D(x)|\,dx+n^{-1/\alpha}\int_{n^{1/\alpha}}^{\infty}|D(x)|\,dx+n^{-2/\alpha}\right\},$$

and if in addition D is ultimately of the one sign,

$$C\{\Delta_n+n^{1-2/\alpha}\}\geqq n\left\{n^{-2/\alpha}\int_0^{n^{1/\alpha}}x|S(x)|\,dx+|S(n^{1/\alpha})|\right.$$
$$\left.+n^{-3/\alpha}\int_0^{n^{1/\alpha}}x^2|D(x)|\,dx+n^{-1/\alpha}\int_{n^{1/\alpha}}^{\infty}|D(x)|\,dx\right\}.$$

The analogue of the Berry-Esséen theorem for convergence to G imposes the difference moment condition

$$\int_{-\infty}^{\infty}|x|\,|F(x)-G(x)|\,dx<\infty, \tag{2}$$

and gives a rate of convergence of $O(n^{1-2/\alpha})$; see Satyabaldina [22, 23]. It follows from Theorem 2 that the distribution with tails given by

$$1-F(x)=c_1x^{-\alpha}+cx^{-2} \quad\text{and}\quad F(-x)=c_2x^{-\alpha}-cx^{-2}$$

for all sufficiently large x, where c is any constant, has the property that $\Delta_n=O(n^{1-2/\alpha})$. However, condition (2) fails to hold for this distribution unless $c=0$.

To demonstrate the utility of our results we shall derive some characterizations of rates of convergence of the type obtained in [4, 5, 8, 15].

Corollary 1. *If $\alpha\leqq\beta<2$, if S is ultimately monotone and D ultimately of the one sign, then*

$$\sum_1^{\infty}n^{\beta/\alpha-2}\Delta_n<\infty \tag{3}$$

if and only if

$$\int_{-\infty}^{\infty} |x|^{\beta-1} |F(x)-G(x)|\, dx < \infty. \tag{4}$$

If $\alpha<\beta<2$ and S and D are both ultimately monotone, then

$$\Delta_n = O(n^{1-\beta/\alpha}) \tag{5}$$

if and only if

$$|F(x)-G(x)|+|F(-x)-G(-x)| = O(x^{-\beta}). \tag{6}$$

If S and D are both ultimately monotone then

$$\Delta_n = O(n^{1-2/\alpha})$$

if and only if

$$\int_0^{\infty} x\,|S(x)|\, dx < \infty \quad \textit{and} \quad |D(x)| = O(x^{-2}).$$

Proof. We prove only that (3)⇒(4) and (5)⇒(6). Firstly, if (3) holds then from Theorem 2,

$$\sum_1^{\infty} \left\{ n^{\beta/\alpha-1} |S(n^{1/\alpha})| + n^{\beta/\alpha-1/\alpha-1} \int_{n^{1/\alpha}}^{\infty} |D(x)|\, dx \right\} < \infty.$$

Making integral approximations to these series we find that

$$\int_1^{\infty} \left\{ y^{\beta-1} |S(y)| + y^{\beta-2} \int_y^{\infty} |D(x)|\, dx \right\} dy < \infty,$$

which is equivalent to (4). Next, if (5) is true then $|S(n^{1/\alpha})| = O(n^{-\beta/\alpha})$, and if D is monotone on (a, ∞),

$$O(n^{-\beta/\alpha}) = n^{-3/\alpha} \int_a^{n^{1/\alpha}} x^2 |D(x)|\, dx \geqq n^{-3/\alpha} |D(n^{1/\alpha})| \int_a^{n^{1/\alpha}} x^2\, dx,$$

from which follows (6).

3. The Case $\alpha = 1$

Throughout this section we impose the condition

$$\int_1^{\infty} |D(x)|\, dx < \infty. \tag{7}$$

There are some fundamental differences between the case $\alpha=1$ and that considered in the previous section. For example, even in the "ideal" situation where $|S(x)|+|D(x)| = O(x^{-2})$, the fastest rate of convergence permissible in general is $O(n^{-1}(\log n)^2)$ and not $O(n^{-1})$, as might perhaps be expected. Indeed, it is possible to derive Edgeworth expansions in which the first term is

$O(n^{-1}(\log n)^2)$; see [11]. However, if the limiting stable law is symmetric then rates of $O(n^{-1})$ may be achieved.

We first state the following analogue of Theorems 1 and 2.

