Z. Wahrscheinlichkeitstheorie verw. Gebiete
33, 269–272 (1976)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Another Quasi-Poisson Plane Point Process

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Rényi (1967) proved that if a point process on a line is such that the number of points occurring in every set consisting of a finite number of half-open intervals has a Poisson distribution with mean equal to the measure of the set, then the process is a Poisson process (and he extended this result to R^n). He asked if the same result is true if the same condition is only known to hold for all sets consisting of a single interval. This was shown to be false by counter examples constructed by Shepp (1967), Moran (1967), and Lee (1968). Later Szász (1970) showed that for every integer $k \geqq 1$, there exists a non-Poisson point process on the line such that for every set of k disjoint half-open intervals $I_1, \ldots, I_k$, the numbers of points falling in the I_i are such independent Poisson variates, and he also extended this result to R^n. In a previous note (Moran (1975)) I have constructed a plane point process which is not a Poisson process, but which is such that the number of points falling in any convex Borel set has a Poisson distribution with mean equal to its area. In the present note I show that for any integer $k \geqq 1$, there exists a non-Poisson point process in the plane which is such that the numbers of points falling inside any k disjoint convex Borel sets are independent Poisson variates with means equal to their areas. This extends the R^2 result of Szász to all convex Borel sets. To do this we first prove a lemma on such convex Borel sets.

Lemma. *Consider a circle C in the plane. Let $C_1, \ldots, C_n$ be n disjoint circles whose centres lie on the circumference of C, and which all have the radius S. Let S be so small, and the centres of the C_i so far apart, that no straight line meets three or more of the C_i. If we now have k disjoint plane convex Borel sets $K_1, \ldots, K_k$, with the property that every C_i is met by at least two of the K_j, then*

$$k \geqq [\tfrac{1}{2}(n+1)]+1, \tag{1}$$

where $[x]$ is the integral part of x.

Proof. Let N be the total number of times a K_j meets a C_i. Then $N \geqq 2n$. We prove the lemma by induction, it being clearly true for $n=2$.

Let the region K_j meet m_j of the circles. Then $\sum_j m_j = N \geqq 2n$. Then either $k \geqq 2n$ in which case (1) is satisfied, or there is a K, K_1 say, for which $m_1 \geqq 2$. Suppose the circles C_i are numbered in cyclic order $C_1, \ldots, C_n$ around the circumference of C, and that K_1 meets the circles with suffices $i_1, \ldots, i_{m_1}$. For simplicity of statement we represent this by a cycle of symbols A, B where we write A in the i-th place if C_i is met by K_1, and B if it is not. Thus the A's and B's form a cycle with alternate runs of A's and B's, and we are concerned with "runs in a ring".

Consider a run of B's of length b so that, for example (identifying C_{i+n} with C_i), C_{i-1} and C_{i+b} are met by K_1, but not $C_i, C_{i+1}, \ldots, C_{i+b-1}$. Then by convexity no circle in the sequence $C_i, C_{i+1}, \ldots, C_{i+b-1}$ can be met by any K which meets a circle in the sequence $C_{i+b+1}, C_{i+b+2}, \ldots, C_{i-2}$. By induction there must be at least $[\frac{1}{2}(b+1)]+1$ K's which meet $C_i, \ldots, C_{i+b-1}$ but none of the $C_{i+b+1}, \ldots, C_{i-2}$. Note that some of these K's may meet C_{i-1} or C_{i+b}.

Now consider a run of a A's, for example $C_i, C_{i+1}, \ldots, C_{i+a-1}$, all of which are met by K_1. C_i and C_{i+a-1} may also be met by K's which meet C_{i-1}, C_{i+a} respectively. Consider the K's which meet $C_{i+1}, \ldots, C_{i+a-2}$ (assuming $a \geqq 3$). Then by convexity and the straight line condition any K which meets one of these cannot meet any of $C_{i+a}, C_{i+a+1}, \ldots, C_{i-1}$. Moreover by convexity, none of these K's can meet more than two of $C_{i+1}, \ldots, C_{i+a-2}$. Thus, in addition to K_1 there are at least $[\frac{1}{2}(a-1)]$ K's which meet these and do not meet $C_{i+a}, \ldots, C_{i-1}$.

