Z. Wahrscheinlichkeitstheorie verw. Gebiete
57, 339–347 (1981)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Interval-Dividing Processes

Sam Gutmann

Northeastern University, Dept. of Mathematics, 360 Huntington Avenue, Boston, MA 02115, USA

Summary. If $\forall n \sum_{\pi} P(X_{\pi_1} < \ldots < X_{\pi_n}) = 1$ and $\forall \pi, n$, $P(X_{\pi_1} < \ldots < X_{\pi_n}) = P(Y_{\pi_1} < \ldots < Y_{\pi_n})$ then $P(n^{-1} \cdot [\delta(Y_1) + \ldots + \delta(Y_n)]$ converges to cnts. law on $R^1) = P(n^{-1} \cdot [\delta(Y_1) + \ldots + \delta(Y_n)]$ converges to a cnts. law on $R^1)$. Thus if $P(X_{\pi_1} < \ldots < X_{\pi_n}) = (n!)^{-1} \forall \pi, n$ then $n^{-1}[\delta(X_1) + \ldots + \delta(X_n)]$ converges a.s. The main result here generalizes this: Let $X^n_{(1)}, X^n_{(2)}, \ldots, X^n_{(n)}$ be the order statistics associated with $X_1, X_2, \ldots, X_n$. Define random variables $Z_1, Z_2, \ldots$ by $\{Z_n = i\} = \{X_n = X^n_{(i)}\}$. Then if $Z_1, Z_2, Z_3, \ldots$ are independent and $P(Z_n \leqq i) \leqq i/n$, and $\{X_i\}$ is bounded, $n^{-1} \cdot [\delta(X_1) + \ldots + \delta(X_n)]$ converges a.s.

§1. Introduction

Let $X_1, X_2, \ldots$ be a sequence of real valued random variables such that $P(X_i = X_j) = 0$ $\forall i, j$. Let $X^n_{(i)}$ be the i^{th} order statistic among $X_1, \ldots, X_n$, i.e. $\{X^n_{(1)}, \ldots, X^n_{(n)}\} = \{X_1, \ldots, X_n\}$ and $X^n_{(1)} < \ldots < X^n_{(n)}$ a.s. For convenience let $X^n_{(0)} = -\infty$ and $X^n_{(n+1)} = +\infty$. Define a sequence $Z_1, Z_2, \ldots$ such that Z_n takes on only the values $1, 2, 3, \ldots, n$ by letting $\{Z_n = i\} = \{X_n = X^n_{(i)}\}$. In other words, Z_n indicates which of the n intervals $(-\infty, X^{n-1}_{(1)})$, $(X^{n-1}_{(1)}, X^{n-1}_{(2)}), \ldots, (X^{n-1}_{(n-1)}, +\infty)$ X_n falls into. Let F_Z denote $\sigma(Z_1, Z_2, \ldots)$; equivalently F_Z is the σ-algebra generated by the events $\{X_i < X_j\}$, $i, j = 1, 2, 3, \ldots$. This definition of the Z-process from a sequence $X_1, X_2, \ldots$ will be used throughout.

The aim of this paper is to relate hypotheses on the distribution of $Z_1, Z_2, \ldots$ to convergence of the empirical distributions $n^{-1} \cdot [\delta(X_1) + \ldots \delta(X_n)]$ as $n \to \infty$. Throughout $\delta(x)$ will denote the point mass at x. The hypotheses on the distribution of $Z_1, Z_2, \ldots$ may be given directly or in terms of the events $\{X_i < X_j\}$.

Recent work on interval-dividing has centered on Kakutani's scheme, in which at each stage the longest remaining interval is divided according to a fixed [1] or random proportion [4, 5]. The results in the present paper seem quite distinct from these. In the Kakutani scheme, the interval the $(n+1)st$

0044-3719/81/0057/0339/$01.80

point falls into is determined by the first n points, i.e. $Z_{n+1} \subset \sigma(X_1, \ldots, X_n)$. In Theorem 2 below, the Z_i will be assumed to form an independent sequence, so the $(n+1)st$ point falls into an interval picked independently of the ordering of the first n points. These situations

$$(Z_{n+1} \subset \sigma(X_1, \ldots, X_n) \quad \text{and} \quad Z_{n+1} \perp \sigma(Z_1, \ldots, Z_n) \subset \sigma(X_1, \ldots, X_n))$$

are of course not mutually exclusive but for the Kakutani scheme with the longest interval divided uniformly at each stage [5], the Z_i are not an independent sequence (Z_2 and Z_3 are not independent for example).