Theorem 3. *If F is in the domain of normal attraction of a stable law of exponent $\alpha=1$, and if condition (7) holds, then for $n\geqq 2$,*

$$\Delta_n \leqq C\left\{n^{-1}\int_0^n x|S(x)|\,dx + n^{-2}\int_0^n x^2|D(x)|\,dx + \int_n^\infty [|S(x)|+|D(x)|]\,dx + n^{-1}(\log n)^2\right\}.$$

If in addition S is ultimately monotone then

$$\Delta_n \leqq C\left\{n^{-1}\int_0^n x|S(x)|\,dx + n\int_n^\infty x^{-1}|S(x)|\,dx + n^{-2}\int_0^n x^2|D(x)|\,dx + \int_n^\infty |D(x)|\,dx + n^{-1}(\log n)^2\right\},$$

and if also D is ultimately of the one sign,

$$C\{\Delta_n + n^{-1}(\log n)^2\} \geqq n^{-1}\int_0^n x|S(x)|\,dx + n|S(n)| + n^{-2}\int_0^n x^2|D(x)|\,dx + \int_n^\infty |D(x)|\,dx.$$

There is no difficulty in obtaining characterizations of the rate of convergence in terms of series conditions and order of magnitude conditions, using the techniques of Sect. 2 (see also [4, 5, 8, 15]). We note here only the following results, which do not follow the usual pattern.

Corollary 2. *Assume condition (7) holds. If S is ultimately monotone and D ultimately of the one sign, then*

$$\sum_1^n n^{-1}\Delta_n < \infty \tag{8}$$

if and only if

$$\int_1^\infty |S(x)|\,dx < \infty \quad \textit{and} \quad \int_1^\infty |D(x)|\log x\,dx < \infty. \tag{9}$$

If in addition D is ultimately monotone then

$$\Delta_n = O(n^{-1}(\log n)^2) \tag{10}$$

if and only if

$$\int_1^x y|S(y)|\,dy = O((\log x)^2) \quad \textit{and} \quad |D(x)| = O(x^{-2}(\log x)^2). \tag{11}$$

Proof. If (8) holds then $\int_1^\infty |S(x)|\,dx + \int_1^\infty y^{-1}\,dy \int_y^\infty |D(x)|\,dx < \infty$, which is equivalent to (9). Conversely, it follows from Theorem 3 that

$$\sum n^{-1} \Delta_n \leqq C \left\{ \int_1^\infty y^{-2}\,dy \int_1^y x\,|S(x)|\,dx + \int_1^\infty dy \int_y^\infty x^{-1}\,|S(x)|\,dx \right.$$
$$\left. + \int_1^\infty y^{-3}\,dy \int_1^y x^2\,|D(x)|\,dx + \int_1^\infty y^{-1}\,dy \int_y^\infty |D(x)|\,dx + C \right\},$$

and the right hand side is finite if (9) holds. If (10) is true then

$$\int_1^n x\,|S(x)|\,dx = O((\log n)^2) \quad \text{and} \quad \int_1^n x^2\,|D(x)|\,dx = O(n(\log n)^2).$$

The last condition implies that $|D(n)| \int_1^n x^2\,dx = O(n(\log n)^2)$, and (11) follows. Conversely, if (11) holds then for large n,

$$O((\log n)^2) = \int_1^n x\,|S(x)|\,dx \geqq |S(n)| \int_1^n x\,dx,$$

and so $|S(n)| = O(n^{-2}(\log n)^2)$. Consequently $\int_n^\infty |S(x)|\,dx = O(n^{-1}(\log n)^2)$, and (10) is now easily proved from Theorem 3.

4. The Case $0<\alpha<1$

Throughout this section we assume that the functions S and D are both ultimately monotone.

Theorem 4. *If F is in the domain of normal attraction of a stable law of exponent α, $0<\alpha<1$, and if S and D are both ultimately monotone, then*

$$\Delta_n \leqq C n \left\{ n^{-2/\alpha} \int_1^{n^{1/\alpha}} x\,|S(x)|\,dx + n^{-1/\alpha} \int_1^{n^{1/\alpha}} |D(x)|\,dx \right.$$
$$\left. + \int_{n^{1/\alpha}}^\infty x^{-1} [|S(x)| + |D(x)|]\,dx + n^{-\min(2,\,1/\alpha)} \right\}$$

and

$$C\{\Delta_n + n^{-\min(1,\,1/\alpha-1)}\} \geqq n \left\{ n^{-2/\alpha} \int_1^{n^{1/\alpha}} x\,|S(x)|\,dx + |S(n^{1/\alpha})| \right.$$
$$\left. + n^{-1/\alpha} \int_1^{n^{1/\alpha}} x\,|dD(x)| + n^{1/\alpha} \int_{n^{1/\alpha}}^\infty x^{-1}\,|dD(x)| \right\}.$$

Cramér's [6, 7] upper bounds on Δ_n are easily deduced from this result. It is also possible to characterize the rate of convergence:

