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A Splitting Technique for Harris Recurrent Markov Chains

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Summary. A technique is presented, which enables the state space of a Harris recurrent Markov chain to be "split" in a way, which introduces into the split state space an "atom". Hence the full force of renewal theory can be used in the analysis of Markov chains on a general state space. As a first illustration of the method we show how Derman's construction for the invariant measure works in the general state space. The Splitting Technique is also applied to the study of sums of transition probabilities.

1. Introduction

Let $X=\{X_n; n=0, 1, \ldots\}$ be a Markov chain (M.C.) on a general measurable state space $(E,\mathscr{E})$. Our notation and terminology follows Revuz (1975) (abbrev. [R]), to which the reader is referred for unexplained notation and terminology. We shall assume throughout this paper that the M.C. X is *recurrent in the sence of Harris* (φ-recurrent in the terminology of Orey (1971)): we assume the existence of a non-trivial $\varphi\in\mathscr{M}_+(\mathscr{E})$ such that

$$P_x[X_n\in A \text{ i.o.}]=1 \qquad \text{for all } x\in E,\ A\in\mathscr{E} \text{ with } \varphi(A)>0.$$

When studying recurrent Markov chains with a denumerable state space, a basic technique is to fix one state, and investigate the properties of the chain by using the independence of the paths between visits to this fixed state. This enables, for example, the full force of renewal theory to be used on the diagonal elements of a Markov chain transitions matrix (cf. Feller (1957)). When the state space is a general measurable space $(E,\mathscr{E})$, this type of argument fails: there does not, in general, exist a single point which is visited with positive probability from any point in the state space.

The main purpose of this paper is to present a technique which enables the state space of a Harris recurrent Markov chain to be *split* in a way which preserves the recurrent character of the chain, and which introduces into the split state space an

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atom: that is a set from the points of which all transitions are identical. On the new state space elementary renewal theorems can be used by considering returns to the atom in a manner analogous to the technique of countable state space chains.

Our splitting technique is in some ways similar to a method used by Griffeath in the context of coupling methods for Markov chains (cf. Theorem 3 of Ch. 3.3. of Griffeath (1976)). According to Griffeath, to every visit of the coupled process to the rectangle $C \times C$, where C is a C-set (for the definition of C-sets see [R], p. 160), is associated a random variable Z taking values 0 and 1 and being independent of the past history of the coupled process. The transition probability function of the coupled process is then modified according to the value of Z. The modification is such that the marginal distribution of the resulting stochastic process is the same as the distribution of the original coupled process.

Our method, however, does not involve the coupling of two copies of the original chain X. Instead, we are interested in a bivariate process, formed by the Markov chain X and an associated random variable taking the values 0 and 1, which essentially indicates in which "half" of the split space the chain is currently taking its value. Again, though, the bivariate process is such that the marginal distribution of X is that of the original process.

In order to split the space in this way, our basic assumption is that for some k, the k-step transition probability P_k is bounded below in a certain way: specifically, we assume the existence of $h \in \mathscr{E}_+$ with $\varphi(h) > 0$ and of a probability measure ν such that for all $x \in E, A \in \mathscr{E}$:

$$P_k(x, A) \geqq h \otimes \nu(x, A). \tag{M}$$

By considering the properties of C-sets mentioned above, it can be seen that this *Minorization Assumption* is in fact automatically satisfied when the σ-algebra $\mathscr{E}$ is countably generated. Since most results can be extended from countably generated to arbitrary σ-algebras using the method of admissible σ-algebras (cf. Orey (1971)) our splitting technique has almost universal applicability.

2. The Splitting Technique

We shall assume for a while that the transition probability P satisfies (M) *with* $k=1$, i.e. we assume that

$$P \geqq h \otimes \nu.$$

We shall construct a Markov chain X^*, such that the original M.C. X is "embedded" in the chain X^*, and such that X^* possesses an atom which is visited infinitely often with probability one.