In §2 it will be shown (Theorem 1) that the probability of the event $\{n^{-1} \cdot [\delta(X_1) + \ldots + \delta(X_n)]$ converges to a continuous law on $R^1\}$ depends only on the numbers $P(X_{\pi_1} < \ldots < X_{\pi_n})$ for all n and permutations π of $\{1, 2, \ldots, n\}$. A corollary is that $n^{-1} \cdot [\delta(X_1) + \ldots + \delta(X_n)]$ converges a.s. if $P(X_{\pi_1} < \ldots < X_{\pi_n}) = (n!)^{-1}$ for all π and n. The familiar example of sequences $X_1, X_2, \ldots$ satisfying this condition, in addition to iid continuous sequences, is the exchangeable case (for which the Glivenko-Cantelli theorem is an immediate consequence of deFinetti's theorem.) A very different example is as follows: there exists a sequence $X_1, X_2, \ldots$ satisfying $P(X_{\pi_1} < \ldots < X_{\pi_n}) = (n!)^{-1}$ such that a.s. $X_1, X_2, \ldots$ is an enumeration of the rationals.

Of course Theorem 1 "generalizes" the Kakutani scheme (or any scheme for which $\lim n^{-1}[\delta(X_1) + \ldots + \delta(X_n)]$ is known to be continuous). If $X_1, X_2, \ldots$ arise from the Kakutani scheme, and $Y_1, Y_2, \ldots$ satisfy $P(Y_{\pi_1} < \ldots < Y_{\pi_n}) = P(X_{\pi_1} < \ldots < X_{\pi_n})$ then $n^{-1}[\delta(Y_1) + \ldots + \delta(Y_n)]$ converges almost surely. But the numbers $P(X_{\pi_1} < \ldots < X_{\pi_n})$ seem hard to compute (at least exactly) for the Kakutani scheme.

Convergence to a possibly discontinuous law is discussed in §3. Though, as examples show, the event $\{n^{-1} \cdot [\delta(X_1) + \ldots + \delta(X_n)]$ converges to a law on $R^1\}$ is not in F_Z, a sufficient condition for convergence based only on the events $\{X_i < X_j\}$ can be found. Theorem 2 applies this condition to show that if $\{X_i\}$ is bounded and $Z_1, Z_2, \ldots$ is an independent sequence with $P(Z_n \leqq i) \leqq i/n$ then $n^{-1}[\delta(X_1) + \ldots + \delta(X_n)]$ converges a.s. This extends the $P(X_{\pi_1} \ldots X_{\pi_n}) = (n!)^{-1}$ condition, for which $P(Z_n \leqq i) = i/n$. Some discussion of the limits obtained in Theorem 2 follows the proof.

§2. The Continuous Limit Case

The lemma below refers to a sequence of distinct reals, not random variables. Let $x^n_{(i)}$ be defined analogously to $X^n_{(i)}$, that is $\{x^n_{(1)}, \ldots, x^n_{(n)}\} = \{x_1, \ldots, x_n\}$ and $x^n_{(1)} < \ldots < x^n_{(n)}$; let $x^n_{(0)} = -\infty$ and $x^n_{(n+1)} = +\infty$. For any set A let $\mu_\infty(A)$ denote $\lim_{n \to \infty} n^{-1} \cdot \sum_{i=1}^{n} I(x_i \in A)$ whenever the limit exists; here "I" denotes "indicator function of".

Lemma 1. *Let $x_1, x_2, \ldots$ be a sequence of distinct reals. Then $n^{-1} \cdot [\delta(x_1) + \ldots + \delta(x_n)]$ converges to a continuous law if and only if*

(1) $\mu_\infty(-\infty, x_k)$ *exists for each k, and*

(2) $\max_{i=0,1,\ldots,n} \mu_\infty(x^n_{(i)}, x^n_{(i+1)}) \to 0$ *as $n \to \infty$.*

Proof. only if: Since the limit of $n^{-1}\cdot[\delta(x_1)+\ldots\delta(x_n)]$ is a continuous law (call this limit law μ), $n^{-1}\cdot\sum_{i=1}^{n} I(x_i\in A)$ converges for every interval A, so (1) holds and each $\mu_\infty(x^n_{(i)}, x^n_{(i+1)})$ is defined. Also $\mu_\infty(A)=\mu(A)$ for every interval A.