Corollary 3. *If* $\alpha \leqq \beta < \min(1, 2\alpha)$ *and S and D are both ultimately monotone then*

$$\sum_1^\infty n^{\beta/\alpha - 2} \Delta_n < \infty \tag{12}$$

if and only if

$$\int_{-\infty}^{\infty} |x|^{\beta-1} |F(x) - G(x)| \, dx < \infty. \tag{13}$$

If in addition $x^\varepsilon |D(x)|$ *is ultimately nonincreasing for some* $\varepsilon > 0$, *and if* $\alpha < \beta < \min(1, 2\alpha)$, *then*

$$\Delta_n = O(n^{1-\beta/\alpha}) \tag{14}$$

if and only if

$$|F(x) - G(x)| + |F(-x) - G(-x)| = O(x^{-\beta}). \tag{15}$$

Proof. The proof is very similar to that of Corollary 1. In proving that (12)⇒(13), note that $\int_1^\infty x^\beta |dD(x)| < \infty$ if and only if $\int_1^\infty x^{\beta-1} |D(x)| \, dx < \infty$, and in showing that (14)⇒(15), observe that if $x^\varepsilon |D|$ is nonincreasing then

$$\begin{aligned} |dD(x)| &= -d[x^{-\varepsilon} \cdot x^\varepsilon |D(x)|] \\ &= \varepsilon x^{-1} |D(x)| \, dx - x^{-\varepsilon} d[x^\varepsilon |D(x)|] \geqq \varepsilon x^{-1} |D(x)| \, dx. \end{aligned}$$

Therefore if (14) holds, $O(x^{1-\beta}) = \int_1^x y |dD(y)| \geqq C |D(x)| \int_1^x dy$, giving (15).

5. The Proofs

We shall procede via a sequence of lemmas. Let ϕ denote the characteristic function of the distribution F, and ψ another characteristic function. The symbols a_1 and a_2 denote real constants with $a_1 > 0$, while b_1 and b_2 stand for real valued functions of a real variable.

Lemma 1. *Suppose* $\psi(t) = \exp\{-|t|^\alpha [a_1 + i a_2 \operatorname{sgn}(t)]\}$ *and*

$$1 - \phi(t) = -\log \psi(t) + b_1(t) + i b_2(t) + O(|t|^\beta)$$

as $t \to 0$, *where* $0 < \alpha < 2$, $\alpha < \beta \leqq 2 \min(1, \alpha)$ *and* $|b_1(t)| + |b_2(t)| = o(|t|^\alpha)$. *Then*

$$\phi(t/n^{1/\alpha})^n = \psi(t)\{1 - n[b_1(t/n^{1/\alpha}) + i b_2(t/n^{1/\alpha})]\} + r_n(t),$$

where for positive constants c, C and ε,

$$|r_n(t)| \leqq C\{|n b_1(t/n^{1/\alpha})|^2 + |n b_2(t/n^{1/\alpha})|^2 + |t|^\beta n^{1-\beta/\alpha}\} e^{-c|t|^\alpha}$$

whenever $|t| < \varepsilon n^{1/\alpha}$.

Proof. Under the conditions of the lemma we have $|1 - \phi(t)| \leqq C|t|^\alpha$ for all t. Choose δ so small that $|1 - \phi(t)| < \frac{1}{2}$ for $|t| < \delta$. Then if $|t| < \delta n^{1/\alpha}$,

$$\begin{aligned} n \log \phi(t/n^{1/\alpha}) &= -n[1 - \phi(t/n^{1/\alpha})] - \tfrac{1}{2} n [1 - \phi(t/n^{1/\alpha})]^2 - \dots \\ &= -n[1 - \phi(t/n^{1/\alpha})] + r_{n1}(t), \end{aligned}$$

where $|r_{n1}(t)| \leqq Cn|t/n^{1/\alpha}|^{2\alpha}$. Consequently $\phi(t/n^{1/\alpha})^n = \psi(t)\,\xi_n(t)$, where

$$\begin{aligned}\xi_n(t) &= \exp\{-n[b_1(t/n^{1/\alpha}) + ib_2(t/n^{1/\alpha})] + r_{n2}(t)\}\\ &= 1 - n[b_1(t/n^{1/\alpha}) + ib_2(t/n^{1/\alpha})] + r_{n3}(t),\end{aligned}$$

and for $|t| < \delta n^{1/\alpha}$,

$$\begin{aligned}|r_{n3}(t)| \leqq C\{&|nb_1(t/n^{1/\alpha})|^2 + |nb_2(t/n^{1/\alpha})|^2 + n|t/n^{1/\alpha}|^\beta\}\\ &\times \exp\{|nb_1(t/n^{1/\alpha})| + |nb_2(t/n^{1/\alpha})| + n|t/n^{1/\alpha}|^\beta\}.\end{aligned}$$

Given $0 < \Delta < a_1$ we may choose δ so small that for all n and $|t| < \delta n^{1/\alpha}$, $|nb_1(t/n^{1/\alpha})| + |nb_2(t/n^{1/\alpha})| + |t|^\beta n^{1-\beta/\alpha} < \Delta|t|^\alpha$, and the proof of Lemma 1 is now easily completed.

