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Some Characterizations of Unimodal Distribution Functions*

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Let F be a distribution function and let $Q_F(l)=0$ for $l<0$ and $Q_F(l)=\sup\{F(x+l)-F\_(x)\colon x\in\mathbb{R}\}$ for $l\geqq 0$ be its Lévy concentration function. This paper has two purposes: to give a characterization of unimodal distribution functions (Theorem 3.5) and a representation theorem for the class of unimodal distribution functions (Theorem 6.2), both in terms of their Lévy concentration functions.

1. Introduction

Let F be a (right continuous) distribution function on $\mathbb{R}=(-\infty,\infty)$. Defining $F\_(-\infty)=F(-\infty)=0$, and $F(\infty)=1$, F may be considered as an u.s.c. (upper semicontinuous) function on the compact space $\bar{\mathbb{R}}=[-\infty,\infty]$, and $F\_$ as a l.s.c. (lower semicontinuous) function on $\bar{\mathbb{R}}$. We say that F is *unimodal* whenever there is an a, called a *mode* of F, such that F is convex on $(-\infty,a)$ and concave on (a,∞) (see, e.g., Lukács ([5], p. 91)). There are many characterizations and properties of unimodal distribution functions, principally given in terms of characteristic functions; for a recent survey, see, e.g., Medgyessi [6].

Let us consider now the Lévy *concentration function* Q_F of F, defined by $Q_F(l)=0$ for $l<0$ and $Q_F(l)=\sup\ \{F(x+l)-F\_(x)\colon x\in\mathbb{R}\}$ for $l\in[0,\infty]$. Q_F is again a distribution function and if F is unimodal, then Q_F is unimodal. The converse is however not true. Further let $x'=\inf\{x\colon F(x)>0\}$ and $x''=\sup\{x\colon F(x)<1\}$; then we say that F is *strictly unimodal* if it is unimodal with mode a and if it is strictly convex on (x',a) and strictly concave on (a,x''). Evidently a is the only mode of F. Hengartner and Theodorescu [3] have shown that F is strictly unimodal if and only if Q_F is strictly unimodal.

In this paper, we give first a characterization of unimodal distribution functions (Theorem 3.5) and then a representation theorem (Theorem 6.2), both

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in terms of concentrations functions. We will use the *pointer* A_F of F, a multivalued function, defined by $A_F(l)=\{x\in\bar{\mathbb{R}}: Q_F(l)=F(x+l)-F_-(x)\}$ for $l\in[0,\infty]$, which has unexpectedly nice properties.

2. Preliminary Results

Denote by $\mathscr{F}$ the set of distribution functions, by $\mathscr{F}_+$ the set of all $F\in\mathscr{F}$ such that $F_-(0)=0$, and by $\tilde{\mathscr{F}}_+$ the set of all subadditive $F\in\mathscr{F}_+$. Identifying $\mathscr{F}$ with the set of probability measures on $\mathbb{R}$, $\mathscr{F}$ is provided with the narrow topology $\sigma(\mathscr{M}_b(\mathbb{R}), C_b(\mathbb{R}))$ (see [1]). Some important properties of concentration functions are listed in the following:

Remark 2.1. (1) Q: $F\mapsto Q_F$ is a continuous convex idempotent surjection from $\mathscr{F}$ onto $\tilde{\mathscr{F}}_+$. (2) $Q_F(0)$ is the greatest jump of F on $\mathbb{R}$ and $Q_F(l)>0$ for $l>0$. (3) Q_F is continuous on $\bar{\mathbb{R}}$ if and only if Q_F is continuous at 0.

For details, see Hengartner and Theodorescu [2].

A *correspondence* A: $L\to X$ is a map from L into the power set of X such that $A(l)\neq\emptyset$ for each $l\in L$. We denote by $D(A)$ the set of all $l\in L$ for which $A(l)$ is a singleton. In the case $D(A)=L$, A can be considered as a map from L into X. If L and X are topological spaces, then we say that A is *cocontinuous* at the point $l_0\in L$, if, for each neighbourhood W of $A(l_0)$ there exists a neighbourhood V of l_0, such that $A\langle V\rangle=\bigcup_{l\in V} A(l)\subset W$. We also say that A is *cocontinuous* on L if A is cocontinuous at each $l\in L$.

From Smithson [8] we take the following:

Remark 2.2. Let A: $L\to X$ be a correspondence and $l_0\in L$. (1) If L is metrizable and X is locally compact, then A is cocontinuous at l_0 if and only if $(l_n)\subset L$, $l_n\to l_0$, $x_n\in A(l_n)$, x_0 cluster point of $(x_n)\Rightarrow x_0\in A(l_0)$. (2) Let X be separated and A cocontinuous on L. The image $A\langle K\rangle$ of each compact subset K of L is a compact subset of X if and only if each $A(l)$ is compact. (3) Let A be cocontinuous on L. The image $A\langle C\rangle$ of each connected subset C of L is a connected subset of X if and only if each $A(l)$ is connected.

From now on F is a fixed distribution function, Q its concentration function, and A its pointer. As in convex analysis, we use the convention $\infty+(-\infty)=-\infty+\infty=\infty$. Obviously, if F is not continuous, then $A(0)$ is the finite nonempty set of all $x\in\mathbb{R}$ such that $Q(0)=F(x)-F_-(x)$, while $A(0)=\bar{\mathbb{R}}$ if F is continuous.

