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On Distribution Functions of Class L

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1. Introduction and Main Results

A distribution function $F(x)$ on the real line is said to be of class L (or L distribution function), if there are a sequence of independent random variables $\{X_n\}_{n\geqq 1}$ and constants $b_n>0$ and a_n such that the distribution of $b_n^{-1}\sum_{p=1}^{n} X_p - a_n$ weakly converges to F as $n\to\infty$ and, for every $\varepsilon>0$,

$$\lim_{n\to\infty}\max_{1\leqq p\leqq n} P(b_n^{-1}|X_p|>\varepsilon)=0.$$

Since K.L. Chung's remark in his translation of the book of Gnedenko and Kolmogorov [4] in 1954, unimodality of L distribution functions has long been an open problem. But, one of the authors [16] recently proved that every L distribution function is unimodal. The purpose of the present paper is to make a deeper analysis of properties of L distribution functions. We will classify L distribution functions into several classes and study each class. One of the main results we will show is that L distributions are strictly unimodal except in one class.

The class of L distributions is a natural family of infinitely divisible distributions including stable distributions. The representation of their characteristic functions was found by Lévy [8] (see also [4]). Let $\phi(t)$ be the characteristic function of a distribution function $F(x)$. Then, $F(x)$ is of class L if and only if

$$\phi(t)=\exp\left\{i\gamma t-\frac{\sigma^2 t^2}{2}+\int_{R_0}\left(e^{itu}-1-\frac{itu}{1+u^2}\right)\frac{k(u)}{u}\,du\right\}, \tag{1.1}$$

where γ is real, $\sigma^2\geqq 0$, $R_0=(-\infty,0)\cup(0,\infty)$, $k(u)$ is nonpositive on $(-\infty,0)$ and nonnegative on $(0,\infty)$, $k(u)$ is non-increasing on each of $(-\infty,0)$ and $(0,\infty)$, and

$$\int_{|u|\leqq 1} u\,k(u)\,du+\int_{|u|>1} u^{-1}k(u)\,du<\infty. \tag{1.2}$$

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Henceforth, let $F(x)$ be an L distribution function with characteristic function $\phi(t)$ of (1.1). We assume right-continuity of $k(u)$ without loss of generality. γ, σ^2, $k(u)$ are thus uniquely determined by F. We denote $\lambda_+ = k(0+)$, $\lambda_- = |k(0-)|$, $\lambda = \lambda_+ + \lambda_-$. These are important characteristics of F. If $0<\lambda<\infty$, then we define N as an integer such that

$$N<\lambda \leqq N+1. \tag{1.3}$$

If

$$\int_{|u|\leqq 1} |k(u)|\, du < \infty, \tag{1.4}$$

then the following expression is more convenient:

$$\phi(t) = \exp\left\{ i\gamma_0 t - \frac{\sigma^2 t^2}{2} + \int_{R_0} (e^{itu}-1) \frac{k(u)}{u} du \right\}, \tag{1.5}$$

where

$$\gamma_0 = \gamma - \int_{R_0} \frac{k(u)}{1+u^2} du. \tag{1.6}$$

A distribution function $G(x)$ is said to be unimodal if, for some a, $G(x)$ is convex on $(-\infty, a)$ and concave on (a, ∞). The point a is called a mode of G. Let $b_1 = \inf\{x\colon G(x)>0\}$ and $b_2 = \sup\{x\colon G(x)<1\}$. We say that $G(x)$ is strictly unimodal if there is a point a such that $G(x)$ is absolutely continuous on $(-\infty, a) \cup (a, \infty)$ and has a density increasing on (b_1, a) and decreasing on (a, b_2). (We are using the words 'increasing' and 'decreasing' in the strict sense). The mode of a unimodal distribution is not necessarily unique, but the mode of a strictly unimodal distribution is unique.

The following two theorems are known.

Theorem 1.1 (Yamazato [16]). *$F(x)$ is unimodal.*

Theorem 1.2 (Zolotarev [17] and Wolfe [13]). *If $\lambda>1$ or $\sigma^2>0$, then $F(x)$ has a continuous density $f(x)$ on $(-\infty, \infty)$. If $\sigma^2=0$ and $0<\lambda\leqq 1$, then $F(x)$ is continuous on $(-\infty,\infty)$ and has a continuous density $f(x)$ on $(-\infty, \gamma_0)\cup(\gamma_0, \infty)$. If* (1.4) *holds, define*

$$h(x) = (x-\gamma_0) f(x) \quad \text{for } x \neq \gamma_0 \text{ and } h(\gamma_0)=0. \tag{1.7}$$

If $\sigma^2=0$ and $0<\lambda\leqq 1$, then h is continuous on $(-\infty,\infty)$, but f is not continuous on $(-\infty,\infty)$. If $\sigma^2=0$ and $1<\lambda<\infty$, then f is a C^{N-1} function on $(-\infty,\infty)$ and C^N on $(-\infty,\gamma_0)\cup(\gamma_0,\infty)$, h is C^N on $(-\infty,\infty)$, but f is not C^N on $(-\infty,\infty)$. If $\lambda=\infty$ or $\sigma^2>0$, then f is C^∞ on $(-\infty,\infty)$.

These are remarkable facts. Other results are found in [13, 14, 15, 16, 17]. From now on, $f(x)$ denotes the density of $F(x)$ in Theorem 1.2.

We say that F is of

type I if $\sigma^2=0,\ \lambda_-=0,\ \lambda_+>0$;

type II if $\sigma^2>0,\ \lambda_-=0,\ \lambda_+>0$;

type III if $\sigma^2=0,\ \lambda_+\geqq\lambda_->0$;

type IV if $\sigma^2>0,\ \lambda_+\geqq\lambda_->0$.

Let $\check{F}(x)$ be the reflection of $F(x)$, that is, $\check{F}(x)=1-F((-x)-)$ (we define distribtuion functions to be right-continuous). If $F(x)$ is non-degenerate and non-Gaussian, then F or $\check{F}$ belongs to one of the above four types. Hence, in order to study properties of L distributions, it is enough to study the above four types. We further subdivide types I and III. In case F is of type I, we say that F is of

type I_1 if $0<\lambda<1$;

type I_2 if $\lambda=1$, $k(u)<1$ for all $u>0$ and $\int_0^1 (1-k(u))\,u^{-1}\,du=\infty$;

type I_3 if $\lambda=1$, $k(u)<1$ for all $u>0$ and $\int_0^1 (1-k(u))\,u^{-1}\,du<\infty$;

type I_4 if $\lambda=1$ and $k(u)=1$ for some $u>0$;

type I_5 if $1<\lambda<\infty$;

type I_6 if $\lambda=\infty$ and $\int_0^1 k(u)\,du<\infty$;

type I_7 if $\lambda=\infty$ and $\int_0^1 k(u)\,du=\infty$.

In case F is of type III, we say that F is of

type III_1 if $0<\lambda<1$;

type III_2 if $\lambda=1$;

type III_3 if $1<\lambda\leqq 2$ and $0<\lambda_-\leqq\lambda_+\leqq 1$;

type III_4 if $1<\lambda\leqq 2$ and $0<\lambda_-<1<\lambda_+$;

type III_5 if $2<\lambda<\infty$;

type III_6 if $\lambda=\infty$ and $\int_{-1}^{1} |k(u)|\,du<\infty$;

type III_7 if $\lambda=\infty$ and $\int_{-1}^{1} |k(u)|\,du=\infty$.

