Z. Wahrscheinlichkeitstheorie verw. Gebiete
43, 319–329 (1978)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Probability Inequalities for Sums of Absolutely Regular Processes and Their Applications

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1. Introduction

Let $\{\xi_j, -\infty<j<\infty\}$ be a (not necessarily strictly stationary) sequence of random variables which are defined on a probability space $(\Omega, \mathscr{A}, P)$. For $a \leqq b$, let $\mathscr{M}_a^b$ denote the σ-algebra of events generated by $\xi_a, \ldots, \xi_b$. As in [8, 14] and [15], we shall say that the sequence is absolutely regular if

$$\beta(n)=\sup_k E\{\sup_{A\in\mathscr{M}_{n+k}^{\infty}} |P(A\mid\mathscr{M}_{-\infty}^k)-P(A)|\}\downarrow 0 \tag{1.1}$$

as $n\to\infty$. Further, we shall say that $\{\xi_i\}$ satisfies the ϕ-mixing condition if

$$\phi(n)=\sup_k \sup_{B\in\mathscr{M}_{-\infty}^k, A\in\mathscr{M}_{n+k}^{\infty}} |P(A\cap B)-P(A)\,P(B)|/P(B)\downarrow 0 \tag{1.2}$$

as $n\to\infty$. Since $\beta(n)\leqq\phi(n)$, so if $\{\xi_i\}$ is ϕ-mixing, then it is absolutely regular (cf. [8]).

Recently, many authors studied limiting behavior of some function of sums of ϕ-mixing sequence of random variables and obtained many fruitful results. But, so far, for ϕ-mixing sequence, the general probability inequalities concerning the probability such as $P(S_n>z)$ and $P(\max_{1\leqq j\leqq n}|S_j|>z)$ are few known.

In this paper, we shall prove some probability inequalities for sums of absolutely regular processes which are powerful to extend a broad class of probability inequalities for sums of independent random variables such as probability inequalities of Nagaev-Fuk type [5]. We shall prove some fundamental theorems in Section 2, and show some applications of them in Sections 3–5.

2. Fundamental Inequalities

In this and following sections, we always assume that $\{\xi_i\}$ is a (not necessarily strictly stationary) absolutely regular sequence of d-dimensional random vectors with $\beta(n)$.

0044-3719/78/0043/0319/$02.20

The following lemma is proved by the method used in the proof of Lemma 1 in [14].

Lemma. *Let δ be some positive number. Let $g(x_1, x_2, \ldots, x_k)$ be a Borel function such that*

$$\int \cdots \int_{R^{dk}} |g(x_1, x_2, \ldots, x_k)|^{1+\delta} dF^{(1)}(x_1, \ldots, x_j) dF^{(2)}(x_{j+1}, \ldots, x_k) \leqq M \tag{2.1}$$

where $x_1, \ldots, x_k$ are d-dimensional vectors and $F^{(1)}$ and $F^{(2)}$ are distribution functions of random vectors $(\xi_{i_1}, \ldots, \xi_{i_j})$ and $(\xi_{i_{j+1}}, \ldots, \xi_{i_k})$, respectively, and $i_1 < i_2 < \cdots < i_k$. If

$$E|g(\xi_{i_1}, \xi_{i_2}, \ldots, \xi_{i_k})|^{1+\delta} \leqq M_1$$

then

$$\begin{aligned} &|Eg(\xi_{i_1}, \xi_{i_2}, \ldots, \xi_{i_k}) \\ &\quad - \int \cdots \int_{R^{dk}} g(x_1, \ldots, x_j, x_{j+1}, \ldots, x_k) dF^{(1)}(x_1, \ldots, x_j) \cdot dF^{(2)}(x_{j+1}, \ldots, x_k)| \\ &\leqq 4 M_1^{1/1+\delta} \{\beta(i_{j+1} - i_j)\}^{\delta/1+\delta}. \end{aligned} \tag{2.2}$$

As a special case, if $g(x_1, x_2, \ldots, x_k)$ is bounded, say, $|g(x_1, x_2, \ldots, x_k)| \leqq M_2$, then we can replace the right-hand side of (2.2) *by* $2M_2 \beta(i_{j+1} - i_j)$.