The subsequent results rely heavily on the following:

Theorem 2.3. (1) A: $[0,\infty]\to\bar{\mathbb{R}}$ *is a point-compact correspondence.* (2) *If* $l\in(0,\infty)$, *then* $A(l)\subset\mathbb{R}$. (3) *If* $l_0\in[0,\infty]$ *and if* Q *is continuous at* l_0, *then* A *is cocontinuous at* l_0.

Proof. (1) Indeed, each $A(l)$ is the nonempty closed set of all maximal points of the u.s.c. function $x\mapsto F(x+l)-F_-(x)$ on the compact space $\bar{\mathbb{R}}$. (2) In this case, $A(l)$ is necessarily contained in the bounded set $\{x: F(x+l)\geqq Q(l)\}\cap\{x: F_-(x)\leqq 1-Q(l)\}$. (3) Let $l_n\to l_0$ and let x_0 be a cluster point of a sequence

$x_n \in A(l_n)$. Since $F(x_0+l_0)-F_-(x_0) \leqq Q(l_0) = \lim_{n\to\infty} Q(l_n) = \lim_{n\to\infty}\sup[F(x_n+l_n)-F_-(x_n)] \leqq F(x_0+l_0)-F_-(x_0) \leqq Q(l_0)$, we have $x_0 \in A(l_0)$ and hence A is cocontinuous at l_0 by Remark 2.2, (1).

Similar results hold for the correspondence $A+I$, where I: $[0,\infty]\to[0,\infty]$ is the identity map.

Lemma 2.4. *Let $0 \leqq l_0 < L \leqq \infty$ and suppose that Q is continuous on $[l_0, L]$. If $-\infty < z = \inf A\langle(l_0, L)\rangle$, then $F_-(z) = F(z)$ or z is no limit point of $A\langle(l_0, L)\rangle$.*

Proof. For $F_-(z) < F(z)$, $z_n > z$, $z_n \downarrow z$, $z_n \in A(l_n)$, $l_n \in (l_0, L)$, $l_n \to l$, we obtain the contradiction $Q(l) = F(z+l) - F_-(z) > F(z+l) - F(z) \geqq \lim\sup[F(z_n+l_n) - F_-(z_n)] = Q(l)$.

The following properties of convex functions will be frequently used and can be found, e.g., in Valentine [9] or in Rockafellar [7]:

Remark 2.5. Let f be a real valued convex function, defined on a interval J of $\mathbb{R}$. (1) f is continuous on Int J. (2) If a, b, c, $d \in J$, $a \leqq b$, $c \leqq d$, and $d-c \geqq b-a$, then $f(d)-f(c) \geqq f(b)-f(a)$. (3) Let $a<b<c$, g: $(a,c)\to\mathbb{R}$ continuous, and g locally convex on $(a,b)\cup(b,c)$; then g is convex on (a,c) if and only if $g(b) \leqq \frac{1}{2}g(b+\varepsilon_n)+\frac{1}{2}g(b-\varepsilon_n)$ for some sequence $\varepsilon_n \downarrow 0$. (4) At any $x \in \mathrm{Int}\, J$, the left derivative $f'_-(x)$ and the right derivative $f'_+(x)$ exist. Moreover, $f'_- \leqq f'_+$, f'_- and f'_+ are nondecreasing functions, f'_- is left continuous, and f'_+ is right continuous. (5) For each $x \in \mathrm{Int}\, J$, $\partial f(x) = [f'_-(x), f'_+(x)]$ is the set of subgradients of f at x, i.e., the set of all $\alpha \in \mathbb{R}$ such that $f(y)-f(x) \geqq \alpha(y-x)$ for each $y \in J$. (6) Let g be convex on J; $f+g$ reaches its minimum on J at $x \in \mathrm{Int}\, J$ if and only if $0 \in \partial f(x)+\partial g(x)$. (7) g: Int $J \to \mathbb{R}$ is convex if and only if g is a primitive of a nondecreasing function h. The set of all nondecreasing functions for which g is a primitive is the set of nondecreasing selections of the subdifferential ∂g of g.

In the next sections, use will be made of the following three technical lemmas.

Lemma 2.6. *Let $l \in (0,\infty)$ and $\varepsilon \in (0,l)$. (1) If $x, y \in A(l)$, $|x-y| \leqq l$, and if Q is concave on $[l-|x-y|, l+|x-y|]$, then Q is affine on $[l-|x-y|, l+|x-y|]$. (2) If Q is strictly concave on a neighbourhood of l and if $l \in D(A)$, then $l \in \mathrm{Int}\, D(A)$. (3) If Q is concave on $(0, 2l)$, and if Q is not affine on $(l-\varepsilon, +\varepsilon)$, and if $Q(l) > \frac{1}{2}$, then $l \in D(A)$. (4) If $Q(l)=1$, then $A(l)$ is convex.*

Proof. (1) $Q(l) \geqq \frac{1}{2}Q(x+l-y)+\frac{1}{2}Q(y+l-x) \geqq \frac{1}{2}F(x+l)-\frac{1}{2}F(y)+\frac{1}{2}F(y+l) -\frac{1}{2}F_-(x) = Q(l)$. (2) Choose $\varepsilon \in (0,l)$ such that Q is strictly concave on $(l-\varepsilon, l+\varepsilon)$. By Theorem 2.3, (3), there exists $\delta \in (0, \varepsilon/2)$ such that $A(k) \subset (A(l)-\varepsilon/4, A(l)+\varepsilon/4)$ whenever $|k-l| < \delta$. For such a k one has diam $A(k) \leqq \varepsilon/2 \leqq k$. By (1) it follows that $k \in D(A)$. (3) Suppose $x, y \in A(l)$, $x \neq y$. By (1), $|x-y| > l$, say $y > x+l$. We obtain the absurd inequality $F(y+l) = Q(l)+F_-(y) \geqq Q(l)+F(x+l) \geqq 2Q(l) > 1$. (4) $Q(l)=1$ and $x \in A(l)$ implies $F_-(x)=0$ and $F(x+l)=1$.