We denote for all $x \in E$, $A \in \mathscr{E}$

$$x_0 = (x, 0), \quad x_1 = (x, 1);$$

$$A_0 = A \times \{0\}, \quad A_1 = A \times \{1\}, \quad A^* = A \times \{0, 1\}.$$

We denote by $\mathscr{E}^*$ the σ-algebra on E^* generated by the sets A_i ($A \in \mathscr{E}$, $i=0,1$). In the following we identify any subset A of E with the subset A^* of E^*. In particular we can write $\mathscr{E} \subset \mathscr{E}^*$. Any measure $\lambda \in \mathscr{M}_+(\mathscr{E})$ is automatically extended to a measure on $\mathscr{E}^*$ by defining its values on the sets $A_i (A \in \mathscr{E}, i=0,1)$

$$\lambda(A_0) = \lambda I_{1-h}(A), \qquad \lambda(A_1) = \lambda I_h(A). \tag{2.1}$$

We call $\lambda \in \mathscr{M}_+(\mathscr{E}^*)$ the *splitting* of $\lambda \in \mathscr{M}_+(\mathscr{E})$ (because $A_0 \cup A_1 = A$ and $A_0 \cap A_1 = \emptyset$, the extension is well defined). A finite λ can be extended in a similar way to $\mathscr{E}^*$. Any $\mathscr{E}$-measurable function f on E is automatically extended to an $\mathscr{E}$-measurable function on $E^* = E x\{0,1\}$ by defining for all $x \in E$

$$f(x_1) = f(x_0) = f(x).$$

Note that $\lambda(f)$ is unambiguously defined:

$$\lambda(f) = \int_E \lambda(dx) f(x) = \int_{E^*} \lambda(dz) f(z).$$

We define a T.P. P^* from $(E^*, \mathscr{E}^*)$ into $(E, \mathscr{E})$ as follows. Let $x \in E$, $A \in \mathscr{E}$ be arbitrary:

$$P^*(x_0, A) = \begin{cases} \nu(A) & \text{for } x \in \{h=1\}, \\ (1-h(x))^{-1}(P - h \otimes \nu)(x, A) & \text{for } x \in \{h<1\}; \end{cases}$$

$$P^*(x_1, A) = \nu(A).$$

We extend P^* to a T.P. on $(E^*, \mathscr{E}^*)$ as follows: for every $z \in E^*$, the measure $P^*(z, \cdot) \in \mathscr{M}_+(\mathscr{E}^*)$ is the splitting as defined by (2.1) of the measure $P^*(z, \cdot) \in \mathscr{M}_+(\mathscr{E})$.

Let $X^* = \{X_n^*\} = \{(X_n, Y_n)\}$, ($X_n \in E$, $Y_n \in \{0,1\}$), be the M.C. with state space (E^*, $\mathscr{E}^*$) and with T.P. P^*. We immediately see that the set E_1, which we shall henceforth denote by B, is an *atom* for the T.P. P^*, satisfying $\varphi(B) = \varphi(h) > 0$. We shall use the obvious notations for X^*; e.g. P_μ^* denotes the canonical probability measure on $(\Omega^*, \mathscr{F}^*) = (E^{*\infty}, \mathscr{E}^{*\infty})$ induced by an initial probability μ on $(E^*, \mathscr{E}^*)$ and the T.P. P^*.

We immediately get from the definitions the following two key results.

Theorem 1. *For any probability measure λ on $(E, \mathscr{E})$, the marginal distribution (w.r.t. P_λ^*) of the first coordinate process $\{X_n\}$ of the M.C. $\{X_n^*\}$ and the distribution (w.r.t. P_λ) of the original M.C. $\{X_n\}$ are identical.*

In particular, for any $A \in \mathscr{E}$, $n \geq 0$,

$$\lambda P_n^* = \lambda P_n.$$

Theorem 2. (i) *The M.C. X^* is Harris recurrent.*

(ii) *The set $B \overset{\text{def}}{=} E_1$ is a recurrent atom for X^*.*

Proof. (i) By the definition of the M.C. X^* and since X is φ-recurrent, we have for all $z \in E^*$, $A \in \mathscr{E}$ with $\varphi(A) > 0$

$$P_z^*[S_A < \infty] = P^*(z, A) + \int_{A^c} P^*(z, dy) P_y[S_A < \infty] = 1. \tag{2.2}$$

Fix $z \in E^*$ and $A \in \mathscr{E}$ with $\varphi(A_1) = \varphi I_h(A) > 0$. Then there exist $\beta > 0$ and $C \subset A$ such that $\varphi(C) > 0$ and $h \geqq \beta$ on C. Denote by S_C^n the instant of the n'th visit of X to C and define for $n \geqq 1$

$$Z_n = Y_{S_C^n} 1_{\{S_C^n < \infty\}}.$$

Then

$$P_z^* [Z_n = 1 | Z_1, \ldots, Z_{n-1}] \geqq \beta.$$

Hence by (2.2) $P_z^* [S_{C_1} < \infty] = P_z^* [Z_n = 1$ for some $n] = 1$, from which we get $P_z^* [S_{A_1} < \infty] = 1$.