Using Lemma, we shall prove some theorems which play fundamental roles to obtain probability inequalities for sums of absolutely regular sequence of random vectors. Put

$$S_n = \sum_{j=1}^{n} \xi_j, \qquad S_0 = 0 \tag{2.3}$$

and denote the length of a vector $\mathbf{x}$ by $\|x\|$.

Theorem 1. *The following inequalities hold for any positive number z and any positive integer $m (\leqq n)$;*

(i) *for any integer $d (\geqq 1)$*

$$P(\|S_n\| \geqq z) \leqq \sum_{j=1}^{m} P(\|Y_j + Y_{j+m} + \cdots + Y_{j+k_j m}\| \geqq m^{-1} z) + 4n\beta(m), \tag{2.4}$$

(ii) *for $d = 1$*

$$P(S_n \geqq z) \leqq \sum_{j=1}^{m} P(Y_j + Y_{j+m} + \cdots + Y_{j+k_j m} \geqq m^{-1} z) + 4n\beta(m). \tag{2.5}$$

Here, for each $j (1 \leqq j \leqq m)$ $k_j = k_{n,j}$ is the largest integer for which $j + k_j m \leqq n$ and $\{Y_j\}$ are independent random vectors defined on the probability space $(\Omega, \mathscr{A}, P)$ such that each Y_i has the same df as that of ξ_i.

Proof. Let

$$S_n^{(j)} = \xi_j + \xi_{j+m} + \cdots + \xi_{j+k_j m} \qquad (j = 1, \ldots, m). \tag{2.6}$$

We note that

$$[n/m] \leqq k_j \leqq k_1 \qquad (j=1, \dots, m)$$

where $[s]$ denotes the largest integer m such that $m \leqq s$. Then

$$P(\|S_n\| \geqq z) \leqq P\left(\sum_{j=1}^{m} \|S_n^{(j)}\| \geqq z\right) \leqq \sum_{j=1}^{m} P(\|S_n^{(j)}\| \geqq m^{-1} z). \tag{2.7}$$

For each $j (1 \leqq j \leqq m-1)$, let A_j be the Borel subset of the $k_j d$-dimensional Euclidean space $R^{k_j d}$ defined by

$$A_j = \{(x_1, \dots, x_{k_j}) : \|x_1 + \dots + x_{k_j}\| \geqq m^{-1} z\}$$

where $x_i \in R^d (i=1, \dots, k_j)$ and put

$$g_j(x_1, \dots, x_{k_j}) = \begin{cases} 1 & \text{if } (x_1, \dots, x_{k_j}) \in A_j \\ 0 & \text{otherwise.} \end{cases}$$

Then, $|g_j(x_1, \dots, x_{k_j})| \leqq 1$ and so it follows from Lemma (with $M_2 = 1$) that

$$\begin{aligned} &P(\|\xi_j + \xi_{j+m} + \dots + \xi_{j+k_j m}\| \geqq m^{-1} z) \\ &\quad = E g_j(\xi_j, \xi_{j+m}, \dots, \xi_{j+k_j m}) \\ &\quad \leqq \int \cdots \int_{R^{k_j d}} g_j(x_1, \dots, x_{k_j}) \, dF_j(x_1) \dots dF_{j+k_j m}(x_{k_j}) + 2 k_j \beta(m) \\ &\quad = P(\|Y_j + \dots + Y_{j+k_j m}\| \geqq m^{-1} z) + 2 k_j \beta(m). \end{aligned} \tag{2.8}$$

Therefore, (2.4) follows from (2.7) and (2.8). The proof of (2.5) is similar and so is omitted. Thus, we have the theorem.