Lemma 2.7. *Let $l' > l > 0$, $x \in A(l)$, and $y \in A(l')$. (1) If $x \notin A(l')$ or $y \notin A(l)$, and if F is concave on $[x+l, \infty) \cap (-\infty, y+l']$, then $y < x$. (2) If $y+l' \notin A(l)+l$ or $x+l \notin A(l)+l'$, and if F is concave on $[y,\infty)\cap(-\infty, x]$, then $y+l' > x+l$.*

Proof. The proofs of these four statements being similar, we restrict ourselves to the first one. Indeed, suppose $y>x\notin A(l')$ and let F be concave on the indicated set. We have $F(y+l)-F(x+l)>F(y+l')-F(x+l')\geqq F_-(y)-F_-(x)$ and hence the contradiction $Q(l)\geqq F(y+l)-F_-(y)>F(x+l)-F_-(x)=Q(l)$.

Lemma 2.8. *Let* $0<l_0<l_1\leqq\infty$, $J=A\langle(l_0,l_1)\rangle$, $K=(A+I)\langle(l_0,l_1)\rangle$. *Suppose that* Q *is concave (respectively strictly concave) on* (l_0,l_1) *and that* A *is convex valued on* (l_0,l_1). *Then* F *is convex (respectively strictly convex) on* $\operatorname{Int} J$ *and concave (respectively strictly concave) on* $\operatorname{Int} K$.

Proof. By Theorem 2.3, (2), (3) and Remark 2.2, (3), J and K are intervals of $\mathbb{R}$. Let $l_0<\gamma<\delta<l_1$ and $[p,q]=A\langle[\gamma,\delta]\rangle$. For any $x,y,z\in(p,q)$, such that $y<x<z$, $z-x<\gamma-l_0$ and $x-y<l_1-\delta$, there exists $l\in[\gamma,\delta]$ such that $x\in A(l)$, and hence $F(x+l)-F_-(x)=Q(l)\geqq(z-x)(z-y)^{-1}\,Q(x-y+l)+(x-y)\,(z-y)^{-1}\,Q(l-z+x)$ $\geqq(z-x)\,(z-y)^{-1}\,[F(x+l)-F_-(y)]+(x-y)\,(z-y)^{-1}\,[F(x+l)-F_-(z)]$. We obtain $F_-(x)\leqq(z-x)\,(z-y)^{-1}\,F_-(y)+(x-y)\,(z-y)^{-1}\,F_-(z)$. This means that F_- is locally convex on (p,q). Therefore $F=F_-$, F is convex on (p,q), and F is convex on $\operatorname{Int} J=\bigcup_{\gamma,\delta}\operatorname{Int} A\langle[\gamma,\delta]\rangle$. The above inequalities are strict when Q is strictly concave. The proof of the dual statement for K is similar.

3. Unimodal Distribution Functions

Let $a_-=\inf\{x\in\mathbb{R}\colon F \text{ concave on } (x,\infty)\}$, $a_+=\sup\{x\in\mathbb{R}\colon F \text{ convex on } (-\infty,x)\}$. Let $\mathscr{U}$ denote the set of all unimodal distribution functions. Clearly, $F\in\mathscr{U}$ if and only if $a_-\leqq a_+$; in this case, $[a_-,a_+]$ is the set of all modes of F and $a_-=a_+$ or F is continuous. Further, let $\mathscr{U}_a$ be the set of all $F\in\mathscr{U}$ with mode a. Then $\mathscr{U}_a$ is a convex subset of $\mathscr{U}$ and $\mathscr{U}$ and $\mathscr{U}_a$ are closed in $\mathscr{F}$ with respect to the narrow topology. This last statement follows from Lukács [5], Theorem 4.5.4, p. 97.

Clearly, the next result must hold:

Theorem 3.1. *Let* $F\in\mathscr{U}$, $l>0$, *and* $x\in A(l)$. *Then* $x\leqq a_+$ *and* $x+l\geqq a_-$.

Proof. Suppose $a_+<x\in A(l)$; since $Q(l)=F(x+l)-F(x)\leqq F(a_++l)-F(a_+)\leqq Q(l)$, F is affine on the interval $(a_+,x+l)$. By Remark 2.5, (3) for any $\varepsilon>0$ sufficiently small, we have $\frac{1}{2}F(a_++\varepsilon)+\frac{1}{2}\,F(a_+-\varepsilon)<F(a_+)$, and hence we obtain the contradiction $Q(l)\geqq F(a_++l-\varepsilon)-F(a_+-\varepsilon)>F(a_++l-\varepsilon)-2F(a_+)+F(a_++\varepsilon)=F(a_++l)-F(a_+)=Q(l)$. The relation $x+l\geqq a_-$ is proved analogously.

Corollary 3.2. *If* $F\in\mathscr{U}$, *then* $Q\in\mathscr{U}$, *with smallest mode* 0.

Proof. Given $l_1,l_2>0$ and $\lambda\in(0,1)$, choose $x_1\in A(l_1)$ and $x_2\in A(l_2)$. We have $\lambda\,Q(l_1)+(1-\lambda)\,Q(l_2)\leqq F(\lambda\,x_1+(1-\lambda)\,x_2+\lambda\,l_1+(1-\lambda)\,l_2)-F_-(\lambda x_1+(1-\lambda)\,x_2)$ $\leqq Q(\lambda\,l_1+(1-\lambda)\,l_2)$ and hence $Q\in\mathscr{U}_0$. Obviously, 0 is the smallest mode (cf. Remark 2.1, (2)).