Similarly $P_z^* [S_{A_0} < \infty] = 1$ for all $z \in E^*$ and $A \in \mathscr{E}$ with $\varphi(A_0) > 0$.

(ii) Follows from (i) and from the fact that $\varphi(B) = \varphi(h) > 0$. □

We define a transition kernel Q as follows:

$$Q = P \sum_{n=0}^{\infty} (P - h \otimes v)^n.$$

It is easily seen that Q has, in terms of the split chain X^*, the following probabilistic interpretation: for all $\lambda \in \mathscr{M}_+(\mathscr{E})$, $f \in \mathscr{E}_+$,

$$\begin{aligned} \lambda Q f &= E_\lambda^* \left[\sum_{n=1}^{S_B} f(X_n^*) \right] \\ &= \lambda U_B^* f, \quad (\text{cf. [R], p. 48}). \end{aligned} \tag{2.3}$$

Denote in the following by s a fixed point in the atom $B = E_1$, and let

$$\begin{aligned} a_\lambda(n) &= P_\lambda^* [S_B = n] = \lambda P (P - h \otimes v)^{n-1} h, \\ a(n) &= P_s^* [S_B = n] = v (P - h \otimes v)^{n-1} h, \\ \psi_f(n) &= E_s^* [f(X_n^*)] 1_{\{S_B \geqq n\}}] = v (P - h \otimes v)^{n-1} f, \\ u(n) &= P_n^*(s, B) = \begin{cases} 1 & \text{for } n = 0, \\ v P_{n-1} h & \text{for } n \geqq 1. \end{cases} \end{aligned}$$

The sequence $\{u(n);\ n = 0, 1, \ldots\}$ is a *renewal sequence* satisfying the renewal equation

$$u(n) = \delta(n) + a * u(n), \quad (\delta(0) = 1,\ \delta(n) = 0 \text{ for } n \geqq 1).$$

By using the fact that B is an atom, we obtain the following important *first-entrance-last-exit decomposition*

$$\lambda P_n f = E_\lambda^* [f(X_n^*) 1_{\{S_B \geqq n\}}] + a_\lambda * \psi_f * u(n). \tag{2.4}$$

In order to avoid unessential technicalities *we shall assume for the rest of the paper that the M.C. X is aperiodic* (see [R], p. 163).

If k in (M) is allowed to be greater than 1, we can always consider the *k-step* M.C. ${}_kX = \{X_{nk};\ n = 0, 1, \ldots\}$ and construct its split chain in the manner described above. The following lemma is needed when turning from the original M.C. X to the M.C. ${}_kX$.

Lemma 2.1. ${}_mX$ *is Harris recurrent for all* $m \geqq 1$.

Proof. Fix $A \in \mathscr{E}$ with $\varphi(A) > 0$. Let $\mathscr{B} \subset \mathscr{E}$ be an admissible σ-field (see Orey (1971)) such that $A \in \mathscr{B}$ and X with state space $(E, \mathscr{B})$ remains aperiodic. Let $C \in \mathscr{B}$ with $\varphi(C) > 0$ be a C-set (w.r.t. X on $(E, \mathscr{B})$). By the aperiodicity, there exists $N \geqq 1$ such that

$$\inf\{P_n(y, A);\ y \in C,\ N \leqq n < N + m\} > 0.$$

By using an obvious "geometric trial argument" we can conclude that the chain ${}_mX$ eventually visits the set A (P_x-a.s. for any $x \in E$). □

As an illustration of the Splitting Technique, we shall present a new *constructive proof* for the existence of an *invariant measure* (Harris (1956)).

Theorem 3. *The measure*

$$m = \nu \sum_{n=0}^{\infty} (P_k - h \otimes \nu)^n$$

is invariant for X *and satisfies* $m(h) = 1$.