Theorem 2. *Let D be the subset of R^d defined by*

$$D = \{(x_1, \dots, x_d) : x_i \leqq c_i \ (i=1, \dots, d)\}.$$

Further, for any s, let

$$D_s = \{(x_1 + s, \dots, x_d + s) : (x_1, \dots, x_d) \in D\}.$$

If $E\|\xi_j\|^r \leqq M_0$ for some $r > 0$ and for all j, then

$$\begin{aligned} &P\left(\sum_{i=1}^{k} Z_{n,i} \in D_{-\varepsilon}\right) - P(n^{-1/r} \|S_n'\| \geqq \varepsilon) - 2k\beta(q) \leqq P(n^{-1/r} S_n \in D) \\ &\qquad \leqq P\left(\sum_{i=1}^{k} Z_{n,i} \in D_{\varepsilon}\right) + P(n^{-1/r} \|S_n'\| \geqq \varepsilon) + 2k\, \beta(q) \end{aligned} \tag{2.9}$$

for any $\varepsilon > 0$ and for all n sufficiently large. Here, $Z_{n.i}$ $(i=1, \dots, k)$ are independent random vectors such that for each i $(1 \leqq i \leqq k)$ $Z_{n,i}$ has the same df as that of η_i defined by

$$\eta_i = n^{-1/r} \sum_{j=1}^{p} \xi_{(i-1)(p+q)+j} \tag{2.10}$$

and

$$S'_n = \sum_{i=1}^{k} \sum_{j=1}^{q} \xi_{k(p+q)+j} \tag{2.11}$$

and p, q, k are integers such that $1 \leqq q < p < n/2$, $k = [n/(p+q)]$.

Proof. We note that for any $\varepsilon > 0$

$$P\left(\sum_{i=1}^{k} \eta_i \in D_{-\varepsilon}\right) - P(n^{-1/r}\|S'_n\| \geqq \varepsilon) \leqq P(n^{-1/r} S_n \in D)$$

$$\leqq P\left(\sum_{i=1}^{k} \eta_i \in D_{\varepsilon}\right) + P(n^{-1/r}\|S'_n\| \geqq \varepsilon). \tag{2.12}$$

Using the method in the proof of Theorem 1 we have

$$P\left(\sum_{i=1}^{k} \eta_i \in D_{-\varepsilon}\right) \geqq P\left(\sum_{i=1}^{k} Z_{n,i} \in D_{-\varepsilon}\right) - 2k\,\beta(q) \tag{2.13}$$

and

$$P\left(\sum_{i=1}^{k} \eta_i \in D_{\varepsilon}\right) \leqq P\left(\sum_{i=1}^{k} Z_{n,i} \in D_{\varepsilon}\right) + 2k\,\beta(q). \tag{2.14}$$

Thus, from (2.12)–(2.14), we have the theorem.

For the distribution of the maximum of sums, the following theorem holds.

Theorem 3. *Let z be any positive number. Then, for any positive integer* $m(\leqq n)$, *the following inequalities hold:*

(i) *for any integer* $d(\geqq 1)$

$$P(\max_{1 \leqq i \leqq n} \|S_i\| \geqq z) \leqq \sum_{j=1}^{m} P(\max_{1 \leqq i \leqq n} \|T_{n,i}^{(j)}\| \geqq m^{-1} z) + 4n\,\beta(m), \tag{2.15}$$

(ii) *for* $d = 1$

$$P(\max_{1 \leqq i \leqq n} S_n \geqq z) \leqq \sum_{j=1}^{m} P(\max_{1 \leqq i \leqq n} T_{n,i}^{(j)} \geqq m^{-1} z) + 4n\,\beta(m). \tag{2.16}$$

Here, for each $i(\geqq 1)$

$$T_{n,i}^{(j)} = Y_j + \cdots + Y_{j+k_{i,j}m} \qquad (j = 1, \dots, m) \tag{2.17}$$

and $k_{i,j}$ *and* $\{Y_j\}$ *are the ones defined in Theorem* 1.

Proof. We shall prove (2.15). Define $S_i^{(j)}$ by (2.6). Then

$$P(\max_{1 \leqq i \leqq n} \|S_i\| \geqq z) = P\left(\max_{1 \leqq i \leqq n} \left\|\sum_{j=1}^{m} S_i^{(j)}\right\| \geqq z\right) \leqq \sum_{j=1}^{m} P(\max_{1 \leqq i \leqq n} \|S_i^{(j)}\| \geqq m^{-1} z).$$

For each $j (1 \leqq j \leqq m)$, let B_j be the Borel subset of the $k_{n,j}d$-dimensional Euclidean space defined by

$$B_j = \{(x_1, \dots, x_{k_{n,j}}) \colon \max_{1 \leqq i \leqq n} \|x_j + x_{j+m} + \cdots + x_{j+k_{i,j}m}\| \geqq m^{-1} z\}$$

and put

$$h_j(x_1, \dots, x_{k_{n,j}}) = \begin{cases} 1 & \text{if } (x_1, \dots, x_{k_{n,j}}) \in B_j \\ 0 & \text{otherwise.} \end{cases}$$