Now, let us state our main theorems. We write, for instance, $F\in\mathrm{I}_1$ in the meaning that F is of type I_1. $f(x)$ is said to be log-concave if $f(x)>0$ and $\log f(x)$

is concave. If $f(x)$ is absolutely continuous, we denote by $f^*(x)$ the almost-everywhere derivative of $f(x)$.

Theorem 1.3. (i) $F \in \bigcup_{j=1}^{6} \mathrm{I}_j \Rightarrow f(x)=0$ *on* $(-\infty, \gamma_0)$.

(ii) $F \in \bigcup_{j=1}^{4} \mathrm{I}_j \Rightarrow f(x)$ *is absolutely continuous on* (γ_0, ∞).

(iii) $F \in \bigcup_{j=1}^{3} \mathrm{I}_j \Rightarrow f^*(x)<0$ a.e. *on* (γ_0, ∞).

(iv) $F \in \bigcup_{j=1}^{2} \mathrm{I}_j \Rightarrow f(\gamma_0+)=\infty$.

(v) $F \in \mathrm{I}_3 \Rightarrow f(\gamma_0+)<\infty$.

(vi) $F \in \mathrm{I}_4 \Rightarrow$ *Let* $\beta=\sup\{u>0: k(u)=1\}$. $f(x)=\text{const}$ *on* $(\gamma_0, \gamma_0+\beta]$. $f^*(x)<0$ a.e. *on* $(\gamma_0+\beta, \infty)$.

(vii) $F \in \bigcup_{j=5}^{6} \mathrm{I}_j \Rightarrow f(x)$ *is continuous on* $(-\infty, \infty)$, C^1 *on* (γ_0, ∞). *There is a point* $a \in (\gamma_0, \infty)$ *such that* $f'(x)$ *is positive on* (γ_0, a), *zero at* a, *and negative on* (a, ∞). $f(x)$ *is* log-*concave on* $(\gamma_0, a]$.

(viii) $F \in \bigcup_{j=1}^{2} \mathrm{III}_j \Rightarrow f(x)$ *is absolutely continuous on* $(-\infty, \gamma_0) \cup (\gamma_0, \infty)$. $f^*(x)$ *is positive* a.e. *on* $(-\infty, \gamma_0)$ *and negative* a.e. *on* (γ_0, ∞). $f(\gamma_0-)=\infty, f(\gamma_0+)=\infty$.

(ix) $F \in \mathrm{III}_3 \Rightarrow f(x)$ *is continuous on* $(-\infty, \infty)$ *and* C^1 *on* $(-\infty, \gamma_0) \cup (\gamma_0, \infty)$. $f'(x)$ *is positive on* $(-\infty, \gamma_0)$ *and negative on* (γ_0, ∞).

(x) $F \in \mathrm{III}_4 \Rightarrow f(x)$ *is continuous on* $(-\infty, \infty)$ *and* C^1 *on* $(-\infty, \gamma_0) \cup (\gamma_0, \infty)$. *There is a point* $a \in (\gamma_0, \infty)$ *such that* $f'(x)$ *is positive on* $(-\infty, \gamma_0) \cup (\gamma_0, a)$, *zero at* a, *and negative on* (a, ∞). $f'(\gamma_0-)=\infty, f'(\gamma_0+)=\infty$.

(xi) $F \in \mathrm{I}_7 \cup \mathrm{II} \cup \mathrm{III}_5 \cup \mathrm{III}_6 \cup \mathrm{III}_7 \cup \mathrm{IV} \Rightarrow f(x)$ *is* C^1 *on* $(-\infty, \infty)$. *There is a point* a *such that* $f'(x)$ *is positive on* $(-\infty, a)$, *zero at* a, *and negative on* (a, ∞).

(xii) $F \in \mathrm{I}_7 \cup \mathrm{II} \Rightarrow f(x)$ *is* log-*concave on* $(-\infty, a]$.

The following two theorems are immediate consequences of Theorem 1.3. Theorem 1.5 shows that Theorem 4 of [17] is incorrect.

Theorem 1.4. *F is strictly unimodal if and only if neither F nor* $\check{F}$ *is of type* I_4.

Theorem 1.5. *F has unbounded density if and only if F or* $\check{F}$ *belongs to* $\mathrm{I}_1 \cup \mathrm{I}_2 \cup \mathrm{III}_1 \cup \mathrm{III}_2$.

We are interested in how f behaves in a neighborhood of γ_0. Let $f^{(n)}$ be the n-th derivative of f; $f^{(0)}=f$. For $F \in \bigcup_{j=1}^{5} \mathrm{I}_j$, we describe behaviors of $f^{(n)}$ for $n=0, \ldots, N$, which is an extension of a result of [15]. For $F \in \bigcup_{j=1}^{5} \mathrm{III}_j$, we describe behaviors of $f^{(N)}$. In general, when $\lambda<\infty$, we define a constant c and two functions $K(x)$ and $L(x)$ of $x \neq 0$ as follows:

$$c=\exp\left\{\lambda\int_0^1(e^{-u}-1)\,u^{-1}\,du+\lambda\int_1^\infty e^{-u}u^{-1}\,du-\int_1^\infty(k(u)-k(-u))\,u^{-1}du\right\}, \tag{1.8}$$

$$K(x)=\exp\int_{|x|}^1(\lambda-k(u)+k(-u))\,u^{-1}\,du, \tag{1.9}$$

$$L(x)=\int_{|x|}^1 K(u)\,u^{-1}\,du. \tag{1.10}$$

$K(x)$ and $L(x)$ are slowly varying as $x\to 0$.

Theorem 1.6. *If* $F\in\bigcup_{j=1}^5 \mathrm{I}_j$, *then, for* $n=0,\dots,N$,

$$f^{(n)}(x)\sim c\,\Gamma(\lambda-n)^{-1}(x-\gamma_0)^{\lambda-n-1}K(x-\gamma_0) \quad \textit{as } x\downarrow\gamma_0. \tag{1.11}$$

Theorem 1.7. *Suppose that* $F\in\bigcup_{j=1}^5 \mathrm{III}_j$.

(i) *If* $N<\lambda<N+1$, *then*

$$\lim_{x\downarrow\gamma_0}\frac{f^{(N)}(x)}{(x-\gamma_0)^{\lambda-N-1}K(x-\gamma_0)}=\frac{c\sin\lambda_+\pi}{\Gamma(\lambda-N)\sin\lambda\pi}, \tag{1.12}$$

$$\lim_{x\uparrow\gamma_0}\frac{f^{(N)}(x)}{(\gamma_0-x)^{\lambda-N-1}K(\gamma_0-x)}=\frac{(-1)^N c\sin\lambda_-\pi}{\Gamma(\lambda-N)\sin\lambda\pi}. \tag{1.13}$$

(ii) *If* $\lambda=N+1$, *then*

$$\lim_{x\to\gamma_0}\frac{f^{(N)}(x)}{L(x-\gamma_0)}=\frac{c}{\pi}\cos\frac{(\lambda_+-\lambda_--N)\pi}{2}. \tag{1.14}$$

(iii) *If* $\lambda=N+1$, *then*

$$\lim_{x\downarrow\gamma_0}\frac{f^{(N)}(x)-f^{(N)}(2\gamma_0-x)}{K(x-\gamma_0)}=\frac{c}{2}((-1)^{N+1}\cos\lambda_+\pi+\cos\lambda_-\pi). \tag{1.15}$$

Let us give some remarks on Theorem 1.7. Let $\boldsymbol{N}$ be the set of positive integers. If $F\in\bigcup_{j=1}^5 \mathrm{III}_j$, then we have the following four cases:

(a) $\lambda\notin\boldsymbol{N}$, $\lambda_+\notin\boldsymbol{N}$, $\lambda_-\notin\boldsymbol{N}$;

(b) $\lambda\notin\boldsymbol{N}$, λ_+ or $\lambda_-\in\boldsymbol{N}$;

(c) $\lambda\in\boldsymbol{N}$, $\lambda_+\notin\boldsymbol{N}$, $\lambda_-\notin\boldsymbol{N}$;

(d) $\lambda\in\boldsymbol{N}$, $\lambda_+\in\boldsymbol{N}$, $\lambda_-\in\boldsymbol{N}$.