Proof. (i) If $k = 1$, then according to the construction of Derman (1954), the measure m, defined by

$$m(f) = E_s^* \left[\sum_{n=1}^{S_B} f(X_n^*) \right] = \sum_{n=1}^{\infty} \psi_f(n)$$
$$= \nu \sum_{n=0}^{\infty} (P - h \otimes \nu)^n f$$

is invariant for X^* (hence also for X) and satisfies $m(B) = m(h) = 1$.

(ii) The case $k > 1$ easily follows from (i), from the uniqueness of the invariant measure and from Lemma 2.1. □

We shall henceforth use the notation

$$\mathscr{E}^+ = \{A \in \mathscr{E};\ m(A) > 0\}.$$

The Splitting Technique combined with the renewal theorem of Erdös, Feller and Pollard (1949) provides also (via the decomposition (2.4)) a simple proof for *Orey's convergence theorem* ([R], Theorem 2.8 on p. 169). We leave the details to the reader.

3. Sums of Transition Probabilities

In this section we shall illustrate the use of the Splitting Technique further and study the convergence of sums of transition probabilities (abbrev. S.T.P.). Our results generalize and sharpen some results of Kemeny, Snell and Knapp (1966), Pitman (1974), Cogburn (1975) and Griffeath (1976).

The following two concepts turn out to be useful in this context.

Definition 3.1. (i) Let $f \in \mathscr{L}^1_+(m)$ be arbitrary. A measure $\lambda \in \mathrm{b}\mathscr{M}_+(\mathscr{E})$ is called *f-regular*, provided that

$$\lambda U_A f < \infty \qquad \text{for all } A \in \mathscr{E}^+.$$

(ii) (Cogburn (1975)). A set $F \in \mathscr{E}$ is called *strongly uniform*, provided that

$$\sup_{x \in F} E_x[S_A] < \infty \qquad \text{for all } A \in \mathscr{E}^+.$$

We shall need the following lemma, the proof of which is straightforward and is therefore omitted.

Lemma 3.2. (i) *Denote* $\bar{f} = \sum_{n=0}^{k-1} P_n f$. *λ is f-regular, if and only if it is* $\bar{f}$*-regular w.r.t.* ${}_kX$ *and satisfies* $\lambda(\bar{f} - f) < \infty$.

(ii) *F is strongly uniform, if and only if it is strongly uniform w.r.t.* ${}_kX$. □

By considering the split chain X^* and using (2.4) we easily obtain the following characterization result, which, by using the preceding lemma, easily could be formulated in the case $k > 1$, too.

Proposition 3.3. *Assume that* $k = 1$. *Then*

(i) *λ is f-regular, if and only if* $\lambda Q f$ *is finite;*

(ii) *F is strongly uniform, if and only if* $Q1$ *is bounded on F.* □

The following lemma has also independent interest, since it generalizes the following result of Cogburn: for positive X, $E_x[S_A]$ is finite for all $A \in \mathscr{E}^+$, and m-almost all x.

Lemma 3.4. *For any* $f \in \mathscr{L}^1_+(m)$, $A \in \mathscr{E}^+$, $U_A f$ *is finite m-a.e.*

Proof. Denote $G = \{U_A f = \infty\}$. By using the *resolvent equation* of [R], p. 48, and the fact that $m I_A U_A f = m(f) < \infty$ ([R], p. 54), we obtain

$$P I_{A^c \cap G} U_A f(x) < \infty \qquad \text{for } x \notin G.$$

From this we easily get by induction that

$$m I_A P_n(G) = 0 \qquad \text{for all } n \geqq 0,$$

which implies the assertion. □

The interpretation of Q in (2.3), Proposition 3.3 and Lemma 3.4 lead us to the following result.

Proposition 3.5. (i) *The set* $R_f = \{x \in E; \varepsilon_x$ is f-regular$\}$ *of f-regular states is equal to E up to an m-negligible set.*

(ii) *There exists an increasing sequence* $\{F_n\}$ *of strongly uniform sets such that* $F_\infty = R_1$ *is equal to E up to an m-negligible set.* □

Remark 3.6. In the special case $f \equiv 1$ this sharpens Cogburn's (1975) Theorem 3.1 by ruling out the possibility that $m(R_1) = 0$. □

After these preliminaries we are able to formulate and prove our S.T.P.-results. The first result (Theorem 4) generalizes Theorem 9.15 of Kemeny, Snell and Knapp (1966) to general state space. The assumption that $k=1$ is not restrictive, since again by considering the k-step chain ${}_kX$ and using Lemma 3.2, we could easily formulate and prove the theorem for general k. We shall use the following notations:

$$\mathrm{b}\mathcal{M}_0(\mathcal{E})=\{\lambda\in\mathrm{b}\mathcal{M}(\mathcal{E});\ \lambda(E)=0\},$$
$$\mathcal{L}_0^1(m)=\{f\in\mathcal{L}^1(m);\ m(f)=0\}.$$

Theorem 4. *Assume that* $k=1$.