Lemma 2. *Suppose* $\psi(t) = \exp\{-|t|[a_1 + ia_2 \operatorname{sgn}(t)\log|t|]\}$ *and* $1 - \phi(t) = -\log\psi(t) - i\mu t + b_1(t) + ib_2(t) + O(|t|^\beta)$ *as* $t \to 0$, *where* $1 < \beta \leqq 2$, $-\infty < \mu < \infty$, $|b_1(t)| = o(|t|)$ *and* $|b_2(t)| = o(|t|\log|t|^{-1})$. *Then*

$$\phi(t/n)^n \exp[-it(\mu + a_2 \log n)] = \psi(t)\{1 - n[b_1(t/n) + ib_2(t/n)]\} + r_n(t),$$

where for positive constants c, C and ε,

$$\begin{aligned}|r_n(t)| \leqq C\{&|nb_1(t/n)|^2 + |nb_2(t/n)|^2 + |t|^\beta n^{1-\beta}\\ &+ t^2 n^{-1}[(\log n)^2 + (\log|t|)^2]\}\, e^{-c|t|}\end{aligned}$$

whenever $|t| < \varepsilon n$.

Proof. We have $|1 - \phi(t)| < C|t|\log|t|^{-1}$ for $|t| < \frac{1}{2}$. Choose $\delta < \frac{1}{2}$ so small that $|1 - \phi(t)| < \frac{1}{2}$ for $|t| < \delta$. Then if $|t| < \delta n$, $n\log\phi(t/n) = -n[1 - \phi(t/n)] + r_{n1}(t)$ where $|r_{n1}(t)| \leqq Cn[|t/n|\log|n/t|]^2$. Consequently

$$\begin{aligned}n\log\phi(t/n) = &-|t|[a_1 + ia_2 \operatorname{sgn}(t)\log|t|] - n[b_1(t/n) + ib_2(t/n)]\\ &+ it(\mu + a_2 \log n) + r_{n2}(t)\end{aligned}$$

where $|r_{n2}(t)| \leqq Cn\{|t/n|^\beta + [|t/n|\log|n/t|]^2\}$. Therefore

$$\phi(t/n)^n \exp[-it(\mu + a_2 \log n)] = \psi(t)\,\xi_n(t),$$

where

$$\begin{aligned}\xi_n(t) &= \{\cos[nb_2(t/n)] - i\sin[nb_2(t/n)]\}\exp\{-nb_1(t/n) + r_{n2}(t)\}\\ &= \{1 - inb_2(t/n) + r_{n3}(t)\}\{1 - nb_1(t/n) + r_{n4}(t)\},\end{aligned}$$

$|r_{n3}(t)| \leqq C|nb_2(t/n)|^2$ and

$$|r_{n4}(t)| \leqq C\{|nb_1(t/n)|^2 + |r_{n2}(t)|\}\exp\{|nb_1(t/n)| + |r_{n2}(t)|\}.$$

Given $0 < \Delta < a_1$ we may choose $\delta > 0$ so small that for all n and $|t| < \delta n$, $|nb_1(t/n)| + |r_{n2}(t)| \leqq \Delta|t|$, and the proof is now easily completed.

Let F_n denote the distribution with characteristic function $\phi(t/n^{1/\alpha})^n$ (in the case of Lemma 1) or $\phi(t/n)^n \exp(-it(\mu + a_2 \log n))$ (in the case of Lemma 2), and G the distribution with characteristic function ψ. Set $\Delta_n = \sup_x |F_n(x) - G(x)|$.

The following result is easily deduced from Lemmas 1 and 2, and the smoothing inequality for characteristic functions; see [21, Theorem 2, p. 109]. The constant c appearing below is a little less than that above.