In Cases (a) and (c), the right-hand sides of (1.12), (1.13), and (1.14) do not vanish and these describe the exact order of $f^{(N)}(x)$ as x approaches γ_0. In Case (b), (i) gives the exact order only on one side of γ_0. In Case (d), the right-hand side of (1.14) vanishes but the right-hand side of (1.15) is $(-1)^{\lambda-}c$. We have not succeeded in describing the exact order in Cases (b) and (d).

In Section 2, we will give an integro-differential equation satisfied by L distribution functions. In Section 3, a theorem on convolution of two unimodal distributions will be proved. Using these results as essential tools, we will prove the major part of Theorem 1.3 in Section 4. More information on the location of modes in case of types II, III, and IV will be included in Section 4. Section 5 contains the proof of Theorems 1.6 and 1.7 and completion of the proof of Theorem 1.3. A result on asymptotic behavior near γ_0 of $f(x)$ of type I_6 is also given in Section 5. Location of modes in case of types I and II will be discussed in Section 6 as an application of the integro-differential equation of Section 2. Our proof of Theorems 1.3–1.7 does not presuppose Theorems 1.1 and 1.2. Besides certain results found in the standard references [2] and [4], the only thing we use without proof is a general property of log-concavity (Lemma 4.5). Since Theorem 1.3 is stronger than Theorem 1.4 plus Theorem 1.3(vi), our argument gives an alternative proof of Theorem 1.1. Also, Theorem 1.2 is proved and refined by our Lemmas 2.5, 2.6 and Theorems 1.6, 1.7.

2. Integro-Differential Equation for L Distribution Functions

Let $F(x)$ be a non-degenerate L distribution function with characteristic function (1.1).

Theorem 2.1. *If* $\lambda>1$ *or* $\sigma^2>0$, *then*

$$\begin{aligned}(x-\gamma)f(x)&=\int_{R_0}(F(x-u)-F(x)+f(x)\arctan u)\,dk(u)-\sigma^2 f'(x)\\ &=\int_{R_0}\left(f(x-u)-\frac{f(x)}{1+u^2}\right)k(u)\,du-\sigma^2 f'(x)\end{aligned}\tag{2.1}$$

for every x. *If* $\lambda\leqq 1$ *and* $\sigma^2=0$, *then* (2.1) *holds for* $x\neq\gamma_0$. *Ignore the term* $-\sigma^2 f'(x)$ *in* (2.1) *when* $\sigma^2=0$. (f *is not always* C^1 *in case* $\sigma^2=0$.)

Lemma 2.1.

$$\lim_{|u|\to\infty} k(u)\log|u|=0, \tag{2.2}$$

$$\int_{|u|>1}\log|u|\,dk(u)>-\infty, \tag{2.3}$$

$$\lim_{u\to 0} u^2 k(u)=0, \tag{2.4}$$

$$\int_{0<|u|\leqq 1} u^2\,dk(u)>-\infty. \tag{2.5}$$

If (1.4) *holds, then*

$$\lim_{u \to 0} u\, k(u) = 0, \tag{2.6}$$

$$\int_{0<|u|\leqq 1} |u|\, dk(u) > -\infty. \tag{2.7}$$

Proof. For $1 < u_1 < u_2$,

$$k(u_2)\log u_2 - k(u_1)\log u_1 = \int_{u_1}^{u_2} u^{-1}\, k(u)\, du + \int_{u_1}^{u_2} \log u\, dk(u). \tag{2.8}$$

This shows that

$$k(u_2)\log u_2 - k(u_1)\log u_1 \leqq \int_{u_1}^{u_2} u^{-1}\, k(u)\, du. \tag{2.9}$$

Let θ_1 and θ_2 be the lower and upper limits of $k(u)\log u$ as $u \to \infty$, respectively. If $\theta_1 < \theta_2 = \theta_1 + \varepsilon$, then, by (1.2), we can find u_1 and u_2 such that $\int_{u_1}^{u_2} u^{-1}\, k(u)\, du < \varepsilon/2$ and $k(u_2)\log u_2 - k(u_1)\log u_1 > \varepsilon/2$, which contradicts (2.9). Hence $\theta_1 = \theta_2$. If $\theta_1 = \theta_2 > 0$, then $u^{-1}\, k(u) > 2^{-1}\, \theta_1 (u \log u)^{-1}$ for large u, contradicting (1.2). Hence $\theta_1 = \theta_2 = 0$. Considering $u \to -\infty$ in the same manner, we get (2.2). (2.3) follows from (1.2), (2.2), and (2.8). Similarly, (2.4) and (2.5) are proved by (1.2) and

$$u_2^2\, k(u_2) - u_1^2\, k(u_1) = 2\int_{u_1}^{u_2} u\, k(u)\, du + \int_{u_1}^{u_2} u^2\, dk(u) \tag{2.10}$$

for $0 < u_1 < u_2$ or $u_1 < u_2 < 0$. (2.6) and (2.7) are proved by (1.4) and

$$u_2\, k(u_2) - u_1\, k(u_1) = \int_{u_1}^{u_2} k(u)\, du + \int_{u_1}^{u_2} u\, dk(u) \tag{2.11}$$

for $0 < u_1 < u_2$ or $u_1 < u_2 < 0$.

The following expression of $\phi(t)$ is essentially the same as Urbanik [11].

Lemma 2.2.

$$\phi(t) = \exp\left\{ i\gamma t - \frac{\sigma^2 t^2}{2} - \int_{R_0} \left(\int_0^{tu} \frac{e^{iv}-1}{v}\, dv - i\, t \arctan u \right) dk(u) \right\}. \tag{2.12}$$

Proof. For each $t \neq 0$,

$$\int_0^u \left(e^{itv} - 1 - \frac{i\, t\, v}{1+v^2} \right) \frac{dv}{v} \sim -\frac{t^2 u^2}{4} \quad \text{as } |u| \to 0$$

and

$$\left| \int_{[-u,\,-1]\cup[1,\,u]} \left(e^{itv} - 1 - \frac{i\, t\, v}{1+v^2} \right) \frac{dv}{v} \right| \leqq 4 \log u + |t|\, \pi$$

for $u>1$. Hence, Lemma 2.1 and integration by parts rewrite (1.1) as

$$\phi(t)=\exp\left\{i\gamma t-\frac{\sigma^2 t^2}{2}-\int_{R_0}\left(\int_0^u\left(e^{itv}-1-\frac{itv}{1+v^2}\right)\frac{dv}{v}\right)dk(u)\right\}.$$

This is identical with (2.12).