Assume that the max in (2) does not converge to 0. Then there must exist $\varepsilon>0$ and a sequence of *nested* intervals A_n of the form $(x^n_{(i_n)}, x^n_{(i_n+1)})$ such that for all n, $\mu(A_n)>\varepsilon$. (A compactness argument shows that the intervals may be chosen to be nested: Let B_n be the union of all intervals of the form $[x^n_{(i)}, x^n_{(i+1)}]$ such that $\mu(x^n_{(i)}, x^n_{(i+1)})>\varepsilon$; adjoin $\pm\infty$ to compactify R. If the max in (2) is greater than ε for all n, then each B_n is nonempty. Of course $B_n\supset B_{n+1}$ $\forall n$. Choose $x\in\bigcap_n B_n$. Then $x\in[x^n_{(i_n)}, x^n_{(i_n+1)}]$ for some i_n; pick the leftmost interval if there is a choice of two. Then $A_n=(x^n_{(i_n)}, x^n_{(i_n+1)})$ is a nested sequence with $\mu(A_n)>\varepsilon$.) Note that $A_n\cap\{x_1, x_2, \ldots, x_n\}=\phi$. The only possibilities for $\bigcap_n A_n$ are ϕ, a singleton, or an interval. Since by countable additivity $\mu(\bigcap_n A_n)\geqq\varepsilon$, $\bigcap_n A_n$ must be an interval. But $[\bigcap_n A_n]\cap\{x_1, x_2, \ldots\}=\phi$ since $A_n\cap\{x_1, \ldots, x_n\}=\phi$, so $m^{-1}\cdot\sum_{i=1}^{m} I(x_i\in\bigcap_n A_n)=0$ for every m, contradicting $\mu(\bigcap_n A_n)\geqq\varepsilon$ and establishing (2).

if: Since the max in (2) converges to 0, $\mu_\infty(x^n_{(1)}, x^n_{(n)})$ converges to 1. Given ε choose n such that $\mu_\infty(x^n_{(1)}, x^n_{(n)})>1-\varepsilon/2$. Since

$$m^{-1}\cdot\sum_{i=1}^{m} I(x_i\in(x^n_{(1)}, x^n_{(n)}))\to\mu_\infty(x^n_{(1)}, x^n_{(n)}),$$

for m large enough $m^{-1}\cdot[\delta(x_1)+\ldots+\delta(x_m)](x^n_{(1)}, x^n_{(n)})>1-\varepsilon$. Thus $\{m^{-1}\cdot[\delta(x_1)+\ldots+\delta(x_m)]\}$ is a tight sequence. It suffices to show that $\mu_\infty(-\infty, a)$ exists for every a. By (1), $\mu_\infty(-\infty, a)$ exists when $a\in\{x_1, x_2, \ldots\}$, so assume $a\notin\{x_1, x_2, \ldots\}$. For each n there exists i_n such that $x^n_{(i_n)}<a<x^n_{(i_n+1)}$. Since

$$\begin{aligned}&(-\infty, x^n_{(i_n)})\subset(-\infty, a)\subset(-\infty, x^n_{(i_n+1)}),\ \mu_\infty(-\infty, x^n_{(i_n)})\\ &\leqq\liminf_{m\to\infty} m^{-1}\cdot[\delta(x_1)+\ldots+\delta(x_m)](-\infty, a)\\ &\leqq\limsup_{m\to\infty} m^{-1}\cdot[\delta(x_1)+\ldots+\delta(x_m)](-\infty, a)\leqq\mu_\infty(-\infty, x^n_{(i_n+1)}).\end{aligned}$$

Let $n\to\infty$. Then $\mu_\infty(-\infty, x^n_{(i_n+1)})-\mu_\infty(-\infty, x^n_{(i_n)})=\mu_\infty(x^n_{(i_n)}, x^n_{(i_n+1)})\to 0$ by condition (2). So

$$\begin{aligned}&\liminf_{m\to\infty} m^{-1}\cdot[\delta(x_1+\ldots\delta(x_m)](-\infty, a)\\ &\quad=\limsup_{m\to\infty} m^{-1}\cdot[\delta(x_1)+\ldots\delta(x_m)](-\infty, a)\end{aligned}$$

and $\mu_\infty(-\infty, a)$ exists. This completes the proof of the lemma. □

Now let $X_1, X_2, \ldots$ be a sequence of random variables with $P(X_i=X_j)=0$. Using Lemma 1, the event $\{n^{-1}\cdot[\delta(X_1)+\ldots\delta(X_n)]$ converges to a continuous law$\}$ can be shown to be measurable with respect to the σ-algebra F_Z, i.e. the

σ-algebra generated by the events $\{X_i < X_j\}$, $i, j = 1, 2, 3, \ldots$. This is done as follows: When condition (1) is written as $\{\lim_{n\to\infty} n^{-1} \cdot \sum_{i=1}^{n} I(X_i < X_K) \text{ exists}\}$ then it is transparently F_Z-measurable.