Corollary 3.3. *If* $F\in\mathscr{U}$, *then* $A(l)$ *is a convex set for each* $l\in[0,\infty]$ *and* $A(l)\subset[a_--l,a_+]$ *for* $0<l<\infty$.

Proof. By Lemma 2.6, (4) it suffices to consider the case $Q(l)<1$. For $l=0$, we have $A(0)=\mathbb{R}$ or $A(0)=\{a_-\}=\{a_+\}$. Now let $0<l<\infty$. From Remark 2.5, (2) and Theorem 3.1 it follows easily that $A(l)\subset[a_- - l, a_+]$. But in this interval $Q(l)$ is the maximum of the continuous and concave function $x\mapsto F(x+l)-F_-(x)$.

Corollary 3.4. *If $F\in\mathscr{U}$, then Q is continuous on $(0,\infty]$. Moreover (1) A is cocontinuous on $(0,\infty]$; (2) if $B\subset(0,\infty]$ is connected, then $A\langle B\rangle$ is connected; (3) if $C\subset(0,\infty]$ is compact, then $A\langle C\rangle$ compact.*

Proof. Indeed, apply successively Remark 2.5, (1), Theorem 2.3, and Remark 2.2, (2), (3).

Our first main result is contained in:

Theorem 3.5. *$F\in\mathscr{U}$ and only if $Q\in\mathscr{U}$ and A is convex valued for $l>0$.*

Proof. The necessity of these conditions follows from the Corollaries 3.2 and 3.3. Conversely, assume that $Q\in\mathscr{U}$ and that each $A(l)$, $l>0$, is a convex set. By Corollary 3.4, (2), $J=A\langle(0,\infty)\rangle$ and $K=(A+I)\langle(0,\infty)\rangle$ are intervals in $\mathbb{R}$. By Lemma 2.4, for $x=\inf J$, we have $F(x)=0$ or $J=\{x\}$. Indeed, for $F_-(x)>0$, $y>x$, $F(y)>1-F_-(x)$, $Q(l)\geqq F(y)$, we get the contradiction $A(l)\cap(-\infty,x)\neq\emptyset$. Similarly, $F_-(y)=1$ or $K=\{y\}$ if $y=\sup K$. Finally, $J\cup K$ contains the set $\{x\colon 0<F(x)<1\}$. Indeed, assume that $\sup J=\alpha<\beta=\inf K$; we obtain a point $x\in A((\beta-\alpha)/2)$ and hence the contradiction $x\leqq\alpha$, $\beta>x+(\beta-\alpha)/2\in K$. From Lemma 2.8 it now follows that F is convex on J - and thus on $(-\infty,\alpha)$ - and F is concave on K - and thus on (β,∞).

We give now an alternative short proof of the result of Hengartner and Theodorescu [3]:

Corollary 3.6. *F is strictly unimodal if and only if Q is strictly unimodal. Moreover, in this case $D(A)\supset(0,L)$, where $L=\sup\{l\colon Q(l)<1\}$.*

Proof. In view of Theorem 3.5, Lemma 2.8, and Lemma 2.6, (4), it suffices to show that $(0,L)\subset D(A)$ whenever Q is strictly unimodal. Only the case $L>0$ is of interest. By Lemma 2.6, (2), (3), $D(A)$ contains a largest nonempty set of the form (l_0,L), $l_0\geqq 0$.

Put $z_0=\inf A\langle(l_0,L)\rangle$, $z_1=\sup A\langle(l_0,L)\rangle$, $z_2=\inf(A+I)\langle(l_0,L)\rangle$ and $z_3=\sup(A+I)\langle(l_0,L)\rangle$. We may suppose $z_0<z_1$. By the Lemmas 2.8 and 2.4, F is strictly convex on $[z_0,z_1]$, strictly concave on $[z_2,z_3]$, $F=0$ on $(-\infty,z_0]$, and $F=1$ on $[z_3,\infty)$. By Lemma 2.7, the function A is nonincreasing on (l_0,L), and $A+I$ is nondecreasing on (l_0,L). Now assume that $l_0>0$; by Lemma 2.6, (1), $A(l_0)$ consists of a finite number of points with mutual distances $>l_0$. From Theorem 2.3 it follows that there exist a sequence $l_n\downarrow l_0$ and a point $x\in A(l_0)$ such that $A(l_n)\uparrow x$. Hence $x=z_1$. Let x' be another point of $A(l_0)$. We have either $x'<z_1-l_0$ and then $Q(l_0)=F(x'+l_0)-F_-(x')<F(z_1)-F(z_1-l_0)\leqq Q(l_0)$, or $x'>z_1+l_0$, and then $Q(l_0)=F(x'+l_0)-F(x')<F(z_2+l_0)-F_-(z_2)\leqq Q(l_0)$.

We shall call $F\in\mathscr{U}$ *monomodal* if it has a unique mode and *polymodal* if it has more than one mode, i.e., if $a_-<a_+$. Note that every discontinuous $F\in\mathscr{U}$ is monomodal.

The three following examples show the nonvalidity of some weaker forms of Theorem 3.5 and of Corollary 3.6. More precisely: (a) If Q is continuous, and if A is convex valued, then F is not necessarily unimodal. (b) If Q is monomodal and if F is discontinuous, then F is not necessarily unimodal. (c) If Q is monomodal, and if F is continuous, then F is not necessarily unimodal.