Then, as in the proof of Theorem 1, we have

$$\begin{aligned} P(\max_{1 \leq i \leq n} \|\xi_j + \xi_{j+m}, \dots, \xi_{j+k_{i,j}m}\| \geq m^{-1} z) &= E h_j(\xi_j, \xi_{j+m}, \dots, \xi_{j+k_{i,j}m}) \\ &\leq \int_{R^{k_{i,j}d}} \cdots \int h_j(x_1, \dots, x_{k_{i,j}}) \, dF_j(x_1) \dots dF_{j+k_{i,j}m}(x_{k_{i,j}}) + 2k_{n,j}\beta(m) \\ &= P(\max_{1 \leq i \leq n} \|T_{n,i}^{(j)}\| \geq m^{-1} z) + 2k_{n,j}\beta(m). \end{aligned}$$

From (2.18) and (2.19), we have (2.15). The proof of (2.16) is similar and so is omitted.

3. Further Inequalities

(I) Bernstein's Inequalities

For absolutely regular sequences of bounded random vectors with $\beta(n)$, the followings hold:

Theorem 4. *Let $\{\xi_i\}$ be a strictly stationary, absolutely regular sequence of d-dimensional random vectors such that $\|\xi_i\| \leq M_0$ and $E\xi_i = 0$. Then, for the normalized sum $n^{-\frac{1}{2}} S_n$, the following inequalities hold when n is sufficiently large:*

(i) *If $d=1$, then for $0 < r < (\sigma_0^2/M_0) n^{\frac{1}{2}}$*

$$P(n^{-\frac{1}{2}}|S_n| \geq r) \leq 2m \exp\left\{ -\frac{(m^{-\frac{1}{2}} r)}{2\sigma_0^2} \left(1 - \frac{M_0 r}{\sigma_0^2 (mn)^{\frac{1}{2}}}\right)\right\} + 4n\beta(m) \tag{3.1}$$

and for $r \geq (\sigma_0^2/M_0) n^{\frac{1}{2}}$

$$P(n^{-\frac{1}{2}}|S_n| \geq r) \leq 2m \exp\left\{ -\frac{r n^{\frac{1}{2}}}{2M_0 m^{\frac{1}{2}}}\right\} + 4n\beta(m) \tag{3.2}$$

where $\sigma_0^2 = \mathrm{Var}(\xi_1) > 0$.

(ii) *If $d \geq 1$, then*

$$P(\|n^{-\frac{1}{2}} S_n\| \geq r) \leq 2mM_1 \exp\left\{ -\frac{r^2}{8e^2 M_0^2 m}\right\} + 4n\beta(m) \tag{3.3}$$

where M_1 is a constant depending only on M_0 and $E\|\xi_1\|^2$ (cf. Theorem in [11]).

Corollary. *Let $\{\xi_i\}$ be a strictly stationary, absolutely regular sequence of zero-one-valued random variables such that*

$$P(\xi_i = 0) = 1 - P(\xi_i = 1) = 1 - z, \quad 0 < z < 1. \tag{3.4}$$

If $\beta(n) = O(e^{-\gamma n})$ *for some* $\gamma > 0$, *then*

$$P(|S_n - nz| \geqq t) \leqq M(\log n)\, e^{-h} + 4n\,\beta(c \log n) \tag{3.5}$$

for all $t > 0$, *where* c *is a positive number and*

$$h = t^2 (\log n)^{-2} \left[2\{nz(1-z) + (t/3 \log n) \max(z, 1-z)\}\right]^{-1}. \tag{3.6}$$

The proofs of Theorem 4 and its corollary are easily obtained from Bernstein's inequalities and Theorem 1.

Using this corollary, from Bahadur's result in [1] we can obtain analogous results to Sen's ones on Bahadur's representation in [12].

(II) An Estimate for Tail Probabilities of Sums

Using Theorem 17.11 in [3] and Theorem 1, we have an estimate for tail probabilities of sums.