To see that condition (2) is F_Z-measurable, note (as above) that $\mu_\infty(-\infty, X_k)$ is F_Z-measurable. Now $\mu_\infty(-\infty, X^n_{(i)})$ is also F_Z-measurable, since

$$\mu_\infty(-\infty, X^n_{(i)}) = \sum_{k=1}^{n} \mu_\infty(-\infty, X_k) \cdot I(X_k = X^n_{(i)})$$

and $I(X_k = X^n_{(i)}) = \bigcup_A I(X_j < X_k \text{ iff } j \in A)$ where the union is taken over all $A \subset \{1, 2, \ldots, n\}$ with $i-1$ elements. Since $\mu_\infty(X^n_{(i)}, X^n_{(i+1)}) = \mu_\infty(-\infty, X^n_{(i+1)}) - \mu_\infty(-\infty, X^n_{(i)})$ this shows that condition (2) is F_Z-measurable. A restatement of this fact is given in the theorem below.

Theorem 1. *Let $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ be two sequences of random variables, such that $P(X_i = X_j) = 0$, and $P(X_{\pi_1} < \ldots < X_{\pi_n}) = P(Y_{\pi_1} < \ldots < Y_{\pi_n})$ for every n and every permutation π of $\{1, 2, \ldots, n\}$. Then $P(n^{-1} \cdot [\delta(X_1) + \ldots + \delta(X_n)]$ converges to a continuous law$) = P(n^{-1} \cdot [\delta(Y_1) + \ldots \delta(Y_n)]$ converges to a continuous law).*

The conclusion is immediate since the joint law of the random variables $\{I(X_i < X_j)\}\ i, j = 1, 2, \ldots$ must equal that of $\{I(Y_i < Y_j)\}\ i, j = 1, 2, \ldots$

Corollary. *Let $X_1, X_2, \ldots$ be any sequence of random variables such that $P(X_{\pi_1} < \ldots < X_{\pi_n}) = (n!)^{-1}$ for all n and permutations π. Then $n^{-1} \cdot [\delta(X_1) + \ldots + \delta(X_n)]$ converges almost surely to a continuous law.*

Proof. Let $Y_1, Y_2, \ldots$ be any sequence of continuous iid random variables and apply the Glivenko-Cantelli theorem and Theorem 1 above. □

Define the Komogorov-Smirnov statistic as $\sup_x |F_n(x) - F(x)|$ for any sequence $X_1, X_2, \ldots$ such that the empirical distribution function $n^{-1}[\delta(X_1) + \ldots \delta(X_n)]\ (-\infty, x] =: F_n(x)$ converges to a (possibly random) distribution function $F(x)$. This statistic can can be written as $\max_{\substack{i-1,\ldots,n \\ j=i-1,i}} |j/n - \mu_\infty(-\infty, X^n_{(i)})|$ and hence is also F_Z-measurable. So the distribution of the Kolmogorov-Smirnov statistic is the same for any sequence satisfying $P(X_{\pi_1} < \ldots < X_{\pi_n}) = (n!)^{-1}$ as it is for an iid sequence of continuous random variables.

§3. Possibly Discontinuous Limits

The simple first example below shows that the event $\{n^{-1} \cdot [\delta(X_1) + \ldots \delta(X_n)]$ converges to a law on $R^1\}$ is not an F_Z-measurable event.

Example 1. Let $x_1 < x_2 < x_3 < \ldots$ and $y_1 < y_2 < y_3 < \ldots$ be two sequences of reals such that $x_n \to x < \infty$ and $y_n \to +\infty$. If $X_1, X_2, X_3, \ldots$ is a sequence of random variables such that $P(X_i = x_i \forall i) = 1/2 = P(X_i = y_i \forall i)$ then $P(Z_i = i \forall i) = 1$. [Recall the definition of the Z-process in §1.] So F_Z is trivial, but $\{n^{-1} \cdot [\delta(X_1) + \ldots \delta(X_n)]$ converges to a law on $R^1\} = \{X_i = x_i \forall i\}$ which has probability 1/2.

As the next example shows, the difficulty is not only in the escape of mass to infinity.