Example 3.7. Let $\alpha, \beta \geqq 0$, $\alpha+\beta=1$, and $\beta \leqq \alpha/2$. Define

$$F(x)=\begin{cases} \frac{1}{2}\alpha x^2 & \text{for } 0\leqq x\leqq 1,\\ -\frac{1}{2}\alpha x^2+2\alpha x-\alpha & \text{for } 1\leqq x\leqq 2,\\ \alpha & \text{for } 2\leqq x\leqq 3,\\ \alpha+\frac{1}{2}\beta(x-3)^2 & \text{for } 3\leqq x\leqq 4,\\ 1-\frac{1}{2}\beta(5-x)^2 & \text{for } 4\leqq x\leqq 5. \end{cases}$$

We compute

$$A(l)=\begin{cases} 1-l/2 & \text{for } 0\leqq l\leqq 2,\\ [2-l,0] & \text{for } 2\leqq l\leqq 3,\\ (l-3)\,\beta/(\alpha-\beta), & \text{for } 3\leqq l\leqq(4\alpha-\beta)/\alpha,\\ (5-l)\,\beta & \text{for } (4\alpha-\beta)/\alpha\leqq l\leqq 5,\\ [5-l,0] & \text{for } l\geqq 5, \end{cases}$$

$$Q(l)=\begin{cases} \alpha l(1-l/4) & \text{for } 0\leqq l\leqq 2,\\ \alpha & \text{for } 2\leqq l\leqq 3,\\ (\alpha+\frac{1}{2}\alpha\beta(l-3)^2)/(\alpha-\beta) & \text{for } 3\leqq l\leqq(4\alpha-\beta)/\alpha,\\ 1-\frac{1}{2}\alpha\beta(5-l)^2 & \text{for } (4\alpha-\beta)/\alpha\leqq l\leqq 5,\\ 1 & \text{for } l\geqq 5, \end{cases}$$

$a_-=(4\alpha-\beta)/\alpha$, $a_+=0$.

Example 3.8. For $s=1$, 2 we define

$$F_s(x)=\begin{cases} 0 & \text{for } x<0,\\ x/4 & \text{for } 0\leqq x\leqq 1,\\ 1/4 & \text{for } 1\leqq x<2,\\ (x+2)/8 & \text{for } 4\leqq x<6, \end{cases}$$

$$F_1(x)=\tfrac{1}{4}(4x-7)^{\frac{1}{2}} \quad \text{for } 2\leqq x<4,$$

$$F_2(x)=\begin{cases} (x-2)/8+\frac{1}{4} & \text{for } 2\leqq x<3,\\ (x-2)/8+\frac{1}{2} & \text{for } 3\leqq x<4. \end{cases}$$

We find $A_s(4)=\{0,2\}$,

$$Q_s(l)=\begin{cases} 0 & \text{for } l<0,\\ (l+2)/8 & \text{for } 2\leqq l<6,\\ 1 & \text{for } l\geqq 6, \end{cases}$$

$Q_1(l)=\frac{1}{4}[(4l+1)^{\frac{1}{2}}-1]$, and $Q_2(l)=(l+2)/8$ for $0\leqq l<2$.

4. Larboard and Starboard

Let $F\in\mathscr{U}$. A more detailed description of the behaviour of the pointer of F can be obtained by introducing the *larboard function* b and the *starboard function* t of F, defined by $b(l)=\inf A(l)$, $t(l)=\sup A(l)$ for $l\in(0,\infty]$, $b(0)=b_+(0)$, $t(0)=t_+(0)$. Obviously $A(l)=[b(l), t(l)]$ for $l>0$.

Lemma 4.1. (1) *If* $l\in[0,\infty]$, *then* $b(l)\in[a_- -l, a_-]$ *and* $t(l)\in[a_+ -l, a_+]$. (2) $a_- =b(0)$, $a_+ =t(0)$, $b(\infty)=-\infty$, $t(\infty)=x'$.

Proof. (1) Apply Corollary 3.3 and Remark 2.5, (2). (2) Apply (1).

Theorem 4.2. (1) *b and t are nonincreasing on* $[0,\infty]$ *and absolutely continuous on* $[0,\infty)$. (2) $b+I$ *and* $t+I$ *are nondecreasing on* $[0,\infty]$ *and absolutely continuous on* $[0,\infty)$.

Proof. In virtue of Theorem 3.1, the monotonicity of these functions is an immediate consequence of Lemma 2.7. Now observe that $l_1\geqq l_2$ implies $l_1-l_2\geqq b(l_2)-b(l_1)\geqq 0$. Hence b is absolutely continuous. Analogously for t, $b+I$, and $t+I$.

Note that from Theorem 4.2 and Corollary 3.6 it follows that $L\in D(A)$, whenever $L\in(0,\infty)$ and F is strictly unimodal.

The following result is complementary to Theorem 4.2:

Lemma 4.3. (1) *b is continuous at $l=\infty$ if and only if $x'=-\infty$ or $x''<\infty$.* (2) *t is continuous at $l=\infty$.* (3) *$b+I$ is continuous at $l=\infty$.* (4) *$t+I$ is continuous at $l=\infty$ if and only if $x''=\infty$ or $x'>-\infty$.*

Proof. (1) Let $\lim_{l\to\infty} b(l)=-\infty$ and let $F(x)=0$. Choose $l>0$ such that $b(l)<x$. We have $Q(l)=F(b(l)+l)-F(b(l))\leqq F(x+l)-F_-(x)\leqq Q(l)$ and $F(b(l)+l)=F(x+l)$. This shows that F is constant on $(x+l,\infty)$. Conversely, put $\hat{x}=\lim_{l\to\infty} b(l)$. Since $1\geqq \lim_{l\to\infty}(Q(l)+F_-(b(l))\geqq 1+F_-(\hat{x})$, we have $\hat{x}=-\infty$ whenever $x'=-\infty$. In the case $x', x''\in\mathbb{R}$ we have $\hat{x}\leqq\lim_{l\to\infty}(x''-l)=-\infty$. (2) Since A is cocontinuous at $l=\infty$, t is u.s.c. and hence left continuous at $l=\infty$. (3, 4) Similar proofs.