Lemma 2.3. *$\phi(t)$ is C^1 on $(-\infty,0)\cup(0,\infty)$ and*

$$\phi'(t)=\phi(t)\left\{i\gamma-\sigma^2 t-t^{-1}\int_{R_0}(e^{itu}-1-it\arctan u)\,dk(u)\right\}. \tag{2.13}$$

Proof. (2.13) is obtained from (2.12), if we change the order of integration and differentiation. Since

$$\begin{aligned}&|t^{-1}(e^{itu}-1-it\arctan u)|\\ &\quad\leqq\left|\int_0^u(e^{itv}-1)\,dv\right|+\left|\int_0^u(1-(1+v^2)^{-1})\,dv\right|\leqq 2^{-1}|t|\,u^2+3^{-1}|u|^3,\end{aligned} \tag{2.14}$$

the change is justified for $t\neq 0$ by (2.5). The right-hand side of (2.13) is continuous in $t\neq 0$.

Lemma 2.4.

$$|\phi(t)|\leqq\exp(-2^{-1}\sigma^2 t^2)\quad \textit{for all } t. \tag{2.15}$$

If $0<\lambda<\infty$, then there is a constant M such that, for $|t|\geqq 1$,

$$|\phi(t)|\leqq M\,|t|^{-\lambda}K(|t|^{-1}). \tag{2.16}$$

If $0<\lambda\leqq\infty$, then, for each $\alpha<\lambda$,

$$|\phi(t)|=o(|t|^{-\alpha})\quad \textit{as } |t|\to\infty. \tag{2.17}$$

Proof. (2.15) is obvious. Let $0<\lambda<\infty$. To prove (2.16), we may assume $\sigma^2=\gamma_0=0$. Since $\phi(-t)=\overline{\phi(t)}$, we may further assume $t\geqq 1$. Let

$$l(u)=k(u)-k((-u)-). \tag{2.18}$$

We have

$$\begin{aligned}|\phi(t)|&=\exp\int_0^\infty(\cos tu-1)\,u^{-1}\,l(u)\,du\leqq\exp\int_{1/t}^1(\cos tu-1)\,u^{-1}\,l(u)\,du\\ &=\exp\left\{-\lambda\log t+\int_{1/t}^1(\lambda-l(u))\,u^{-1}\,du+\int_{1/t}^1(\cos tu)\,u^{-1}\,l(u)\,du\right\}\\ &=t^{-\lambda}K(t^{-1})\exp\left\{l(t^{-1})\int_1^t v^{-1}\cos v\,dv+\int_{1/t}^1\left(\int_{tu}^t v^{-1}\cos v\,dv\right)dl(u)\right\}.\end{aligned}$$

$\int_1^t v^{-1}\cos v\,dv$ is bounded in t. $\int_{tu}^t v^{-1}\cos v\,dv$ is bounded in $u\in[1/t,1]$ uniformly in t. Hence we get (2.16). (2.17) follows from (2.16). If $\lambda=\infty$, then, for $\beta>\alpha>0$, let $k_\beta(u)=\beta\wedge k(u)$ for $u>0$, $k_\beta(u)=(-\beta)\vee k(u)$ for $u<0$, and use

$$|\phi(t)|\leqq\left|\exp\int_{-\infty}^{\infty}(e^{itu}-1)\,u^{-1}k_\beta(u)\,du\right|$$

to obtain (2.17).

Lemma 2.5. *For any x,*

$$F(x)-F(0)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-ixt}-1}{-it}\phi(t)\,dt, \tag{2.19}$$

and F is continuous on $(-\infty,\infty)$. If $1<\lambda<\infty$, then, F is C^N on $(-\infty,\infty)$ and

$$F^{(n)}(x)=f^{(n-1)}(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}(-it)^{n-1}e^{-ixt}\phi(t)\,dt \tag{2.20}$$

for $n=1,\dots,N$. If $\lambda=\infty$ or $\sigma^2>0$, then F is C^∞ on $(-\infty,\infty)$ and (2.20) holds for all $n\geqq 1$.

Proof. (2.19) follows from Lévy's inversion formula. Note that the integrand in (2.19) is integrable (Lemma 2.4). If $\lambda>n$ or $\sigma^2>0$, then $|t|^{n-1}|\phi(t)|$ is integrable (Lemma 2.4), and (2.19) implies (2.20).

Lemma 2.6. *Suppose $0<\lambda<\infty$ and $\sigma^2=0$. $F(x)$ has a continuous density $f(x)$ on $(-\infty,\gamma_0)\cup(\gamma_0,\infty)$. Define $h(x)$ by (1.7). Then $h(x)$ is continuous on $(-\infty,\infty)$ and*

$$h(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-itx}-e^{-i\gamma_0 t}}{-it}\phi(t)\,dt\int_{R_0}(e^{itu}-1)\,dk(u). \tag{2.21}$$

$h(x)$ is C^N on $(-\infty,\infty)$ and, for $n=1,\dots,N$,

$$h^{(n)}(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}(-it)^{n-1}e^{-itx}\phi(t)\,dt\int_{R_0}(e^{itu}-1)\,dk(u). \tag{2.22}$$

Proof. By (2.19),

$$\begin{aligned}F(x+\gamma_0)-F(\gamma_0)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-ixt}-1}{-it}\phi(t)\,e^{-i\gamma_0 t}\,dt\\&=\frac{\operatorname{sgn}x}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-is}-1}{-is}\phi\left(\frac{s}{x}\right)e^{-i\gamma_0 s/x}\,ds\end{aligned} \tag{2.23}$$

for $x\neq 0$. Since

$$\gamma_0=\gamma+\int_{R_0}\arctan u\,dk(u), \tag{2.24}$$

Lemma 2.3 shows that

$$(\phi(t)\, e^{-i\gamma_0 t})' = -\phi(t)\, e^{-i\gamma_0 t}\, t^{-1} \int_{R_0} (e^{itu}-1)\, dk(u)$$

for $t \neq 0$. Hence

$$\frac{d}{dx}\left(\phi\left(\frac{s}{x}\right) e^{-i\gamma_0 s/x}\right) = \frac{1}{x}\,\phi\left(\frac{s}{x}\right) e^{-i\gamma_0 s/x} \int_{R_0} (e^{ius/x}-1)\, dk(u) \tag{2.25}$$

for $x \neq 0$, $s \neq 0$. By Lemma 2.4, there is an $\varepsilon > 0$ such that, as $|s| \to \infty$, the right-hand side of (2.25) is $o(|s|^{-\varepsilon})$ uniformly in x on any compact set off the origin. Hence the extreme right member of (2.23) is continuously differentiable in $x \neq 0$, and

$$f(x+\gamma_0) = \frac{1}{2\pi|x|} \int_{-\infty}^{\infty} \frac{e^{-is}-1}{-is}\,\phi\left(\frac{s}{x}\right) e^{-i\gamma_0 s/x}\, ds \int_{R_0} (e^{ius/x}-1)\, dk(u)$$

for $x \neq 0$. This is (2.21). The other assertion follows from (2.21) and Lemma 2.4.

Proof of Theorem 2.1. The proof consists of two parts. The first part gives the proof under the assumption that $\lambda > 2$ or $\sigma^2 > 0$. The second part gives the proof when $0 < \lambda < \infty$ and $\sigma^2 = 0$.