If there are r runs of A's of lengths $a_1, \ldots, a_r$, and r runs of B's of length $b_1, \ldots, b_r$, we must have

$$\sum a_i + \sum b_i = n,$$

and the total number of K's must be at least

$$1+\sum_i \{1+[\tfrac{1}{2}(b_i+1)]\} + \sum_i [\tfrac{1}{2}(a_i-1)] \geqq 1+r+\tfrac{1}{2}\sum_i b_i + \sum_i (\tfrac{1}{2}a_i - 1)$$
$$\geqq 1+\tfrac{1}{2}n. \tag{2}$$

Since k is an integer, (1) is satisfied and the lemma proved.

We can now construct a non-Poisson process satisfying the required conditions for any *given* integer k. We do this by an elaboration of the method used in Moran (1975). Divide the whole plane into unit squares and suppose that there is a Poisson point process on the plane with the expectation of the number of points in any unit square equal to unity. We now modify the distribution inside each one of the squares in a manner independent of what happens in the others. In each square take a circle with radius $\frac{1}{4}$, say, with its centre at the centre of the square. On the circumference choose $n=2k+1$ points equally spaced apart and with centre at each of these points construct a smaller circle of radius δ. Choose δ small enough so that these circles lie inside the square, are disjoint, and are such that no straight line meets more than two of them.

Within each of the small circles construct a rectangle of sides of length $\delta, \frac{3}{4}\delta$, centered on the centre of the circle. Divide this rectangle into twelve equal squares which are numbered in a natural coordinatewise way as (11), (12), ... (14), ... (34) so that (11), (14), (31), and (34) are the corner squares. We now carry out a procedure similar to that used in the previous paper.

We now modify the distribution of the Poisson process defined on the whole plane and we do this independently for each unit square. We leave unaltered the distributions inside the squares numbered (12), (13), (21), (24), (32), and (33) inside each circle. Consider the joint distribution of the points inside the squares (11), (14), (22), (23), (31), (34). We shall only make modifications to the 64 cases where the numbers of points in these squares take the value 0 and 1. The probabilities of each of these events are all greater than some positive constant $\varepsilon > 0$. We leave unaltered the joint distribution of the numbers of points in the squares

(11), (14), (31), (34), but we make the distribution if the numbers of points in (22), (23) dependent on them.

Let n_{ij} be the number of points in the square (ij). We leave the probabilities

$$p(n_{22}=1, n_{23}=1 \mid n_{11}, n_{14}, n_{31}, n_{34}),$$
$$p(n_{22}=0, n_{23}=0 \mid n_{11}, n_{14}, n_{31}, n_{34})$$

unaltered. We add to

$$p(n_{22}=1, n_{23}=0 \mid n_{11}, n_{14}, n_{31}, n_{34})$$

the quantity

$$\varepsilon Z(2n_{11}-1)(2n_{24}-1)(2n_{31}-1)(2n_{34}-1) \tag{3}$$

where the n_{ij} in these expressions are all 0 or 1. Here Z is a mixing random variable taking the values ± 1 with probabilities $\frac{1}{2}$. We also subtract the expression in (3) from

$$p(n_{22}=0, n_{23}=1 \mid n_{11}, n_{14}, n_{31}, n_{34}).$$

Considered by itself for a particular *rectangle* this procedure does not alter the joint distribution of the numbers of points in the squares. However for each rectangle inside a circle C_i we use a different Z which we denote as Z_i. We take $Z_1, \ldots, Z_{n-1}$ as independent random variables and put $Z_n = Z_1 \ldots Z_{n-1}$. Then any set of $n-1$ Z's are a set of independent random variables, but the whole set $(Z_1, \ldots, Z_n)$ is not independent. Having determined the numbers of points inside each square we suppose that each of them is independently distributed uniformly over the square which contains it.