Lemma 3. *There exist positive constants C, c and ε such that under the conditions of Lemma 1 (note that this implies $\alpha<\beta\leqq 2\min(1,\alpha)$),*

$$\Delta_n \leqq Cn \int_0^{\varepsilon n^{1/\alpha}} t^{-1}\{|b_1(t/n^{1/\alpha})|+|b_2(t/n^{1/\alpha})|\}\, e^{-ct^\alpha}\, dt + O(n^{1-\beta/\alpha}),$$

and under the conditions of Lemma 2,

$$\Delta_n \leqq C\int_0^{\varepsilon n} t^{-1}\{|nb_1(t/n)|+|nb_2(t/n)|+|nb_2(t/n)|^2\}\, e^{-ct}\, dt + O(n^{1-\beta}+n^{-1}(\log n)^2).$$

Let $\psi(t)$ have the meaning it does in Lemmas 1 or 2, and set $B(t)=t(1-t)[\psi(t)]^{-1}$ if $0<t<1$; 0 otherwise, and $\hat{B}(x)=\int_0^1 e^{itx} B(t)\, dt$.

Lemma 4. *For any α, $0<\alpha<2$, we have $\sup_x |\hat{B}(x)|<\infty$ and $\int_{-\infty}^{\infty} |\hat{B}(x)|\, dx<\infty$.*

Proof. We treat only the case of Lemma 1. It is clear that $\hat{B}$ is bounded, and so it suffices to prove that $|\hat{B}(x)|=O(|x|^{-1-\varepsilon})$ as $x\to\infty$, for some $\varepsilon>0$. Integrating by parts we find that

$$\hat{B}(x)=(i/x)\int_0^1 [1-2t+\alpha t^{\alpha-1}(a_1+ia_2)]\exp[t^\alpha(a_1+ia_2)+itx]\, dt.$$

Another integration by parts will prove that

$$\int_0^1 (1-2t)\exp[t^\alpha(a_1+ia_2)+itx]\, dt=O(x^{-1}).$$

To handle the remainder, note that if the complex valued function ξ is periodic of period 1 and satisfies

$$\int_0^1 |\xi(t+h)-\xi(t)|\, dt=O(h^\varepsilon)$$

as $h\to 0$, where $0<\varepsilon\leqq 1$, then $\int_0^1 \xi(t)\, e^{itx}\, dt=O(x^{-\varepsilon})$. (Apply the results of [22, p. 46].) This condition is easily checked for $\varepsilon=\min(\alpha,1)$ and $\xi(t)=t^{\alpha-1}\exp[t^\alpha(a_i+ia_2)]$ on (0,1), extended by periodicity to $(-\infty,\infty)$. This completes the proof.

Let ψ_n denote the characteristic function of the distribution F_n.

Lemma 5. *For any α, $0<\alpha<2$, we have*

$$\left|\int_0^1 \{\psi_n(t)[\psi(t)]^{-1}-1\}(1-t)\, dt\right|\leqq C\Delta_n.$$

Proof. Let $\{\xi_{nm}, m \geqq 1\}$ and $\{\xi_m, m \geqq 1\}$ be sequences of characteristic functions of variables with finite means, and having the property that $\xi_{nm} \to \psi_n$ and $\xi_m \to \psi$ as $m \to \infty$. Let G_{nm} and G_m be the respective distribution functions. Since $G_{nm} - G_m$ is integrable we may integrate by parts in the relation

$$\psi_{nm}(t) - \psi_m(t) = \int_{-\infty}^{\infty} e^{itx}\, d[G_{nm}(x) - G_m(x)],$$

obtaining

$$[\psi_{nm}(t) - \psi_m(t)]/it = -\int_{-\infty}^{\infty} e^{itx}[G_{nm}(x) - G_m(x)]\, dx.$$

Note also that $\overline{B(t)} = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{itx}\, \overline{\hat{B}(x)}\, dx$, the bar denoting conjugation. Applying Parseval's equality to this pair of Fourier transforms we see that

$$i \int_{-\infty}^{\infty} t^{-1}[\psi_{nm}(t) - \psi_m(t)]\, B(t)\, dt$$
$$= \int_{-\infty}^{\infty} [G_{nm}(x) - G_m(x)]\, \hat{B}(x)\, dx,$$

and letting $m \to \infty$ and applying Lemma 4 we may deduce Lemma 5.

In order to apply the preceding lemmas we must derive suitable versions of the functions b_1 and b_2. Note that S and D are functions of bounded variation on (a, ∞) for any $a > 0$, and that we assume $E(X_1) = 0$ if $1 < \alpha < 2$.

Lemma 6. *Fix $a > 0$, and define b_1 by either of the formulae*

$$b_1(t) = -\int_a^{\infty} (1 - \cos tx)\, dS(x) \quad \text{or} \quad b_1(t) = t \int_a^{\infty} \sin tx\, S(x)\, dx,$$

the last integral converging in the Riemann sense.