5. The Subdifferential of a Unimodal Distribution Function

Let $F\in\mathscr{U}$. In the proof of Corollary 3.3 it was pointed out that $A(l)$ is the set of all maximum points of the concave function $F(\cdot+l)-F_-$ on $[a_- -l, a_+]$. By Remark 2.5, (6), the convex function $F_- -F(\cdot+l)$ has a minimum at an interior point x of this interval if and only if $0\in\partial F_-(x)+\partial(-F(x+l))$. Hence:

Lemma 5.1. $A(l)\cap(a_- -l, a_+)=\{x\in(a_- -l, a_+)\colon \partial F(x+l)\cap\partial F(x)\neq\emptyset\}$ *for* $l\in(0,\infty)$.

Lemma 5.2. *Let* $l\in(0,\infty)$ *and* $x\in A(l)$. (1) *If* $x\in[a_- -l, a_+)$, *then* $\partial Q(l)\subset\partial F(x)$. (2) *If* $x\in(a_- -l, a_+]$, *then* $\partial Q(l)\subset\partial F(x+l)$.

Proof. The first statement follows from the inequalities

$$\alpha(l-k)\leqq Q(l)-Q(k)\leqq F(x+l)-F(x)-F(x+l)+F(x+l-k)=F(x+l-k)-F(x),$$

valid for any $k>\min(0, x+l-a_+)$ and any $\alpha\in\partial Q(l)$. The second statement is proven similarly.

Proposition 5.3. *Let* $l\in(0,\infty)$. (1) $A(l)\cap(a_- - l, a_+)=\{x\in(a_- - l, a_+): \partial Q(l) =\partial F(x)\cap\partial F(x+l)\}$. (2) *If* $A(l)\cap(a_- - l, a_+)=\emptyset$, *then* $a_-=a_+$ *and either* (α) $A(k) =\{a_- - k\}$ *and* $\partial Q(k)=\partial F(b(k))$ *for any* $k\in(0,l]$, *or* (β) $A(k)=\{a_+\}$ *and* $\partial Q(k) =\partial F(b(k)+k)$ *for any* $k\in(0,l]$.

Proof. (1). In virtue of the preceding two lemmas, it suffices to prove that $\alpha\in\partial Q(l)$ whenever $\alpha\in\partial F(x)\cap\partial F(x+l)$ and $x\in A(l)\cap(a_- - l, a_+)$. This follows from the inequalities

$$\begin{aligned}Q(l+k)-Q(l)&=F(b(l+k)+l+k)-F(b(l+k))-F(x+l)+F(x)\\&\leqq\alpha(b(l+k)+l+k-(x+l)-\alpha(b(l+k)-x)=\alpha l,\end{aligned}$$

for $k>-l$. (2α) By Lemma 4.1, (1) and by Theorem 4.2, (2) we have $a_-=a_+$ and $A(k)=\{a_- - k\}$ for $0<k\leqq l$. Therefore, $Q(k)=F(a_-)-F(a_- - k)$ and $\partial Q(k) =\partial F(b(k))$. (2$\beta$) Similar proof.

We are now in a position to formulate a criterion for polymodal distributions:

Theorem 5.4. *Let* $F\in\mathscr{U}$. *The following relations are equivalent:* (1) F *is polymodal.* (2) $a_-<a_+$ *and* Q *is affine on* $[0, a_+ - a_-]$, *with the same slope as* F *on* $[a_-, a_+]$. (3) Q *is affine on some segment* $[0, l_0]$, $l_0>0$, *and* F *is continuous.* (4) Q *is polymodal.*

Proof. Only the proof of (3)⇒(1) needs attention. Let $\alpha\in\mathbb{R}$ and $\partial Q(l)=\{\alpha\}$ for $l\in(0, l_0)$, $l_0>0$. According to Proposition 5.3 we have the following three possibilities. (i) $A(l)=\{a_- - l\}$ for some $l\in(0, l_0)$. In this case, $a_+=a_-=a$ and F is affine with slope α on $[a-l, a]$. For $x\in(a-l, a)$ and $\beta\in\partial F(x+l)$, we have

$$Q(l)\geqq F(x+l)-F(x)\geqq\beta(x+l-a)+\alpha(a-x)\geqq Q(l)+(\beta-\alpha)(x+l-a),$$

and hence $\beta\leqq\alpha$. From this it follows that $F(y)\leqq F(a)+\alpha(y-a)$ for $y\geqq a$, and hence that F is concave on $(a-l,\infty)$ in contradiction to the statement $a_-=a_+$. (ii) The case $A(l)=\{a_+\}$ for some $l\in(0, l_0)$ is treated similarly. (iii) $A(l)\cap(a_- - l, a_+)\neq\emptyset$ for each $l\in(0, l_0)$. Since $A\langle(0, l_0)\rangle\supset(b(l_0), a_+)$, $\{\alpha\}=\partial F(x)\cap\partial F(x+l)$ and F is affine with slope α on $[b(l_0), t(l_0)+l_0]$.