Example 2. Let $n(\cdot)$ and $m(\cdot)$ be two increasing sequences of positive integers such that $\{n(k)\}_{k=1}^{\infty}$ and $\{m(k)\}_{k=1}^{\infty}$ are disjoint and have union $\{1,2,3,\ldots\}$. Further assume that $\lim_{k\to\infty} n(k)/k$ does not exist. Now let $x_1, x_2, \ldots$ be a sequence of reals such that $x_{n(1)} < x_{n(2)} < x_{n(3)} < \ldots$ and $x_{m(1)} > x_{m(2)} > x_{m(3)} > \ldots$ and such that $\lim_{k\to\infty} x_{n(k)} = \lim_{k\to\infty} x_{m(k)}$. Let $y_1, y_2, \ldots$ be a sequence with $y_{n(1)} < y_{n(2)} < \ldots$, $y_{m(1)} > y_{m(2)} > \ldots$ and $\lim_{k\to\infty} y_{n(k)} < \lim_{k\to\infty} y_{m(k)}$. If $X_1, X_2, \ldots$ are random variables with $P(X_i = x_i \,\forall i) = 1/2 = P(X_i = y_i \,\forall i)$, then each event $\{X_i < X_j\}$ has probability 0 or 1. Each Z_i is a constant and F_Z is the trivial σ-algebra. But $n^{-1}\cdot[\delta(X_1) + \ldots + \delta(X_n)]$ converges (to $\delta(\lim x_{n(k)})$) if and only if $X_i = x_i \,\forall i$.

There is another application for Example 2. Consider the sequence $y_1, y_2, \ldots$ given above. For each $j, m^{-1}\cdot \sum_{i=1}^{m} I(y_i < y_j)$ converges as $m\to\infty$, to 0 if $j = n(k)\,\exists k$, and to 1 if $j = m(k)$. But $n^{-1}\cdot[\delta(y_1) + \ldots + \delta(y_n)]$ does not converge. Thus condition (1) of Lemma 1 is not sufficient for the convergence of $n^{-1}\cdot[\delta(y_1) + \ldots + \delta(y_n)]$ even if $\{y_i\}$ is bounded. The purpose of the next lemma is to provide a sufficient condition for convergence to a possibly discontinuous law, based only on hypotheses on the ordering of the points.

Lemma 2. *Let $x_1, x_2, x_3, \ldots$ be distinct points in a bounded interval. Then* I *and* II *are equivalent.*

I. *$n^{-1}\cdot[\delta(y_1) + \ldots + \delta(y_n)]$ converges to a law on R^1 for every bdd set $\{y_i\}$ satisfying $y_i < y_j$ iff $x_i < x_j$.*

II. (1) $\lim_{n\to\infty} n^{-1}\cdot \sum_{i=1}^{n} I(x_i < x_k) =: \mu_\infty(-\infty, x_k)$ *exists for all k,*

and

(3) $\lim_{n\to\infty} n^{-1}\cdot \sum_{i=1}^{n} I(\mu_\infty(-\infty, x_i) < r)$ *exists for every rational r.*

The only part of this lemma that will be applied in the sequel is II ⇒ I, i.e. that II, based only on the ordering of $x_1, x_2, \ldots$, is sufficient for the convergence of $n^{-1}\cdot[\delta(x_1) + \ldots + \delta(x_n)]$. The equivalence of I and II shows that II is actually the best possible sufficient condition based only on ordering.

Pf. II ⇒ I. It is enough to show that (1) and (3) imply the convergence of $n^{-1}\cdot[\delta(x_1) + \ldots + \delta(x_n)]$, since if (1) and (3) hold for $x_1, x_2, \ldots$ then they must hold for any sequence $y_1, y_2, \ldots$ satisfying $y_i < y_j$ iff $x_i < x_j$. The assumption of boundedness has eliminated tightness considerations, so it suffices to prove that

$$n^{-1}\cdot[\delta(x_1) + \ldots + \delta(x_n)](-\infty, a) = n^{-1}\cdot \sum_{i=1}^{n} I(x_i < a)$$

converges for each a. Assume, to the contrary, that for some a and rational r

$$\liminf_{n\to\infty} n^{-1}\cdot \sum_{i=1}^{n} I(x_i < a) < r < \limsup_{n\to\infty} n^{-1}\cdot \sum_{i=1}^{n} I(x_i < a).$$

If $x_k < a$, then

$$n^{-1} \cdot \sum_{i=1}^{n} I(x_i < x_k) \leqq n^{-1} \cdot \sum_{i=1}^{n} I(x_i < a),$$

so

$$\mu_\infty(-\infty, x_k) \leqq \liminf_{n \to \infty} n^{-1} \cdot \sum_{i=1}^{n} I(x_i < a) < r.$$

If $x_k \geqq a$, then

$$n^{-1} \cdot \sum_{i=1}^{n} I(x_i < x_k) \geqq n^{-1} \cdot \sum_{i=1}^{n} I(x_i < a),$$

so

$$\mu_\infty(-\infty, x_k) \geqq \limsup_{n \to \infty} n^{-1} \cdot \sum_{i=1}^{n} I(x_i < a) > r.$$

Hence $x_k < a$ iff $\mu_\infty(-\infty, x_k) < r$, and

$$n^{-1} \cdot \sum_{i=1}^{n} I(x_i < a) \equiv n^{-1} \cdot \sum_{i=1}^{n} I(\mu_\infty(-\infty, x_i) < r).$$

But the righthand side converges by (3) and the left-hand side was assumed *not* to converge. The contradiction proves that II$\Rightarrow$I.