Theorem 5. *Let* $\{\xi_n\}$ *be a strictly stationary, absolutely regular d-dimensional random vectors having zero means and* $E\|\xi_1\|^s < \infty$ *for some integer* $s \geqq 3$. *Let*

$$\begin{aligned} &V = \operatorname{cov}(\xi_1), \qquad \lambda_* = \textit{smallest eigenvalue of } V, \\ &\lambda^* = \textit{largest eigenvalue of } V, \qquad \rho_r = E\|\xi_1\|^s \\ &\Delta_s = \inf_{0 \leqq \varepsilon \leqq 1} \left[\varepsilon \lambda_*^{-\frac{s}{2}} \int_{\{\|x\| \leqq \lambda^{*\frac{1}{2}} \varepsilon n^{\frac{1}{2}}\}} \|x\|^s\, dF(x) + \lambda_*^{-\frac{s}{2}} \int_{\{\|x\| \leqq \lambda^{\frac{1}{2}} \varepsilon n^{\frac{1}{2}}\}} \|x\|^s\, dF(x)\right]. \end{aligned} \tag{3.7}$$

If $\beta(n) = O(e^{-\gamma n})$ *for some* $\gamma > 0$, *then for any* $\delta > 0$

$$\begin{aligned} &\sup_{a \geqq ((s-2+\delta)\log n)^{\frac{1}{2}}} \{a^s (\log n)^{-1}\, P(\|n^{-\frac{1}{2}} S_n\| \geqq \lambda^{*\frac{1}{2}}\, a (\log n)^{\frac{1}{2}})\} \\ &\quad \leqq M n^{-(s-2)/2} (\Delta_s + o(1)). \end{aligned} \tag{3.8}$$

(III) Remark. Using Lemma and Theorems 1–2 we can easily obtain many other probability inequalities for partial sums of absolutely regular process such as Nagaev-Fuk type inequalities (see [5]).

4. Central Limit Problems

Theorem 6. *Let* $\{\xi_i\}$ *be a (not necessarily, strictly stationary) absolutely regular sequence of random variables with* $\beta(n)$. *Assume that* $E\xi_i^2 \leqq M_0$ *for all i. Assume that there are functions* $p = p(n)$, $q = q(n)$ *and* $k = k(n)$ *satisfying the following conditions for every* $\varepsilon > 0$: *as* $n \to \infty$

(i) $p \to \infty$, $k = [n/(p+q)] \to \infty$, $k\beta(q) \to 0$ *and* $p^{-1} q^2 \to 0$, (4.1)

(ii) $\sum_{j=1}^{k} \int_{|x| \geqq \varepsilon} dV_{nj}(x) \to 0$, (4.2)

$$\text{(iii)}\quad \sum_{j=1}^{k}\{\int_{|x|<\varepsilon} x^2\,dV_{nj}(x)-(\int_{|x|<\varepsilon} x\,dF_{nj}(x))^2\}\to\sigma^2, \tag{4.3}$$

$$\text{(iv)}\quad \sum_{j=1}^{k}\int_{|x|<\varepsilon} x\,dV_{nj}(x)\to a \tag{4.4}$$

where $\varepsilon>0$ is arbitrary and

$$V_{nj}(x)=P\left((kp)^{-\frac{1}{2}}\sum_{i=1}^{p}\xi_{(j-1)(p+q)+i}\leqq x\right)\qquad (j=1,\dots,k). \tag{4.5}$$

Then the distribution of $n^{-\frac{1}{2}}S_n$ will converge weakly to the normal distribution $N(a,\sigma^2)$ (cf. Theorem 18.4.1 in [7]).