First Part. By $xt = s$ in (2.19), we get

$$F(x) - F(0) = \frac{\operatorname{sgn} x}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-is}-1}{-is}\,\phi\left(\frac{s}{x}\right) ds \qquad \text{for } x \neq 0. \tag{2.26}$$

By Lemma 2.3,

$$\frac{d}{dx}\left(\phi\left(\frac{s}{x}\right)\right) = \frac{1}{x}\,\phi\left(\frac{s}{x}\right)\left\{\int_{R_0}\left(e^{ius/x} - 1 - \frac{is}{x}\arctan u\right) dk(u) - \frac{i\gamma s}{x} + \frac{\sigma^2 s^2}{x^2}\right\} \tag{2.27}$$

for $x \neq 0, s \neq 0$. Noting (2.14) and Lemma 2.4, we can find an $\varepsilon > 0$ such that, as $|s| \to \infty$, the right-hand side of (2.27) is $o(|s|^{-\varepsilon})$ uniformly in x on any compact set off the origin. Here the assumption $\lambda > 2$ or $\sigma^2 > 0$ is made use of. Hence we can differentiate (2.26) under the integral sign. Thus

$$f(x) = \frac{1}{2\pi|x|} \int_{-\infty}^{\infty} \frac{e^{-is}-1}{-is}\,\phi\left(\frac{s}{x}\right)\{\cdots\}\, ds,$$

where the quantity in the braces is that of (2.27). Hence

$$f(x) = \frac{1}{2\pi x} \int_{-\infty}^{\infty} \frac{e^{-itx}-1}{-it}\,\phi(t)\left\{\int_{R_0} (e^{itu} - 1 - it\arctan u)\, dk(u) - i\gamma t + \sigma^2 t^2\right\} dt$$

for $x \neq 0$. By Fubini's theorem,

$$\begin{aligned} x f(x) = {} & \frac{1}{2\pi} \int_{R_0} dk(u) \int_{-\infty}^{\infty} \frac{e^{-itx}-1}{-it}(e^{itu} - 1 - it\arctan u)\,\phi(t)\, dt \\ & + \frac{\gamma}{2\pi} \int_{-\infty}^{\infty} (e^{-itx}-1)\,\phi(t)\, dt - \frac{\sigma^2}{2\pi} \int_{-\infty}^{\infty} (-it)(e^{-itx}-1)\,\phi(t)\, dt. \end{aligned}$$

By Lemma 2.5, this relation shows that

$$R(x)=R(0) \quad \text{for all } x, \tag{2.28}$$

if we define

$$R(x)=(x-\gamma)f(x) - \int_{R_0} (F(x-u)-F(x)+f(x)\arctan u)\,dk(u)+\sigma^2 f'(x). \tag{2.29}$$

We claim that $R(x)=0$. By (2.20) for $n=1,2, f(x)$ and $f'(x)$ tend to zero as $|x|\to\infty$ (use the Riemann-Lebesgue theorem). We have

$$F(x-u)-F(x)+f(x)\arctan u=\int_0^u (f(x)(1+v^2)^{-1}-f(x-v))\,dv, \tag{2.30}$$

$$|f(x)(1+v^2)^{-1}-f(x-v)|\leqq|f(x)-f(x-v)|+v^2(1+v^2)^{-1}f(x) \leqq M(|v|\wedge 1) \tag{2.31}$$

with M independent of x and v, noting that f' is bounded. Hence, using (2.5), we see that the integral in (2.29) tends to zero as $|x|\to\infty$. Since $f(x)$ is nonnegative and integrable, we can choose a sequence $x_n\to\infty$ such that $x_n f(x_n)\to 0$. It follows that $R(x_n)\to 0$ as $n\to\infty$. Combining this fact with (2.28), we get $R(x)=0$ as claimed. This proves the first equality in (2.1). The second equality follows from (2.5), (2.30), (2.31), and Fubini's theorem.

Second Part. Assume $0<\lambda<\infty$ and $\sigma^2=0$. Using Lemma 2.6, Fubini's theorem, and (2.19), we can rewrite (2.21) as

$$(x-\gamma_0)f(x)-\int_{R_0}(F(x-u)-F(x))\,dk(u)=-\int_{R_0}(F(\gamma_0-u)-F(\gamma_0))\,dk(u) \tag{2.32}$$

for $x\neq\gamma_0$. As $|x|\to\infty$, the integral in the left-hand side tends to zero. There is a sequence $x_n\to\infty$ such that $(x_n-\gamma_0)f(x_n)\to 0$, because $f(x)$ is nonnegative and integrable. Hence the right-hand side of (2.32) must vanish. This proves the first equality in (2.1) for $x\neq\gamma_0$ with the last term omitted. We obtain the second equality for $x\neq\gamma_0$, using (2.30) and Fubini's theorem, since k is bounded. If $\lambda>1$, then $f(x)$ is bounded and continuous (Lemma 2.5), and the equalities hold also at γ_0. Proof of Theorem 2.1 is complete.

Corollary 2.1. *If $\lambda>2$ or $\sigma^2>0$, then*

$$(x-\gamma)f'(x) = -f(x)+\int_{R_0}(f(x-u)-f(x)+f'(x)\arctan u)\,dk(u)-\sigma^2 f''(x) \tag{2.33}$$

for every x. If $1<\lambda<\infty$ and $\sigma^2=0$, then $f(x)$ is C^1 on $(-\infty,\gamma_0)\cup(\gamma_0,\infty)$ and

$$(x-\gamma_0)f'(x)=(\lambda-1)f(x)+\int_{R_0} f(x-u)\,dk(u) \quad \textit{for } x\neq\gamma_0. \tag{2.34}$$

If $0<\lambda\leqq 1$ *and* $\sigma^2=0$, *then* $f(x)$ *is absolutely continuous on* $(-\infty,\gamma_0)\cup(\gamma_0,\infty)$ *and* $f^*(x)$ *satisfies*

$$(x-\gamma_0)f^*(x)=(\lambda-1)f(x)+\int_{u\neq 0,x-\gamma_0} f(x-u)\,dk(u) \tag{2.35}$$

for almost every x.

Proof. If $\lambda>3$ or $\sigma^2>0$, then (2.33) is an obvious consequence of (2.1), for f has two bounded continuous derivatives (Lemma 2.5). If $2<\lambda\leqq 3$ and $\sigma^2=0$, then (2.33) holds with the term $-\sigma^2 f''(x)$ omitted, since f has a bounded continuous derivative in this case. Suppose that $0<\lambda<\infty$ and $\sigma^2=0$. It follows from (2.1) that

$$(x-\gamma_0)f(x)=\lambda F(x)+\int_{R_0} F(x-u)\,dk(u) \quad \text{for } x\neq\gamma_0. \tag{2.36}$$