(i) *For any* $\lambda\in\mathrm{b}\mathcal{M}_0(\mathcal{E})$, $f\in\mathcal{L}^1(m)$, *such that* $|\lambda|$ *is* $|f|$*-regular and* $\sum_{n=1}^{\infty}\lambda P_n h$ *converges,*

$$\sum_{n=1}^{\infty}\lambda P_n f=\lambda Q f+\frac{m(f)}{m(h)}\sum_{n=1}^{\infty}\lambda P_n h.$$

(ii) *For any* $\lambda\in\mathrm{b}\mathcal{M}(\mathcal{E})$, $f\in\mathcal{L}_0^1(m)$, *such that* $|\lambda|$ *is* $|f|$*-regular and* $\sum_{n=1}^{\infty}\nu P^n f$ *converges,*

$$\sum_{n=1}^{\infty}\lambda P_n f=\lambda Q f+\lambda(E)\sum_{n=1}^{\infty}\nu P_n f.$$

Proof. We prove only (i), since the proof of (ii) is the "dual" of (i). We sum the decomposition equation (2.4) over n. The r.h.s. then converges to the desired limit

$$\lambda U_B^* f+\frac{m(f)}{m(B)}\sum_{n=1}^{\infty}\lambda P_n^*(B)=\lambda Q f+\frac{m(f)}{m(h)}\sum_{n=1}^{\infty}\lambda P_n h. \quad \square$$

It follows directly from the definitions that a function $f\in\mathcal{L}_+^1(m)$ is *special* (see [R], p. 182), if and only if all $\lambda\in\mathrm{b}\mathcal{M}_+(\mathcal{E})$ are f-regular. We thus obtain the following corollary.

Corollary. *If* $\sum_{n=1}^{\infty}\nu P^{nk} f$ *converges for every charge* f, *then the M.C.* X *is normal* (see [R], p. 242).

For the rest of this section we shall assume that X *is positive*, i.e. m is finite. We shall be concerned with the total variation norm (notation $\|\cdot\|$) convergence of the sums

$$\sum_{n=1}^{N}\lambda P_n, \quad (\lambda\in\mathrm{b}\mathcal{M}_0(\mathcal{E})), \tag{3.1}$$

and of the $\mathcal{L}^1(m)$-norm (notation $\|\cdot\|_m$) convergence of the sums

$$\sum_{n=1}^{N}P_n f, \quad (f\in\mathcal{L}_0^1(m)). \tag{3.2}$$

Theorem 5. (i) *If* $\lambda\in\mathrm{b}\mathcal{M}_0(\mathcal{E})$ *is such that* $|\lambda|$ *is* 1*-regular, then* $\sum_{n=1}^{\infty}\|\lambda P_n\|$ *is finite.*

(ii) *If* $f\in\mathscr{L}_0^1(m)$ *is such that* m *is* $|f|$*-regular, then* $\sum_{n=1}^{\infty}\|P_n f\|_m$ *is finite.*

Proof. Again we prove only (i). By Lemma 3.2 we can restrict ourselves to the case $k=1$. Take the supremum over $f\in\mathscr{U}$ and after that sum over n in both sides of the decomposition equation (2.4) to obtain

$$\sum_{n=1}^{\infty}\|\lambda P_n\|\leqq|\lambda|Q1+\frac{m(E)}{m(h)}\sum_{n=1}^{\infty}|a_\lambda*u|(n).$$

The former term in the r.h.s. is finite by Proposition 3.3 and the latter by the renewal Theorem 6.11 of Pitman (1974). □

Remark 3.7. By choosing $\lambda=\varepsilon_x-\varepsilon_y$ $(x,y\in E)$ in (i) and taking into account Proposition 3.5 we get as a corollary Griffeath's (1976) Theorem 3.3. Note that in Griffeath's theorem the assumption of the existence of a strongly uniform set is unnecessary by our Proposition 3.5.