Proof. Using the method in the proof of Theorem 1, we have that for any $\varepsilon>0$

$$\begin{aligned}
&P(|S'_n|\geqq 2\varepsilon n^{\frac{1}{2}})\\
&\quad=P\left(\left|\sum_{j=1}^{k-1}\sum_{i=1}^{q}\xi_{(j-1)(p+q)+p+i}+\sum_{i=(k-1)(p+q)+p+1}^{n}\xi_i\right|\geqq 2\varepsilon n^{\frac{1}{2}}\right)\\
&\quad\leqq P\left(\left|\sum_{j=1}^{k-1}\sum_{i=1}^{q}\xi_{(j-1)(p+q)+p+i}\right|\geqq \varepsilon n^{\frac{1}{2}}\right)+P\left(\left|\sum_{i=(k-1)(p+q)+p+1}^{n}\xi_i\right|\geqq\varepsilon n^{\frac{1}{2}}\right)\\
&\quad\leqq P\left(\left|\sum_{j=1}^{k-1}Y'_j\right|\geqq\varepsilon n^{\frac{1}{2}}\right)+4k\beta(q)+P\left(\left|\sum_{i=(k-1)(p+q)+p+1}^{n}\xi_i\right|\geqq\varepsilon n^{\frac{1}{2}}\right)
\end{aligned} \tag{4.6}$$

where Y'_j are i.i.d. random variables such that

$$P(Y'_j\leqq y)=P\left(\sum_{i=1}^{q}\xi_{(j-1)(p+q)+p+i}\leqq y\right)\qquad (j=1,\dots,k-1).$$

Since $EY_j'^2\leqq M_0q^2$, so from (i)

$$P\left(\left|\sum_{j=1}^{k-1}Y'_j\right|\geqq\varepsilon n^{\frac{1}{2}}\right)\leqq\frac{1}{\varepsilon^2 n}\sum_{j=1}^{k}EY_j'^2\leqq M\frac{kq^2}{n}\to 0. \tag{4.7}$$

It is obvious from (i) that $k\beta(p)\to 0$ as $n\to\infty$. On the other hand, if necessarily, repeating the above procedure, we can prove that

$$P\left(\left|\sum_{i=(k-1)(p+q)+p+1}^{n}\xi_i\right|\geqq\varepsilon n^{\frac{1}{2}}\right)\to 0\qquad\text{as } n\to\infty. \tag{4.8}$$

Using the notation in Theorem 2, from (2.9) we have that for every $\varepsilon>0$

$$\begin{aligned}
&|P(S_n\leqq xn^{\frac{1}{2}})-\Phi_a(x)|\\
&\quad\leqq\max\left\{\left|P\left(\sum_{i=1}^{k}Z_{n,i}\leqq x-2\varepsilon\right)-\Phi_a(x)\right|,\left|P\left(\sum_{i=1}^{k}Z_{n,i}\leqq x+2\varepsilon\right)-\Phi_a(x)\right|\right\}\\
&\qquad+P(|S'_n|\geqq 2\varepsilon n^{\frac{1}{2}})+2k\beta(q)
\end{aligned} \tag{4.9}$$

where

$$\Phi_a(x)=\int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}\,\sigma}e^{-\frac{(t-a)^2}{2\sigma^2}}dt.$$

Since

$$P(Z_{n,j}\leqq z)=V_{n,j}(z)\qquad (j=1,\ldots,k)$$

so from condition (ii)–(iv) and Theorem 15, Chapter 4 in [9]

$$P\left(\sum_{i=1}^{k}Z_{n,i}\leqq z\right)\to\Phi_a(z)\qquad\text{as } n\to\infty. \tag{4.10}$$

As $\Phi_a(z)$ is continuous, $\varepsilon>0$ is arbitrary and by assumption $k\beta(q)\to 0$ as $n\to\infty$, so from (4.7)–(4.9) we have the theorem.

Corollary. *Let $\{\xi_i\}$ be a (not necessarily strictly stationary) absolutely regular sequence of random variables with $E\xi_i=0$ and $E\xi_i^2\leqq M_0<\infty$ $(i=0, \pm 1, \pm 2,\ldots)$. Suppose that there exists a positive number σ^2 such that as $n\to\infty$*

$$E\left(\sum_{i=1}^{m}\xi_{j+i}\right)^2=n\sigma^2(1+o(1)) \tag{4.11}$$

and

$$\lim_{N\to\infty}\overline{\lim_{n\to\infty}}\int_{|z|>N}z^2\,dF_n^{(j)}(z)=0 \tag{4.12}$$

hold uniformly for all j $(j=0,1,2,\ldots)$, where for each j $F_n^{(j)}(z)$ denotes the df of the random variable $n^{-\frac{1}{2}}\sum_{i=1}^{n}\xi_{j+i}$. Then

$$\frac{1}{n^{\frac{1}{2}}\sigma}\sum_{i=1}^{n}\xi_i\xrightarrow{\mathscr{D}}N(0,1) \tag{4.13}$$

(*cf.* [10]).