(i) *Assume* $\int_a^{\infty} |D(x)|\, dx < \infty$, *and set* $b_2(t) = t \int_a^{\infty} (1 - \cos tx)\, D(x)\, dx$. *Then*

$$1 - \phi(t) = \begin{cases} |t|^{\alpha}[a_1 + i a_2 \operatorname{sgn}(t)] + b_1(t) + i b_2(t) + O(t^2) & \text{if } 1 < \alpha < 2, \\ |t|\,[a_1 + i \operatorname{sgn}(t)(-\mu + a_2 \log|t|)] + b_1(t) + i b_2(t) + O(t^2) & \text{if } \alpha = 1, \end{cases}$$

where μ, a_1 and a_2 are as in Sect. 1.

(ii) *Assume* $0 < \alpha < 1$, *and set* $b_2(t) = \int_a^{\infty} \sin tx\, dD(x)$. *Then*

$$1 - \phi(t) = |t|^{\alpha}[a_1 + i a_2 \operatorname{sgn}(t)] + b_1(t) + i b_2(t) + O(|t|).$$

Proof. Suppose $t > 0$. It is easily proved that for $0 < \alpha < 2$ and either definition of b_1,

$$Rl[1 - \phi(t)] = a_1 t^{\alpha} + b_1(t) + O(t^2);$$

for b_2 defined as in (i),

$$-\operatorname{Im} \phi(t) = \begin{cases} a_2 t^{\alpha} + b_2(t) + O(t^3) & \text{if } 1 < \alpha < 2, \\ -\mu t + a_2 t \log t + b_2(t) + O(t^3) & \text{if } \alpha = 1; \end{cases}$$

and for b_2 defined as in (ii),

$$-\operatorname{Im}\phi(t)=a_2 t^\alpha+b_2(t)+O(t)$$

if $0<\alpha<1$. Hence the result.

Lemma 7. *Assume* $1\leqq\alpha<2$ *and* $\int_a^\infty (|S|+|D|)\,dx<\infty$, *and define* $b_1(t)=t\int_a^\infty \sin tx\,S(x)\,dx$ *and* $b_2(t)=t\int_a^\infty (1-\cos tx)\,D(x)\,dx$. *Then*

$$\int_0^\infty t^{-1}|b_1(t/n^{1/\alpha})|\,e^{-ct^\alpha}\,dt\leqq C\left\{n^{-2/\alpha}\int_0^{n^{1/\alpha}} x|S(x)|\,dx+n^{-1/\alpha}\int_{n^{1/\alpha}}^\infty |S(x)|\,dx\right\},$$

and

$$\int_0^\infty t^{-1}|b_2(t/n^{1/\alpha})|\,e^{-ct^\alpha}\,dt\leqq C\left\{n^{-3/\alpha}\int_0^{n^{1/\alpha}} x^2|D(x)|\,dx+n^{-1/\alpha}\int_{n^{1/\alpha}}^\infty |D(x)|\,dx\right\}.$$

This result follows directly from the inequalities

$$|b_1(t)|\leqq t^2\int_a^{n^{1/\alpha}} x|S(x)|\,dx+t\int_{n^{1/\alpha}}^\infty |S(x)|\,dx,$$

$$|b_2(t)|\leqq t^3\int_a^{n^{1/\alpha}} x^2|D(x)|\,dx+2t\int_{n^{1/\alpha}}^\infty |D(x)|\,dx.$$

Lemma 8. *Assume* $0<\alpha<2$ *and* S *is monotone on* (a,∞), *and define* $b_1(t)=-\int_a^\infty (1-\cos tx)\,dS(x)$. *Then for any* $c>0$,

$$\int_0^\infty t^{-1}|b_1(t/n^{1/\alpha})|\,e^{-ct^\alpha}\,dt\leqq C\left\{n^{-2/\alpha}\int_0^{n^{1/\alpha}} x|S(x)|\,dx+\int_{n^{1/\alpha}}^\infty x^{-1}|S(x)|\,dx+n^{-2/\alpha}\right\},$$

and

$$\left|\int_0^1 (1-t)\,b_1(t/n^{1/\alpha})\,dt\right|+n^{-2/\alpha}\geqq C\left\{n^{-2/\alpha}\int_0^{n^{1/\alpha}} x|S(x)|\,dx+|S(n^{1/\alpha})|\right\}.$$