The right-hand side of (2.36) is a function of bounded variation, since F is monotone. Hence, for each $\varepsilon>0$, $f(x)$ is of bounded variation on $(-\infty,\gamma_0-\varepsilon)\cup(\gamma_0+\varepsilon,\infty)$. For $x\neq\gamma_0$, let $\tilde{f}(x)$ be the right-hand side of (2.35) divided by $x-\gamma_0$ (the integral in (2.35) may be $-\infty$). Let $[x_1,x_2]\subset(\gamma_0,\infty)$. $\tilde{f}(x)$ is bounded from above on $[x_1,x_2]$ and we have

$$\begin{aligned}
&\int_{x_1}^{x_2}((x-\gamma_0)\tilde{f}(x)+f(x))\,dx\\
&=\lambda\int_{x_1}^{x_2} f(x)\,dx+\int_{x_1}^{x_2}dx\int_{u\neq 0,x-\gamma_0} f(x-u)\,dk(u)\\
&=(x_2-\gamma_0)f(x_2)-(x_1-\gamma_0)f(x_1)\\
&=\int_{x_1}^{x_2}(x-\gamma_0)\,df(x)+\int_{x_1}^{x_2} f(x)\,dx,
\end{aligned}$$

by using Fubini and (2.36). If $[x_1,x_2]\subset(-\infty,\gamma_0)$, then $\tilde{f}(x)$ is bounded from below on $[x_1,x_2]$ and we get the same identity. Hence $\tilde{f}$ is the a.e. derivative of f. If $1<\lambda<\infty$ and $\sigma^2=0$, then f is continuous on $(-\infty,\infty)$ (Lemma 2.5), C^1 on $(-\infty,\gamma_0)\cup(\gamma_0,\infty)$ (Lemma 2.6), and the right-hand side of (2.34) is continuous. Since this equals the right-hand side of (2.35) almost everywhere, (2.34) holds. The proof is complete.

Remark. In case $\sigma^2=0$, $k(u)=0$ on $(-\infty,0)$ and $k(u)$ is a step function with a finite number of jumps, the equations (2.34) and (2.36) are extensively used by Wolfe [12, 14] and Yamazato [16]. An equation analogous to (2.1) is known for all one-sided infinitely divisible distributions (Steutel [10], p. 86).

3. A Convolution Theorem

We will give a strict sense version of a theorem of Yamazato [16] on convolutions of unimodal distributions. Let $G(x)$ be an absolutely continuous distribution function, $b_G = b(G) = \inf\{x: G(x) > 0\}$, and $g(x)$ be the density of $G(x)$. Let $g^*(x)$ denote the a.e. derivative of $g(x)$ if $g(x)$ is absolutely continuous. We introduce two conditions.

Condition (A). *$b_G > -\infty$, g is absolutely continuous on (b_G, ∞), and, for some $\delta \geqq 0$, $g^*(x) = 0$ a.e. on $(b_G, b_G + \delta)$ and $g^*(x) < 0$ a.e. on $(b_G + \delta, \infty)$.*

Condition (B). *g is C^1 on (b_G, ∞), $\lim_{x \to b_G} g(x) = 0$. There is a point $a_G = a(G) > b_G$ such that $g'(x)$ is positive on (b_G, a_G), zero at a_G, negative and bounded on (a_G, ∞). $g(x)$ is log-concave on $(b_G, a_G]$.*

If G satisfies (A), let $a_G = b_G$.

Theorem 3.1. *Let $G_1(x)$ and $G_2(x)$ be absolutely continuous distribution functions with densities $g_1(x)$ and $g_2(x)$, respectively. Let $G = G_1 * G_2$ and*

$$g(x) = \int_{-\infty}^{\infty} g_1(y) g_2(x-y)\, dy = \int_{-\infty}^{\infty} g_1(x-y) g_2(y)\, dy, \tag{3.1}$$

the density of G. Let

$$b_1 = -b(\check{G}_1), \quad a_1 = -a(\check{G}_1), \quad b_2 = b(G_2), \quad a_2 = a(G_2). \tag{3.2}$$

(i) *If $\check{G}_1$ and G_2 satisfy Condition (A), then $g(x)$ is absolutely continuous on $(-\infty, b_1 + b_2) \cup (b_1 + b_2, \infty)$ and $g^*(x)$ is positive a.e. on $(-\infty, b_1 + b_2)$, negative a.e. on $(b_1 + b_2, \infty)$.*

(ii) *Suppose that $\check{G}_1$ satisfies (B) and G_2 satisfies (A). Then $g(x)$ is continuous on $(-\infty, \infty)$, C^1 on $(-\infty, b_1 + b_2) \cup (b_1 + b_2, \infty)$, and there is a point $a \in (a_1 + b_2, b_1 + b_2)$ such that $g'(x)$ is positive on $(-\infty, a)$, zero at a, and negative on $(a, b_1 + b_2) \cup (b_1 + b_2, \infty)$. If $b_1 < \infty$, then $\lim_{x \downarrow b_1 + b_2} g'(x)$ exists and < 0 and $\limsup_{x \uparrow b_1 + b_2} g'(x) < 0$ (the possibility of $-\infty$ is not excluded).*

(iii) *If $\check{G}_1$ and G_2 satisfy (B), then $g(x)$ is C^1 on $(-\infty, \infty)$, and there is a point $a \in (a_1 + b_2, a_2 + b_1)$ such that $g'(x)$ is positive on $(-\infty, a)$, zero at a, and negative on (a, ∞).*

Corollary 3.1. *Suppose that $\check{G}_1$ satisfies (A) or (B), G_2 satisfies (A) or (B), and $G = G_1 * G_2$. Then G is strictly unimodal. With the use of (3.2), the mode a of G is located as follows:*

$$a = a_1 + b_2 \quad \text{if } a_1 + b_2 = a_2 + b_1; \tag{3.3}$$

$$a \in (a_1 + b_2, a_2 + b_1) \quad \text{if } a_1 + b_2 < a_2 + b_1. \tag{3.4}$$

Lemma 3.1. *Let G_1 be such that $\check{G}_1$ satisfies Condition (B) and $b_1 = -b(\check{G}_1) > 0$, $a(\check{G}_1) = 0$. For $\varepsilon \in (0, b_1)$, define*

$$A_\varepsilon(x) = g_1(x + \varepsilon)/g_1(x), \tag{3.5}$$

$$B_\varepsilon(x, y) = (g_1(x + \varepsilon - y) - g_1(x + \varepsilon))/(g_1(x - y) - g_1(x)). \tag{3.6}$$

Then,

$$A_\varepsilon(x) \text{ is non-increasing on } [0, b_1-\varepsilon); \tag{3.7}$$

$$A_\varepsilon(x) \text{ is decreasing on } (-\varepsilon, 0]; \tag{3.8}$$

$$A_\varepsilon(x) < 1 \quad \text{on } [0, b_1); \tag{3.9}$$

$$A_\varepsilon(x) > 1 \quad \text{on } (-\infty, -\varepsilon]; \tag{3.10}$$

$$B_\varepsilon(x, y) \geqq A_\varepsilon(x) \quad \text{for } 0 < y \leqq x < b_1 - \varepsilon. \tag{3.11}$$

Proof. Since g_1 is log-concave on $[0, b_1)$, (3.7) follows from

$$A'_\varepsilon(x) = \frac{g_1(x+\varepsilon)}{g_1(x)}\left(\frac{g'_1(x+\varepsilon)}{g_1(x+\varepsilon)} - \frac{g'_1(x)}{g_1(x)}\right) \leqq 0 \quad \text{on } [0, b_1-\varepsilon).$$

Since $g_1(x)$ is increasing on $(-\infty, 0)$ and decreasing on $(0, b_1)$, it is easy to see (3.8), (3.9), and (3.10). It follows from (3.7) that

$$g_1(x+\varepsilon-y)/g_1(x-y) \geqq g_1(x+\varepsilon)/g_1(x)$$

for $0 < y \leqq x < b_1 - \varepsilon$, and hence (3.11).