I$\Rightarrow$II. First assume that (1) fails to hold, i.e. for some x_k, $n^{-1} \cdot \sum_{i=1}^{n} I(x_i < x_k)$ does not converge. Let $y_j = x_j$ if $x_j \geqq x_k$, and $y_j = x_j - \varepsilon$ if $x_j < x_k$. Then for any y in the interval $(x_k - \varepsilon, x_k)$,

$$n^{-1} \cdot \sum_{i=1}^{n} I(y_i < y) \equiv n^{-1} \cdot \sum_{i=1}^{n} I(x_i < x_k).$$

Since the right-hand side doesn't converge, $\{y : n^{-1} \cdot [\delta(y_1) + \ldots + \delta(y_n)](-\infty, y)$ converges$\}$ cannot be dense and thus $n^{-1} \cdot [\delta(y_1) + \ldots + \delta(y_n)]$ cannot converge.

Next assume that (1) holds but that (3) fails, i.e. for some $r, n^{-1} \cdot \sum_{i=1}^{n} I(\mu_\infty$ $(-\infty, x_i) < r$ doesn't converge. Let $A = \{x_k : \mu_\infty(-\infty, x_k) < r\}$, and let x denote the least upper bound of A. If $x_k < x$, then $x_k \in A$ by the definition of x. If $x_k > x$ then $x_k \notin A$. Also, no x_k can equal x because for such an x_k, if $i \neq k$, $x_i < x_k$ would hold iff $x_i \in A$ iff $\mu_\infty(-\infty, x_i) < r$; but $n^{-1} \cdot \sum_{i=1}^{n} I(x_i < x_k)$ converges and $n^{-1} \cdot \sum_{i=1}^{n} I(\mu_\infty(-\infty, x_i) < r)$ doesn't. Hence $x_k < x$ iff $x_k \in A$ iff $\mu_\infty(-\infty, x_k) < r$. Now let $y_j = x_j$ if $x_j > x$ and let $y_j = x_j - \varepsilon$ if $x_j < x$. Then $n^{-1} \sum_{i=1}^{n} I(y_i < y)$ cannot converge for any $y \in (x - \varepsilon, x)$, and as before, $n^{-1} \cdot [\delta(y_1) + \ldots + \delta(y_n)]$ cannot converge, contradicting I. □

The corollary in §2 can be restated in terms of the Z-process as follows. Let $X_1, X_2, \ldots$ be any sequence of random variables such that the corresponding Z_n are independent and uniform, i.e. $P(Z_n = i) = 1/n$ for $i = 1, 2, \ldots, n$. Then

$n^{-1}\cdot[\delta(X_1)+\ldots+\delta(X_n)]$ converges a.s. The main result of this section generalizes this result by applying Lemma 2.

Theorem 2. *Let $X_1, X_2, \ldots$ be random variables taking values in a bounded interval, with $P(X_i=X_j)=0\ \forall i,j$. Assume that the corresponding $Z_1, Z_2, \ldots$ are independent and stochastically larger than uniform, i.e. $P(Z_n \leqq i) \leqq i/n$, $i=1,2,\ldots n$. Then $n^{-1}\cdot[\delta(X_1)+\ldots+\delta(X_n)]$ converges a.s.*

Pf. The conclusion will follow by Lemma 2 if it can be shown that $n^{-1}\cdot\sum_{i=1}^{n} I(X_i<X_k)$ converges a.s. as $n\to\infty$ for each k, and $n^{-1}\cdot\sum_{i=1}^{n} I(\mu_\infty(-\infty, X_i)<r)$ converges a.s. as $n\to\infty$ for every rational r. The a.s. convergence of $n^{-1}\cdot\sum_{i=1}^{n} I(X_i<X_k)$ for each k is equivalent to the a.s. convergence, for each N and $K\leqq N$ of $m^{-1}\cdot\sum_{i=1}^{m} I(X_i<X^N_{(K)})$. Convergence will now be proved in this form.