Proof. We use the same notations in the proof of Theorem 7. Let p, q and k are some functions of n which satisfy (4.1). Firstly, we note that from (4.11) and (4.12)

$$\sum_{j=1}^{k}\int_{|x|\geqq\varepsilon}x^2\,dV_{nj}(x)=\sum_{j=1}^{k}k^{-1}\int_{|z|\geqq\varepsilon k^{\frac{1}{2}}}z^2\,dF_p^{((j-1)(p+q))}(z)\to 0$$

as $n\to\infty$.

Thus, (4.2) is satisfied obviously. Further, since $E\xi_i=0$ $(i=1,2,\ldots)$, so by (4.14)

$$\sum_{j=1}^{k}\left|\int_{|x|<\varepsilon}x\,dV_{nj}(x)\right|=\sum_{j=1}^{k}\left|\int_{|x|\geqq\varepsilon}x\,dV_{nj}(x)\right|\leqq\varepsilon^{-1}\sum_{j=1}^{k}\int_{|x|\geqq\varepsilon}x^2\,dV_{nj}(x)\to 0 \tag{4.15}$$

as $n\to\infty$, which implies (4.4) with $a=0$.

Finally, from (4.11) and (4.14)

$$\sum_{j=1}^{k} \int_{|x|<\varepsilon} x^2\, dV_{nj}(x)$$
$$= \sum_{j=1}^{k} \{\int x^2\, dV_{nj}(x) - \int_{|x|\geqq\varepsilon} x^2\, dV_{nj}(x)\}$$
$$= k^{-1} \sum_{j=1}^{k} E \left| p^{-\frac{1}{2}} \sum_{i=1}^{p} \xi_{(j-1)(p+q)+i} \right|^2 - \sum_{j=1}^{k} \int_{|x|\geqq\varepsilon} x^2\, dV_{nj}(x) \to \sigma^2 \tag{4.16}$$

as $n \to \infty$. Hence, from Theorem 6, the desired conclusion follows.

Remark. Let $\{\xi_i\}$ be a strictly stationary, absolutely regular sequence of random variables with $E\xi_i = 0$ and $E|\xi_i|^{4+\delta} < \infty$. Assume that $\beta(n) = O(e^{-\gamma n})$ for some $\gamma > 0$. Then, if we put $p = [n^{\frac{1}{2}}]$, $q = [c \log n]$ $(c\gamma > 1)$ and $k = [n/(p+q)]$, we can easily cheque the conditions (4.11) and (4.12) and so (4.13) holds. (It is known that (4.13) holds under more less restrictive condition (see [6]).) In this case, we can easily show that

$$\left| \operatorname{Var}\left(\sum_{i=1}^{n} \xi_i \right) - n\sigma^2 \right| = O(1) \tag{4.17}$$

and

$$E\left(\sum_{i=1}^{n} \xi_i \right)^4 \leqq K n^2. \tag{4.18}$$

So, using (4.17), (4.18), Theorems 2 and 5, we can prove that

$$\sup_x \left| P\left(\frac{\sum_{i=1}^{n} \xi_i}{\sqrt{n}\,\sigma} < x \right) - \Phi_0(x) \right| = O(n^{-\frac{1}{4}} (\log n)^{\frac{3}{2}}). \tag{4.19}$$

But, Stein (Corollary 3.2 in [13]) showed that the left-hand side of (4.19) is bounded by $A n^{-\frac{1}{2}} (\log n)^2$ under the slightly restrictive moment condition. It seems impossible that we can obtain the Stein's order $n^{-\frac{1}{2}} (\log n)^2$ using the above described method.

5. Convergence Rates in the Law of Large Numbers

In this section, we assume that $\{\xi_n\}$ is a strictly stationary, absolutely regular sequence of random variables with df F and function $\beta(n) = O(e^{-\gamma n})$ for some $\gamma > 0$.