Consider a single rectangle and suppose the corresponding Z is fixed. From the above construction we see that the n_{ij} in any eleven of the twelve squares are independently distributed if the remaining square is a corner square, n_{11}, n_{14}, n_{31}, or n_{34}. Moreover the variables $n_{11}, n_{14}, n_{22}+n_{23}, n_{31}, n_{34}$ are jointly independent, and jointly independent of the remaining n_{ij}. Suppose a convex Borel set, K, overlaps the rectangle. If there is at least one corner square which it does not meet, the contributions to the number of points inside K from the other squares are independent. If, however, it meets all four corner squares it must, by convexity, contain in its interior the whole of the squares (22) and (23). Thus the total number of points inside K must be a Poisson variate with the correct expectation.

We now carry out the same construction in each unit square in the plane, the sets of n Z's in each unit square being completely independent of each other. Finally the whole coordinate system is given a uniform translation (X, Y) in the plane, where X and Y are independent random variables uniformly distributed on the interval (0, 1). The resulting point process is clearly not a Poisson point process and it is possible in principle to set up a sampling procedure which would confirm this from an infinite empirical realisation.

Now consider any k disjoint convex Borel sets $K_1, \ldots, K_k$. The contributions to the numbers of points in the K_i from different unit squares are independent, so we need only consider the contributions from one of the latter. From the lemma and the fact that $n=2k+1$, there is at least one of the C_i, C_1 say, which is met by at most one of the K_j. If C_1 is not met by any of the K_j, the $n-1$ Z's corresponding to the other circles are independently distributed, and the numbers of points in the K's contributed from the other C_s $(s \neq 1)$ are independently distributed in Poisson distributions with the correct expectations.

Now suppose C_1 is met by one only of the K's, K_1 say. Z_1 is dependent on the other Z's so that the distribution of the numbers of points in the squares (11), (14), (22), (23), (31), (34) in C_1 are not independent of what happens in the other circles. However if K_1 does not meet all of the four corner squares (11), (14), (31), (34), the numbers of points in the ones it does meet are distributed independently. On the other hand if it does meet all the four corner squares then by convexity it must cover completely the squares (22) and (23), so that only the sum of the numbers of points in these squares matters. But we have already seen that, given Z, n_{11}, n_{14}, $n_{22}+n_{23}$, n_{31}, n_{34} are jointly independent and independent of Z. Thus the numbers of points in $K_1, \ldots, K_k$ contributed by their intersections with the unit square under consideration are independent Poisson variates with the correct expectations. Thus the result is proved.

This construction is essentially dependent on the fact that k, although arbitrary, is fixed so that the counterexample constructed depends on the value of k. If a point process is such that for *any* finite set of disjoint convex Borel sets, the numbers of points they contain have independent Poisson distributions with means equal to the areas of the sets, then the process is certainly a Poisson process. This follows from the definition of a Poisson process as a point process for which the numbers of points falling in any set of disjoint Borel sets are independent Poisson variates with appropriate mean values, combined with the fact that any measurable set can be approximated arbitrarily closely in Lebesgue measure by a sum of a finite number of convex Borel sets.

Finally it is possible to add one natural condition which ensures that pathological examples of the type described above can not occur. Lee (1968) has shown that if a point process on a line is infinitely divisible, i.e. for all N is representable as the superposition of N independent point processes of identical structure, and if the number of points occurring in any interval has a Poisson distribution, then the process is Poisson. His proof applies equally well in any Euclidean space and thus the process constructed above is not infinitely divisible.

References

1. Lee, P. M.: Some examples of infinitely divisible point processes. Studia Sci. Math. Hungar **3**, 219–224 (1968)
2. Moran, P. A. P.: A non-Markovian quasi-Poisson process. Studia Sci. Math. Hungar **2**, 425–429 (1967)
3. Moran, P. A. P.: A quasi-Poisson point process in the plane. Bull. London Math. Soc. **7** (to appear 1975)
4. Rényi, A.: Remarks on the Poisson process. Studia Sci. Math. Hungar **2**, 119–123 (1967)
5. Shepp, L.: Appendix to J. R. Goldman: Stochastic point processes: limit theorems. Ann. Math. Statist. **38**, 771–779 (1967)
6. Szász, D. O. H.: Once more on the Poisson process. Studia Sci. Math. Hungar **5**, 441–444 (1970)

Received June 11, 1975