Fix N and $K\leqq N$. Let $\xi_i=I(X_{N+i}<X^N_{(K)})$ and let $S_n=\xi_1+\xi_2+\ldots+\xi_n$. Now consider the event $\{S_n=\ell\}$. On this set ℓ of the points $X_{N+1},\ldots,\ldots X_{N+n}$ must be less than $X^N_{(K)}$, so $X^N_{(K)}=X^{N+n}_{(\ell+K)}$. Thus

$$\begin{aligned} &P(\xi_{n+1}=1|S_n=\ell)\\ &\quad=P(X_{n+N+1}<X^N_{(K)}|S_n=\ell)=P(X_{n+N+1}<X^{N+n}_{(\ell+K)}|S_n=\ell)\\ &\quad=P(Z_{n+N+1}\leqq \ell+K|S_n=\ell) \end{aligned}$$

by the definition of the Z-process. But S_n depends only on the values of $Z_{N+1}, Z_{N+2}, \ldots$ and Z_{N+n}, all of which are independent of Z_{N+n+1}. So $P(Z_{n+N+1}\leqq \ell+K|S_n=\ell)=P(Z_{n+N+1}\leqq \ell+K)$. The same reasoning shows that

$$\begin{aligned} &P(\xi_{n+1}=1|S_1=s_1, S_2=s_2, \ldots S_{n-1}=s_{n-1}, S_n=\ell)\\ &\quad=P(Z_{n+N+1}\leqq \ell+K) \end{aligned}$$

and hence that $\{S_n\}$ is a Markov chain with

$$P(S_{n+1}=\ell+1|S_n=\ell)=P(Z_{n+N+1}\leqq \ell+K).$$

Now

$$E(S_{n+1}|S_1,\ldots,S_n)=E(S_{n+1}|S_n)$$

and

$$\begin{aligned} E(S_{n+1}|S_n=\ell)&=E(S_n+\xi_{n+1}|S_n=\ell)=\ell+P(\xi_{n+1}=1|S_n=\ell)\\ &=\ell+P(Z_{n+N+1}\leqq \ell+K)\leqq \ell+(\ell+K)(n+N+1)^{-1}. \end{aligned}$$

So

$$\begin{aligned} &E((S_{n+1}+K)(n+1+N+1)^{-1}|S_n=\ell)\\ &\quad\leqq[\ell+(\ell+K)(n+N+1)^{-1}+K][n+1+N+1]^{-1}\\ &\quad=(\ell+K)(n+N+1)^{-1}. \end{aligned}$$

This shows that $(S_n+K)(n+N+1)^{-1}$ is a supermartingale. (If the Z_i were actually uniform it would be the martingale associated with a Polya urn.) Now

$$(n+N)^{-1}\cdot\sum_{i=1}^{n+N} I(X_i<X_{(K)}^N)=(n+N)^{-1}\cdot\left[\sum_{i=1}^{N} I(X_i<X_{(K)}^N)\right.$$
$$\left.+\sum_{N+1}^{N+n} I(X_i<X_{(K)}^N)\right]=(n+N)^{-1}\cdot[K-1+S_n]$$

which converges as $n\to\infty$ because the supermartingale $(n+N+1)^{-1}\cdot(S_n+K)$ does.

The other step is to show the convergence of $n^{-1}\cdot\sum_{i=1}^{n} I(\mu_\infty(-\infty,X_i)<r)$. Let F_n denote the σ-algebra generated by $Z_{n+1},Z_{n+2},Z_{n+3},\ldots$ and let $T_n=\sum_{i=1}^{n} I(\mu_\infty(-\infty,X_i)<r)$. Rewriting T_n as $\sum_{i=1}^{n} I(\mu_\infty(-\infty,X_{(i)}^n)<r)$ and noting that $\mu_\infty(-\infty,X_{(i)}^n)$ is F_n-measurable shows that T_n is F_n-measurable. Also, $T_n=T_{n+1}-\sum_{i=1}^{n+1} I(\mu_\infty(-\infty,X_{(i)}^{n+1})<r)\,I(Z_{n+1}=i)$ since the term corresponding to the point X_{n+1} must be omitted in going from T_{n+1} to T_n. Now

$$\begin{aligned}E(T_n|F_{n+1})&=E(T_{n+1}|F_{n+1})\\&\quad-\sum_{i=1}^{n+1} E(I(\mu_\infty(-\infty,X_{(i)}^{n+1}<r)\,I(Z_{n+1}=i)|F_{n+1})\\&=T_{n+1}-\sum_{i=1}^{n+1} P(Z_{n+1}=i)\,I(\mu_\infty(-\infty,X_{(i)}^{n+1})<r),\end{aligned}$$

since T_{n+1} is F_{n+1}-measurable, $I(\mu_\infty(-\infty,X_{(i)}^{n+1})<r)$ is F_{n+1}-measurable, but Z_{n+1} is independent of $F_{n+1}=\sigma(Z_{n+2},Z_{n+3},\ldots)$.