Proof. The first inequality follows from the fact that

$$\begin{aligned}&\left|\int_a^\infty dS(x)\int_0^\infty t^{-1}e^{-ct^\alpha}[1-\cos(tx/n^{1/\alpha})]\,dt\right|\\
&\quad=\left|n^{-1/\alpha}\int_a^\infty S(x)\,dx\int_0^\infty e^{-ct^\alpha}\sin(tx/n^{1/\alpha})\,dt\right|+O(n^{-2/\alpha})\\
&\quad=\left|\int_a^\infty x^{-1}S(x)\,dx\int_0^\infty \alpha ct^{\alpha-1}e^{-ct^\alpha}[1-\cos(tx/n^{1/\alpha})]\,dt\right|+O(n^{-2/\alpha})\\
&\quad\leqq C\int_a^\infty x^{-1}|S(x)|\min\{1,(x/n^{1/\alpha})^2\}\,dx+O(n^{-2/\alpha}).\end{aligned}$$

To prove the second, observe that since the function $f(z)=z^{-2}(\cos z-1+\frac{1}{2}z^2)$ does not change sign on $(0,\infty)$, and satisfies $f(z)\geqq C\min(1,z^2)$, then

$$\begin{aligned}&\left|\int_a^\infty dS(x)\int_0^1(1-t)[1-\cos(tx/n^{1/\alpha})]\,dt\right|\\&\quad=\left|\int_a^\infty f(x/n^{1/\alpha})\,dS(x)\right|\geqq C_1\int_a^\infty \min\{1,(x/n^{1/\alpha})^2\}\,|dS(x)|\\&\quad\geqq C_2\left\{(1-\varepsilon)^{-2}n^{-2/\alpha}\int_a^{(1-\varepsilon)n^{1/\alpha}} x^2\,|dS(x)|+\int_{(1-\varepsilon)n^{1/\alpha}}^\infty |dS(x)|\right\}\end{aligned}$$

for any $\varepsilon\in(0,\frac{1}{2})$. Choose $\varepsilon=\varepsilon(n)$ in this range so that $(1-\varepsilon)\,n^{1/\alpha}$ is a continuity point of S. Then the last written integral on the right hand side equals $|S((1-\varepsilon)\,n^{1/\alpha})|\geqq|S(n^{1/\alpha})|$, while the first equals

$$2\int_a^{(1-\varepsilon)n^{1/\alpha}} x\,|S(x)|\,dx-(1-\varepsilon)^2\,n^{2/\alpha}\,|S((1-\varepsilon)\,n^{1/\alpha})|+O(1).$$

Therefore the lower bound is not less than

$$\tfrac{1}{2}C_2\left\{(1-\varepsilon)^{-2}\int_0^{(1-\varepsilon)n^{1/\alpha}} x|S(x)|\,dx+|S(n^{1/\alpha})|\right\}+O(n^{-2/\alpha}),$$

and since ε may be chosen arbitrarily close to zero, the proof is complete.

Lemma 9. *Assume* $1\leqq\alpha<2$, *D does not change sign on* (a,∞), *and* $\int_a^\infty |D|\,dx<\infty$. *Define* $b_2(t)=t\int_a^\infty(1-\cos tx)\,D(x)\,dx$. *Then*

$$\left|\int_0^1(1-t)\,b_2(t/n^{1/\alpha})\,dt\right|+n^{-2/\alpha}\geqq C\left\{n^{-3/\alpha}\int_0^{n^{1/\alpha}} x^2|D(x)|\,dx+n^{-1/\alpha}\int_{n^{1/\alpha}}^\infty |D(x)|\,dx\right\}.$$

This result may be proved as above, with the function f replaced by $f(z)=\int_0^1 t(1-t)(1-\cos tz)\,dt=z^{-3}[z(\cos z-1+\frac{1}{2}z^2)-2(\sin z-z+\frac{1}{6}z^3)]$.

Lemma 10. *Assume* $0<\alpha<1$ *and D is monotone on* (a,∞), *and define* $b_2(t)=\int_a^\infty \sin tx\,dD(x)$. *Then for any* $c>0$,

$$\begin{aligned}&\int_0^{n^{1/\alpha}} t^{-1}|b_2(t/n^{1/\alpha})|\,e^{-ct^\alpha}\,dt\\&\qquad\leqq C\left\{n^{-1/\alpha}\int_a^{n^{1/\alpha}} |D(x)|\,dx+\int_{n^{1/\alpha}}^\infty x^{-1}|D(x)|\,dx+n^{-1/\alpha}\right\},\end{aligned}$$

while

$$\left|\int_0^1(1-t)\,b_2(t/n^{1/\alpha})\,dt\right|\geqq C\left\{n^{-1/\alpha}\int_a^{n^{1/\alpha}} x|dD(x)|+n^{1/\alpha}\int_{n^{1/\alpha}}^\infty x^{-1}|dD(x)|\right\}.$$