It follows that Q and F have the same *modal length* $a_+ - a_-$, whenever $F\in\mathscr{U}$.

6. A Representation Theorem

Let f: $[0,\infty)\to\mathbb{R}$ be a continuous nonincreasing function and let $\alpha=f(0)$ and $\beta =f_-(\infty)$. For each $x\in(\beta,\alpha)$, $f^{-1}(x)$ is a nonempty closed segment of $\mathbb{R}$. With the conventions $f_-(0)=\alpha$, $f_+(\infty)=\beta$, two generalized right inverses of f, f^s: $(-\infty,\alpha]\to[0,\infty]$, and f^i: $(-\infty,\alpha]\to[0,\infty]$, are defined by $f^i(x)=\inf f^{-1}(x)$, $f^s(x)=\sup(f_-)^{-1}(x)$ for $x\geqq\beta$, $f^i=f^s=\infty$ on $(-\infty,\beta)$, the sup and inf being taken in $[0,\infty]$.

The following statements are readily verified:

Remark 6.1 (1) If $\beta\leqq x<y\leqq\alpha$, then $f^s(y)<f^i(x)\leqq f^s(x)$, $f^i(\alpha)=0$, and $f^s(\beta)=\infty$. (2) If $\beta\leqq x\leqq\alpha$, then $f_+(f^i(x))=f_+(f^s(x))=x$. (3) If $f^i(x)<f^s(x)$, then f is constant on $(f^i(x), f^s(x))$. (4) $f^i=(f^i)_+=(f^s)_+$ on $(-\infty,\alpha)$ and $f^s=(f^s)_-$ on $(-\infty,\alpha]$. (5) For a continuous nondecreasing function g: $(0,\infty)\to\mathbb{R}$, the generalized inverse functions, defined by $g^i(x)=(-g)^i(-x)$, $g^s(x)=(-g)^s(-x)$, have similar properties.

Our second main result is a representation theorem for unimodal distribution functions that reflects the following formal computation. Let $F\in\mathscr{U}_a$ and let f be a selection of the pointer of F. We have

$$Q(l)=F(f(l)+l)-F(f(l))=\max\{F(x+l)-F_-(x): x\in\mathbb{R}\},$$
$$F'(f(l)+l)=F'(f(l)), Q'(l)=F'(f(l))$$

and

$$F(x)=\int_{-\infty}^{x} F'(z)\,dz=\int_{\infty}^{f^s(x)} F'(f(l))\,df(l)$$
$$=-\int_{f^s(x)}^{\infty} Q'(l)\,df(l) \quad \text{for} \quad x<a.$$

Using notations like $fg(x)$ in stead of $(f\circ g)(x)$ or $f(g(x))$, we have:

Theorem 6.2. (1) *Let F be a numerical function and $a\in\mathbb{R}$. Then $F\in\mathscr{U}_a$ if and only if F is of the form*

$$F(x)=\begin{cases}-\int_{f^s(x)}^{\infty}\eta(l)\,df(l) & \text{for } x<a,\\ Qg^i(x)+F_-fg^i(x) & \text{for } x\geqq a,\end{cases} \tag{6.1}$$

where $Q\in\tilde{\mathscr{F}}_+\cap\mathscr{U}$, η is the left derivative of Q and f: $[0,\infty)\to\mathbb{R}$ is a function such that (i) *f is continuous, nonincreasing, and $f(0)=a$;* (ii) *$g=f+I$ is nondecreasing;* (iii) *η is df-integrable.* (2) *In this case, f is necessarily a selection of A_F, $Q=Q_F$, and*

$$F(x)=\begin{cases}\int_{-\infty}^{x}\eta f^i(z)\,dz & \text{for } x<a,\\ 1-\int_{x}^{\infty}\eta g^i(z)\,dz & \text{for } x\geqq a.\end{cases} \tag{6.2}$$

Proof. The proof is given in a number of steps.

(α) Let f satisfy the conditions (i)–(iii) and let F be defined by (6.1). Notice that the bound $f^s(x)$ in (6.1) may be replaced by any $\lambda\in[f^i(x), f^s(x)]$. Since the measure dz on $(f_-(\infty), a)$ is the image of the measure $-df$ on $(0, \infty)$ under the map f, we have

$$F(x) = -\int_{f^s(x)}^{\infty} \eta(l)\,df(l) = \int_{f^i f_-(\infty)}^{f^i(x)} \eta(l)\,df(l)$$
$$= \int_{f^i f_-(\infty)}^{f^i(x)} \eta f^i f(l)\,df(l) = \int_{f_-(\infty)}^{x} \eta f^i(z)\,dz$$
$$= \int_{-\infty}^{x} \eta f^i(z)\,dz \qquad \text{for } x < a.$$

Similarly,

$$F(x) = Q g^i(x) - \lim_{y \uparrow f g^i(x)} \int_{f^s(y)}^{\infty} \eta(l)\,df(l)$$
$$= Q g^i(x) - \int_{f^s f g^i(x)}^{\infty} \eta(l)\,df(l) = 1 - \int_{g^i(x)}^{\infty} \eta(l)\,dg(l)$$
$$= 1 - \int_{x}^{\infty} \eta g^i(z)\,dz \qquad \text{for } x \geqq a.$$

(β) From the representation (6.2) and Remark 2.5, (7), we conclude that $F \in \mathscr{U}_a$. Moreover

$$F(a) = 1 - \int_a^{\infty} \eta g^i(z)\,dz = 1 - \int_0^{\infty} \eta(l)\,dl - \int_0^{\infty} \eta(l)\,df(l) = Q(0) + F_-(a)$$

implies $Q(0) = Q_F(0)$.