The random variables $I(\mu_\infty(-\infty,X_{(i)}^{n+1})<r)$ are *decreasing* with i, since $X_{(1)}^{n+1}<X_{(2)}^{n+2}<\ldots<X_{(n+1)}^{n+1}$. Thus the weighted sum

$$\sum_{i=1}^{n+1} P(Z_{n+1}=i)\,I(\mu_\infty(-\infty,X_{(i)}^{n+1})<r)$$

is less than

$$\sum_{i=1}^{n+1} (n+1)^{-1}\cdot I(\mu_\infty(-\infty,X_{(i)}^{n+1})<r)$$

since Z_{n+1} is stochastically larger than uniform. Using this inequality in the above expression for $E(T_n|F_{n+1})$ produces

$$\begin{aligned}E(T_n|F_{n+1})&\geqq T_{n+1}-\sum_{i=1}^{n+1}(n+1)^{-1}\cdot I(\mu_\infty(-\infty,X_{(i)}^{n+1})<r)\\&=T_{n+1}-(n+1)^{-1}\cdot T_{n+1}=n(n+1)^{-1}\cdot T_{n+1}.\end{aligned}$$

Dividing by n, $E(n^{-1}\cdot T_n|F_{n+1})\geqq(n+1)^{-1}\,T_{n+1}$. Thus $n^{-1}\cdot T_n$ is a reversed submartingale and must converge as $n\to\infty$ [2, p. 333] which was to be shown. □

Perhaps Theorem 2 as stated does not make clear enough the latitude available in choosing the X_i subject to holding the distribution of the Z_i fixed. For example, the X_i can be restricted to $[0,1]$ and then chosen to be the midpoint of the interval dictated by the Z_i. If $Z_1=1$, $Z_2=1$, $Z_3=3$, $Z_4=2$, the corresponding X_i obtained by following this scheme would be $X_1=1/2$, $X_2=1/4$, $X_3=3/4$, $X_4=3/8$. Of course the midpoint can be replaced by any other proportion or a point chosen randomly in the interval. As long as $Z_1, Z_2, Z_3, \ldots$ are independent and $P(Z_n \leqq j) \leqq j/n$, $n^{-1} \cdot [\delta(X_1)+\ldots+\delta(X_n)]$ converges almost surely. In fact the X_i need not even be measurable functions and $\{n^{-1} \cdot [\delta(X_1)+\ldots+\delta(X_n)]$ converges$\}$ still contains a set of probability 1.

It suffices, in Theorem 2, to assume that $\{X_i\}$ be bounded above a.s. (the asymmetry arising because the Z_i are assumed stochastically *larger* than uniform). No mass can escape to $-\infty$ because $\mu_\infty(-\infty, X_{(1)}^n)$ must have a law stochastically smaller than Beta $(1,n)$, the law in the uniform case.

The hypothesis that $\{X_i\}$ be bounded above cannot be dropped, even if it is assumed that $P(Z_n=i) \leqq A/n$ uniformly for some $A>1$. This can be shown using a result of Dubins and Freedman [3]. In these examples, however, EZ_n/n converges to a limit greater than $\frac{1}{2}$. Perhaps $EZ_n/n \to \frac{1}{2}$ is sufficient to guarantee the tightness of $\{n^{-1}[\delta(X_1)+\ldots+\delta(X_n)]\}$.

References

1. Adler, R.L., Flatto, L.: Uniform distribution of Kakutani's interval splitting procedure. Z. Wahrscheinlichkeitstheorie Verw. Gebiete **38**, 253–259 (1977)
2. Doob, J.L.: Stochastic Processes. New York: Wiley 1953
3. Dubins, L.E., Freedman, D.A.: A sharper form of the Borel-Cantelli limma and the strong law. Ann. Math. Statist. **36**, 800–807 (1965)
4. Slud, E.: Entropy and maximal spacings for random partitions. Z. Wahrscheinlichkeitstheorie Verw. Gebiete. **41**, 341–352,
5. Van Zwet, W.R.: A proof of Kakutani's conjecture of random subdivision of longest subintervals. Ann of Probability. **6**, 133–137 (1978)

Received December 19, 1978