The first part of Cogburn's (1975) Theorem 5.3 follows as a corollary from (ii) and Proposition 3.5 by choosing $\lambda=\varepsilon_x-m$. □

The following theorem identifies the limits of (3.1) and (3.2). We formulate it again only in the case $k=1$ the extension to general k being obvious.

Theorem 6. *Assume that* $k=1$.

(i) *For any* $\lambda\in b\mathscr{M}_0(\mathscr{E})$, such that $|\lambda|$ is 1-regular,

$$\lim_{N\to\infty}\left\|\sum_{n=1}^{N}\lambda P_n-\lambda Q(I-1\otimes m)\right\|=0.$$

(ii) *For any* $f\in\mathscr{L}_0^1(m)$ *such that* m *is* $|f|$*-regular,*

$$\lim_{N\to\infty}\left\|\sum_{n=1}^{N}P_n f-(I-1\otimes m)Qf\right\|_m=0.$$

Proof of (i). From (2.4) we get for any $A\in\mathscr{E}$, by summing over n and taking into account Pitman's Theorem 6.11,

$$\left|\sum_{n=1}^{N}\lambda P_n(A)-\lambda Q(I-1\otimes m)(A)\right|$$
$$\leqq\sum_{n=N+1}^{\infty}P_{|\lambda|}[S_B\geqq n]+\sum_{n=N+1}^{\infty}\psi_1*|a_\lambda*u|(n).$$

The former term in the r.h.s. converges to zero by (2.3) and Proposition (3.3), and the latter by Pitman's Theorem 6.11. □

4. A Ratio Limit Theorem for Sums of Transition Probabilities

Finally, we shall study the convergence of the ratio

$$\sum_{n=1}^{N} \lambda P_n f \Big/ \sum_{n=1}^{N} \mu P_n g, \tag{4.1}$$

where λ and μ are probability measures on $(E, \mathscr{E})$, and f and g belong to $\mathscr{L}_+^1(m)$.

Metivier (1972) (see also Neveu (1973) and [R]) has proved that the ratio of (4.1) converges to $m(f)/m(g)$ for any two probability measures λ and μ, and for any two *special functions* f and g with $m(g)>0$. The following theorem is a generalization of this result, since for any probability measure λ and $f \in \mathscr{L}_+^1(m)$ the statement *λ is f-regular* is weaker than *f is special.*

Theorem 7. *For any probability measures λ and μ, and any $f, g \in \mathscr{L}_+^1(m)$ with $m(g)>0$, such that λ is f-regular and μ is g-regular,*

$$\lim_{N\to\infty} \sum_{n=1}^{N} \lambda P_n f \Big/ \sum_{n=1}^{N} \mu P_n g = m(f)/m(g).$$

Proof. It again suffices to consider only the case $k=1$. We denote by $\mathbf{1}$ the sequence $\{1_n;\ n \geqq 0\}$ with $1_n \equiv 1$. By (2.4) and Proposition 3.3

$$\sup_{N \geqq 1} \left| \sum_{n=1}^{N} \lambda P_n f - a_\lambda * \psi_f * u * \mathbf{1}(N) \right| \leqq \lambda Q f < \infty,$$

and similarly for μ and g. Since $\sum_{n=1}^{\infty} \mu P_n g$ is infinite, we have

$$\lim_{N\to\infty} \sum_{n=1}^{N} \lambda P_n f \Big/ \sum_{n=1}^{N} \mu P_n g = \lim_{N\to\infty} a_\lambda * \psi_f * u * \mathbf{1}(N) / a_\mu * \psi_g * u * \mathbf{1}(N).$$

An elementary calculation and Theorem 3 yield

$$\begin{aligned} \lim_{N\to\infty} a_\lambda * \psi_f * u * \mathbf{1}(N)/u * \mathbf{1}(N) &= \sum_{n=1}^{\infty} \psi_f(n) \\ &= m(f)/m(h). \end{aligned}$$

Similar calculation for μ and g leads us to the final assertion. □

There are also other uses of the Splitting Technique. In Nummelin and Tweedie (1976) the Splitting Technique is applied in the study of geometric ergodicity for general state space M.C.'s. The Splitting Technique is extended to Markov renewal processes in Nummelin (1977).

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