Proof of Theorem 3.1. (i) By translation, we may assume $b_1 = b_2 = 0$. Let $x > 0$. Since

$$g(x) = \int_{-\infty}^{0} g_2(x-y) g_1(y)\, dy,$$

$g(x)$ is finite and continuous. We have

$$\begin{aligned} -\int_x^\infty dy \int_y^\infty g_2^*(z) g_1(y-z)\, dz &= -\int_x^\infty g_2^*(z)\, dz \int_{x-z}^{0} g_1(y)\, dy \\ &= -\int_{-\infty}^{0} g_1(y)\, dy \int_{x-y}^{\infty} g_2^*(z)\, dz \\ &= g(x). \end{aligned}$$

Hence, for $x > 0$, $g(x)$ has density $\int_x^\infty g_2^*(z) g_1(x-z)\, dz$, which is finite a.e. and negative. Argument for $x < 0$ is similar.

(ii) We may assume $a_1 = b_2 = 0$. From the second expression of $g(x)$ in (3.1), we see that $g(x)$ is bounded and continuous on $(-\infty, \infty)$. For each $\xi > 0$, we have

$$g(x) = g_2(\xi) \int_{-\infty}^{x} g_1(y)\, dy + \int_\xi^\infty g_2^*(z)\, dz \int_{-\infty}^{x-z} g_1(y)\, dy - \int_0^\xi g_2^*(z)\, dz \int_{x-z}^{x} g_1(y)\, dy, \tag{3.12}$$

because

$$g(x) = \int_\xi^\infty g_1(x-y)\left(g_2(\xi) + \int_\xi^y g_2^*(z)\, dz\right) dy + \int_0^\xi g_1(x-y)\left(g_2(\xi) - \int_y^\xi g_2^*(z)\, dz\right) dy.$$

For $x > b_1$, choosing $\xi < x - b_1$ in (3.12), we see that $g(x)$ is differentiable and

$$g'(x) = \int_{\xi}^{\infty} g_2^*(z)\, g_1(x-z)\, dz. \tag{3.13}$$

Hence, on (b_1, ∞), g is C^1 and $g' < 0$. We have, for each $\varepsilon > 0$,

$$|g_1(x-z) - g_1(x)| \leqq \text{const}\, z \qquad \text{for } x < b_1 - \varepsilon \text{ and } z > 0 \tag{3.14}$$

by Condition (B) of $\check{G}_1$, and

$$\int_0^{\xi} g_2^*(z)\, z\, dz = \int_0^{\xi} z\, dg_2(z) > -\infty, \tag{3.15}$$

which is proved like (2.7) of Lemma 2.1. For $x < b_1$, we can differentiate (3.12) and obtain

$$g'(x) = g_2(\xi)\, g_1(x) + \int_{\xi}^{\infty} g_2^*(z)\, g_1(x-z)\, dz + \int_0^{\xi} g_2^*(z)(g_1(x-z) - g_1(x))\, dz \tag{3.16}$$

for each $\xi > 0$, because (3.14) and (3.15) justify differentiation under the integral sign of the last term of (3.12). Thus g is C^1 on $(-\infty, b_1)$. Since g_1' is bounded continuous on $(-\infty, 0]$, we see that

$$g'(x) = \int_0^{\infty} g_1'(x-y)\, g_2(y)\, dy > 0 \qquad \text{for } x \leqq 0$$

from the second expression of g in (3.1). If $b_1 < \infty$, then

$$\limsup_{x \uparrow b_1} g'(x) < 0,$$

because, choosing $0 < \xi < b_1$ in (3.16) and letting $x \uparrow b_1$, we get

$$g'(x) \leqq g_2(\xi)\, g_1(x) + \int_{\xi}^{\infty} g_2^*(z)\, g_1(x-z)\, dz \to \int_{\xi}^{\infty} g_2^*(z)\, g_1(b_1 - z)\, dz < 0.$$

Also, if $b_1 < \infty$, then

$$\lim_{x \downarrow b_1} g'(x) < 0,$$

since (3.13) shows that, for $x > b_1$,

$$g'(x) = \int_{x-b_1}^{\infty} g_2^*(z)\, g_1(x-z)\, dz,$$

which tends to $\int_0^{\infty} g_2^*(z)\, g_1(b_1 - z)\, dz$ as $x \downarrow b_1$. Now, the proof of (ii) will be complete if we show that

$$\text{if } 0 < x < x + \varepsilon < b_1 \text{ and } g'(x) \leqq 0, \text{ then } g'(x+\varepsilon) < 0. \tag{3.17}$$

Let us prove (3.17). We will make essential use of Lemma 3.1. Let $0<\xi<x$. From (3.16),

$$g'(x+\varepsilon)=g_2(\xi)\,g_1(x+\varepsilon)+\int_\xi^\infty g_2^*(z)\,g_1(x+\varepsilon-z)\,dz$$
$$+\int_0^\xi g_2^*(z)(g_1(x+\varepsilon-z)-g_1(x+\varepsilon))\,dz$$
$$=g_2(\xi)\,A_\varepsilon(x)\,g_1(x)+\int_\xi^\infty g_2^*(z)\,A_\varepsilon(x-z)\,g_1(x-z)\,dz$$
$$+\int_0^\xi g_2^*(z)\,B_\varepsilon(x,z)(g_1(x-z)-g_1(x))\,dz,$$

where A_ε and B_ε are (3.5) and (3.6). We have $A_\varepsilon(x-z)>1>A_\varepsilon(x)$ for $z\geqq x+\varepsilon$ by (3.9), (3.10); $A_\varepsilon(x-z)\geqq A_\varepsilon(x)$ for $x+\varepsilon>z>0$ by (3.7), (3.8); and $B_\varepsilon(x,z)\geqq A_\varepsilon(x)$ for $\xi>z>0$ by (3.11). Hence

$$g'(x+\varepsilon)<A_\varepsilon(x)\left\{g_2(\xi)\,g_1(x)+\int_\xi^\infty g_2^*(z)\,g_1(x-z)\,dz+\int_0^\xi g_2^*(z)(g_1(x-z)-g_1(x))\,dz\right\}$$
$$=A_\varepsilon(x)\,g'(x)\leqq 0.$$

(iii) We may assume $a_1=a_2=0$. By (3.1), g is clearly bounded and continuous. If we are permitted to differentiate (3.1) under the integral sign, we would get

$$g'(x)=\int_{-\infty}^\infty g_2'(x-z)\,g_1(z)\,dz$$

and hence

$$g'(x)=\int_{b_2}^0 g_2'(z)\,g_1(x-z)\,dz+\int_0^\infty g_2'(z)\,g_1(x-z)\,dz. \tag{3.18}$$