(γ) $Q = Q_F$ and $f \subset A$. Indeed, if $l > 0$ and $x = f(l)$, then

$$F(x+l) - F_-(x) = Q g^i(x+l) - \int_{g^i(x+l)}^{\infty} \eta(s)\,df(s) + \int_{f^s(x)}^{\infty} \eta(s)\,df(s)$$
$$= Q(l) + \int_{l}^{g^i(x+l)} \eta(s)\,dg(s) = Q(l).$$

Since $f^s = f^i$ df - a.e., we get $[\eta f^s(x), \eta f^i(x)] \subset \partial F(x)$ for $x < a$ and, similarly, $[\eta g^s(x), \eta g^i(x)] \subset \partial F(x)$ for $x > a$. Assume first that a $-l < x < a$. We have $\eta(l) \in \partial F(x) \cap \partial F(x+l)$ and hence, by Lemma 5.1, $x \in A(l)$ and $Q(l) = Q_F(l)$. Now assume that $x = a$ and hence $f(s) = a$ for $0 < s < l$. We have $F(y+l) - F_-(y) \leqq Q(l)$ for $x \leqq y$ and

$$F(y+l) - F(y) = Q g^i(y+l) - \int_{g^i(y+l)}^{f^s(y)} \eta(l)\,df(l) \leqq Q(l) \qquad \text{for } y < x,$$

since $g^i(y+l) \leqq g^i(x+l) \leqq l \leqq f^s(x) \leqq f^s(y)$.

The case $x + l = a$ is treated similarly.

(δ) Let now $F\in\mathscr{U}_a$. Choose $Q=Q_F$ and $f=\lambda b+(1-\lambda)t$, where $a=\lambda a_- +(1-\lambda)a_+$ (Lemma 4.1). Since dg is a positive measure, we have $-\int_{f^s(x)}^{\infty}\eta(l)\,df(l)\leqq\int_{f^s(x)}^{\infty}\eta(l)\,dl=1-Q(f^s(x))$, and hence f satisfies the conditions (i)-(iii). By the preceding part of the proof, the formulae (6.1) or (6.2) define a function $G\in\mathscr{U}_a$, with $Q_G=Q_F$ and $\eta f^i(x)\in\partial G(x)$. By Lemma 5.2, (1), we have $\partial F(x)\supset\partial Q(f^i(x))$ for each $x\in(-\infty,a)$. From $F(-\infty)=G(-\infty)$ and $\partial F(x)\cap\partial G(x)\neq\emptyset$ for each $x\in(-\infty,a)$, it follows that $F=G$ on $(-\infty,a)$. Similarly, $F=G$ on (a,∞).

As a direct consequence of this result we have the following.

Corrolary 6.3. *$F\in\mathscr{U}_0$ is symmetric if and only if $-l/2\in A_F(l)$ for each $l>0$. Moreover, $F(x)=\frac{1}{2}[1-Q_F(-2x)]$ for $x<0$ and $F(x)=\frac{1}{2}[1+Q_F(2x)]$ for $x\geqq 0$.*

As another application of Theorem 6.2, let $F\in\mathscr{U}_0$ and define

$$G(x)=\begin{cases}-\int\limits_{-\infty}^{x} z\,d\eta f^i(z) & \text{for } x<0,\\ 1+\int\limits_{x}^{\infty} z\,d\eta g^i(z) & \text{for } x>0.\end{cases}$$

It is easily shown that $G(x)-F(x)=-x\eta(f^i(x))$ for $x<0$, $G(x)-F(x)=-x\eta(g^i(x))$ for $x\geqq 0$, and that G is a distribution function such that $G(0)=F(0)$ and $G_-(0)=F_-(0)$. If x is the characteristic function of F and if ψ is the characteristic function of G, then, for each $t\in\mathbb{R}$,

$$\begin{aligned}\chi(t)&=\int\limits_{-\infty}^{0} e^{itz}\eta f^i(z)\,dz+Q(0)+\int\limits_{0}^{\infty} e^{itz}\eta g^i(z)\,dz\\ &=-\int\limits_{-\infty}^{-0}[(e^{itz}-1)/itz]\,z\,d\eta f^i(z)+G(0)-G_-(0)-\int\limits_{0}^{\infty}[(e^{itz}-1)/itz]\,z\,d\eta g^i(z)\\ &=\int\limits_{-\infty}^{\infty}[(e^{itz}-1)/itz]\,dG(z)=t^{-1}\int\limits_{0}^{t}\psi(s)\,ds.\end{aligned}$$

This is the "only if" part of the well known characterization of *A.* Ja. Hinčin (see, e.g., Lukács [5], p. 92): *Let χ be the characteristic function of $F\in\mathscr{F}$. Then $F\in\mathscr{U}_0$ if and only if $\chi(t)=t^{-1}\int_0^t\psi(s)\,ds$, $t\in\mathbb{R}$, where ψ is a characteristic function.*

In its direct form, the theorem of Hinčin roughly amounts to the statement: Let $F\in\mathscr{F}$ be absolutely continuous, except possibly at 0. Then $F\in\mathscr{U}_0$ if and only if F satisfies the differential equation $G(x)=-xF'(x)+F(x)$ a.e., where G is some arbitrary distribution function (for details, see Gnedenko and Kolmogorov [4]). Theorem 6.2 has the advantage of expressing $F\in\mathscr{U}$ strictly in terms of its concentration function and its pointer, i.e., in quantities that may reflect some known or desired properties of F. Moreover, in the "if" part of the theorem, no initial conditions whatever are imposed on F.

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