Proof. The second inequality may be proved as in Lemma 8; we prove only the first. Now

$$|b_2(t)| \leqq t \left| \int_a^{n^{1/\alpha}} x\, dD(x) \right| + \left| \int_{n^{1/\alpha}}^{\infty} \sin tx\, dD(x) \right|,$$

and

$$\left| \int_{n^{1/\alpha}}^{\infty} \sin tx\, dD(x) \right| \leqq t n^{1/\alpha} |D(n^{1/\alpha})| + t \left| \int_{n^{1/\alpha}}^{\infty} D(x) \cos tx\, dx \right|,$$

the last integral existing in the Riemann sense. Using the second mean value theorem we see that the last written term is dominated by $2|D(n^{1/\alpha})|$. For $0<t<n^{-1/\alpha}$ we may also estimate this term as follows:

$$\begin{aligned} t \left| \int_{n^{1/\alpha}}^{\infty} D(x) \cos tx\, dx \right| &\leqq t \int_{n^{1/\alpha}}^{t^{-1}} |D(x)|\, dx + t \left| \int_{t^{-1}}^{\infty} D(x) \cos tx\, dx \right| \\ &\leqq t \int_{n^{1/\alpha}}^{t^{-1}} |D(x)|\, dx + 2|D(t^{-1})|. \end{aligned}$$

Therefore

$$\begin{aligned} I_n \equiv \int_0^{n^{1/\alpha}} t^{-1} |b_2(t/n^{1/\alpha})|\, e^{-ct^\alpha}\, dt \leqq C \Bigg\{ & n^{-1/\alpha} \left| \int_a^{n^{1/\alpha}} x\, dD(x) \right| + |D(n^{1/\alpha})| \\ & + \int_0^1 t^{-1} \left[t \int_1^{t^{-1}} |D(u n^{1/\alpha})|\, du + |D(n^{1/\alpha}/t)| \right] dt \Bigg\}. \end{aligned}$$

Now,

$$\int_0^1 dt \int_1^{t^{-1}} |D(un^{1/\alpha})|\, du + \int_0^1 t^{-1} |D(n^{1/\alpha}/t)|\, dt = 2 \int_{n^{1/\alpha}}^{\infty} x^{-1} |D(x)|\, dx,$$

and consequently

$$I_n \leqq C \left\{ n^{-1/\alpha} \int_a^{n^{1/\alpha}} x |dD(x)| + |D(n^{1/\alpha})| + \int_{n^{1/\alpha}}^{\infty} x^{-1} |D(x)|\, dx \right\}.$$

Integrating by parts and using the continuity point argument of Lemma 8 completes the proof.

Theorem 1 follows from Lemmas 1, 3, 6 and 7, the upper bound in Theorem 2 from Lemmas 1, 3, 6, 7 and 8, the first upper bound in Theorem 3 from Lemmas 2, 3, 6 and 7, the second from Lemmas 2, 3, 6, 7 and 8, and the upper bound in Theorem 4 from Lemmas 1, 3, 6, 8 and 10. Note that under condition (7), $|b_2(t)| = o(|t|)$ in the case $\alpha=1$. The lower bounds on Δ_n are established in essentially the same way, employing Lemma 5 in place of Lemma 3, as well as the lower bounds from Lemmas 8, 9, and 10. The only unusual aspect is the need to prove that

$$\int_0^1 |n b_j(t/n^{1/\alpha})|^2\, dt \leqq C(\Delta_n + \delta_n), \qquad j=1 \text{ and } 2,$$

where $\delta_n = n^{-\min(1, 1/\alpha - 1)}$, $n^{1-2/\alpha}$ or $n^{-1}(\log n)^2$, depending on the theorem to be proved. Let ψ_n have the meaning it does in Lemma 5. Up to terms of order δ_n

the quantities $|nb_1(t/n^{1/\alpha})|$ and $|nb_2(t/n^{1/\alpha})|$ are dominated by $C|\psi_n(t)-\psi(t)|$ uniformly in $0<t<1$, and so it suffices to show that

$$\int_0^1 |\psi_n(t)-\psi(t)|^2\, dt \leqq C\Delta_n.$$

The left hand side is dominated by

$$e\int_{-\infty}^{\infty} |\psi_n(t)-\psi(t)|^2\, e^{-t^2}\, dt = C\int_{-\infty}^{\infty} |f_n(x)-f(x)|^2\, dx,$$

where f_n and f are the densities of the distributions with characteristic functions $\psi_n(t)\, e^{-\frac{1}{2}t^2}$ and $\psi(t)\, e^{-\frac{1}{2}t^2}$, respectively. Writing h for the standard normal density we see that

$$|f_n(x)-f(x)| = \left|\int_{-\infty}^{\infty} [F_n(y)-G(y)]\, h'(x-y)\, dy\right| \leqq C\Delta_n,$$

and the proof is now easily completed.

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