Theorem 7. *Let* $t \geqq 0$. *If*

$$P(|\xi_1| \geqq n) = o(n^{-t-1}) \tag{5.1}$$

and

$$\int_{|x|<n} x\, dF(x) = o(1), \tag{5.2}$$

then for every $\varepsilon>0$

$$P\left(\left|\frac{S_n}{n}\right|\geqq\varepsilon\right)=o(n^{-t}(\log n)^{t+1}). \tag{5.3}$$

If (5.1) *with* $t>0$ *and* (5.2) *hold, then for every* $\varepsilon>0$

$$P\left(\sup_{k\geqq n}\left|\frac{S_k}{k}\right|\geqq\varepsilon\right)=o(n^{-t}(\log n)^{t+1}). \tag{5.4}$$

Proof. (5.3) is obvious from Theorem 27, Chapter 9 in [9] and Theorem 1.

To prove (5.4), let $2^{i_0-1}\leqq n<2^{i_0}$. Let c be a positive integer such that

$$\beta(c\log n)=o(n^{-t-2}). \tag{5.5}$$

From Theorem 3 and (5.5), we have that for every $\varepsilon>0$

$$\begin{aligned}
&n^t(\log n)^{-t-1}P\left(\sup_{k\geqq n}\left|\frac{S_k}{k}\right|\geqq\varepsilon\right)\\
&\leqq n^t(\log n)^{-t-1}P\left(\sup_{i\geqq i_0}\max_{2^{i-1}\leqq k<2^i}\left|\frac{S_k}{k}\right|\geqq\varepsilon\right)\\
&\leqq n^t(\log n)^{-t-1}\sum_{i=i_0}^{\infty}P(\max_{1\leqq k\leqq 2^i}|S_k|\geqq 2^{i-1}\varepsilon)\\
&\leqq n^t(\log n)^{-t-1}\sum_{i=i_0}^{\infty}\left\{\sum_{j=1}^{[c\log 2^i]}P(\max_{1\leqq k\leqq 2^i}|T_{2^i,k}^{(j)}|\geqq 2^{i-1}\varepsilon(c\log 2^i)^{-1})\right.\\
&\qquad\left.+2^{i+2}\beta(c\log 2^i)\right\}\\
&\leqq n^t(\log n)^{-t-1}\sum_{i=i_0}^{\infty}\sum_{j=1}^{[c\log n]}P(|T_{2^i,k}^{(j)}|\geqq 2^{i-1}\varepsilon(c\log 2^i)^{-1})+o(1)\\
&\leqq M\sum_{i=i_0}^{\infty}\left(\frac{\log 2^i}{\log 2^{i_0}}\right)^{t+1}2^{-t(i-i_0)}\frac{1}{\log 2^i}\sum_{j=1}^{[c\log 2^i]}\left(\frac{2^i}{\log 2^i}\right)^t\cdot P(|T_{2^i,k}^{(j)}|\\
&\qquad\geqq 2^{i-1}\varepsilon(c\log 2^i)^{-1})+o(1)\\
&=I_1+o(1),\qquad\text{(say)}.
\end{aligned} \tag{5.6}$$

Let δ be a positive number. We choose i_0 so that

$$n^tP\left(\left|\frac{S_n^*}{n}\right|\geqq\frac{\varepsilon}{2}\right)<\delta \tag{5.7}$$

for $n\geqq 2^{i_0-1}(\log 2^{i_0-1})^{-1}$, where S_n^* is the sum of i.i.d. random variables, each of them having the same df as that of ξ_1. (The existence of such i_0 is easily verified by Theorem 27, Chap. 9 in [9].) Then, from (5.7)

$$I_1\leqq M\sum_{i=i_0}^{\infty}\left(\frac{\log 2^i}{\log 2^{i_0}}\right)2^{-t(i-i_0)}\delta\leqq M\delta\frac{2^{t'}}{2^{t'}-1} \tag{5.8}$$

where $0<t'<t$. Thus, from (5.6) and (5.8)

$$n^t(\log n)^{-t-1} P\left(\sup_{k\geqq n}\left|\frac{S_k}{k}\right|\geqq\varepsilon\right)\to 0$$

which implies (5.4), and the proof is completed.

Acknowledgement. I would like to thank the editor and the referee for their valuable comments and suggestions.

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Received August 1, 1976; In revised form January 1, 1978