Let us show that g is C^1 and (3.18) holds. In fact,

$$g(x)=\int_{b_2}^0 g_1(x-y)\left(g_2(0)-\int_y^0 g_2'(z)\,dz\right)dy+\int_0^\infty g_1(x-y)\left(g_2(0)+\int_0^y g_2'(z)\,dz\right)dy$$
$$=g_2(0)\int_{-\infty}^{x-b_2} g_1(y)\,dy-\int_{b_2}^0 g_2'(z)\,dz\int_{x-z}^{x-b_2} g_1(y)\,dy+\int_0^\infty g_2'(z)\,dz\int_{-\infty}^{x-z} g_1(y)\,dy,$$

from which follows

$$g'(x)=g_2(0)\,g_1(x-b_2)+\int_{b_2}^0 g_2'(z)(g_1(x-z)-g_1(x-b_2))\,dz+\int_0^\infty g_2'(z)\,g_1(x-z)\,dz$$

with the understanding that $g_1(x-b_2)=0$ if $b_2=-\infty$. This shows that g is C^1 on $(-\infty,\infty)$ and (3.18) is true. If $b_1<\infty$, then $g'<0$ on $[b_1,\infty)$, for (3.18) shows

$$g'(x)=\int_{-\infty}^{b_1} g_2'(x-y)\,g_1(y)\,dy.$$

By the symmetry of the assumption, we also see that, if $b_2>-\infty$, then $g'>0$ on $(-\infty, b_2]$. Now, in order to complete the proof, it is enough to verify the following two properties:

$$\text{if } 0\leqq x<x+\varepsilon<b_1 \text{ and } g'(x)\leqq 0, \text{ then } g'(x+\varepsilon)<0; \tag{3.19}$$

$$\text{if } b_2<x-\varepsilon<x\leqq 0 \text{ and } g'(x)\geqq 0, \text{ then } g'(x-\varepsilon)>0. \tag{3.20}$$

By the symmetry of the assumption, it suffices to prove one of these. Suppose $0\leqq x<x+\varepsilon<b_1$ and $g'(x)\leqq 0$. We have, from (3.18),

$$g'(x+\varepsilon)=\int_{b_2\vee(x+\varepsilon-b_1)}^{0} g_2'(z)\,A_\varepsilon(x-z)\,g_1(x-z)\,dz+\int_0^\infty g_2'(z)\,A_\varepsilon(x-z)\,g_1(x-z)\,dz.$$

Note that $A_\varepsilon(x-z)\leqq A_\varepsilon(x)$ for $x+\varepsilon-b_1<z<0$ by (3.7); $A_\varepsilon(x-z)\geqq A_\varepsilon(x)$ for $0<z<x+\varepsilon$ by (3.7), (3.8); $A_\varepsilon(x-z)>1>A_\varepsilon(x)$ for $z\geqq x+\varepsilon$ by (3.9), (3.10). Then,

$$\begin{aligned} g'(x+\varepsilon)&<A_\varepsilon(x)\left(\int_{b_2\vee(x+\varepsilon-b_1)}^{0} g_2'(z)\,g_1(x-z)\,dz+\int_0^\infty g_2'(z)\,g_1(x-z)\,dz\right)\\ &\leqq A_\varepsilon(x)\left(\int_{b_2\vee(x-b_1)}^{0} g_2'(z)\,g_1(x-z)\,dz+\int_0^\infty g_2'(z)\,g_1(x-z)\,dz\right)\\ &=A_\varepsilon(x)\,g'(x)\leqq 0. \end{aligned}$$

This proves (3.19) and the proof of Theorem 3.1 is complete.

4. Strict Unimodality and Related Properties

If

$$\int_{-1}^{0} |k(u)|\,du<\infty, \tag{4.1}$$

then we use

$$\phi_-(t)=\exp\int_{-\infty}^{0} (e^{itu}-1)\,k(u)\,u^{-1}\,du. \tag{4.2}$$

If

$$\int_0^1 k(u)\,du<\infty, \tag{4.3}$$

then let

$$\phi_+(t)=\exp\int_0^\infty (e^{itu}-1)\,k(u)\,u^{-1}\,du. \tag{4.4}$$

Let

$$\phi_1(t)=\phi(t)/\phi_+(t) \tag{4.5}$$

$$\phi_2(t)=\phi(t)/\phi_-(t). \tag{4.6}$$

Denote the distribution corresponding to ϕ_+, ϕ_-, ϕ_1, or ϕ_2 by F_+, F_-, F_1, or F_2, respectively. In strict unimodal case, the mode is denoted by a_+, a_-, a_1, or a_2, respectively. If F is strictly unimodal, the mode is denoted by a. In this section we will prove most of Theorem 1.3 and, simultaneously, give the following results.

Theorem 4.1.

(i) $F\in \mathrm{II}$ *and* (4.3) $\Rightarrow a\in(\gamma_0,\infty)$.

(ii) $F\in \mathrm{III}_4 \Rightarrow a\in(\gamma_0,\gamma_0+a_+)$.

(iii) $F\in \mathrm{III}_5\cup \mathrm{III}_6$ *and* $\lambda_-\leqq 1 \Rightarrow a\in(\gamma_0,\gamma_0+a_+)$.

(iv) $F\in \mathrm{III}_5\cup \mathrm{III}_6$ *and* $\lambda_->1 \Rightarrow a\in(\gamma_0+a_-,\gamma_0+a_+)$.

(v) $F\in \mathrm{III}_7\cup \mathrm{IV}$ *and* (4.1) $\Rightarrow a\in(-\infty,a_2)$.

$F\in \mathrm{III}_7\cup \mathrm{IV}$ *and* (4.3) $\Rightarrow a\in(a_1,\infty)$.

$F\in \mathrm{III}_7\cup \mathrm{IV}$ *and* (1.4) $\Rightarrow a\in(a_1,a_2)$.

First, we give two simple lemmas. Lemma 4.1 applies to all one-sided infinitely divisible distributions. Lemma 4.2 is an extension of Steutel [10], p. 87.

Lemma 4.1. *If* $F\in \bigcup_{j=1}^{6} \mathrm{I}_j$, *then* $F(x)>0$ *for all* $x\in(\gamma_0,\infty)$.

Proof. Given $x_0>\gamma_0$, we can find an $\varepsilon>0$ such that the distribution function $G_1(x)$ with characteristic function

$$\exp\left\{i\gamma_0 t+\int_0^{\varepsilon}(e^{itu}-1)\,u^{-1}k(u)\,du\right\}$$

satisfies $G_1(x_0)>0$, because this distribution weakly converges to the degenerate distribution at γ_0 as $\varepsilon\downarrow 0$. Define G_2 by $F=G_1*G_2$. Then G_2 is a compound Poisson distribution, and hence $G_2(0)-G_2(0-)=G_2(0)>0$. It follows that $F(x_0)>0$.

Lemma 4.2. *If* $F\in \bigcup_{j=1}^{6} \mathrm{I}_j$, *then* $f(x)>0$ *on* (γ_0,∞). *If* $F\in \mathrm{I}_7$, *then* $f(x)>0$ *on* $(-\infty,\infty)$.

Proof. Let

$$\gamma_0=-\infty \qquad \text{for } F\in \mathrm{I}_7. \tag{4.7}$$

Suppose that $f(x_0)=0$ for some $x_0>\gamma_0$. If $F\in \bigcup_{j=1}^{6} \mathrm{I}_j$, then $f(x)>0$ for some $x\in(\gamma_0,x_0)$ by Lemma 4.1. The same is true also for $F\in \mathrm{I}_7$, since the support of F of I_7 is unbounded from below by a general theory of infinitely divisible distributions (Baxter-Shapiro [1]). By continuity (Lemma 2.6), we can find $\gamma_0<x_1<x_2$ such that $f(x_2)=0$ and $f>0$ on (x_1,x_2). By the equation of Theorem 2.1,

$$\int_0^\infty (F(x_2-u)-F(x_2))\,dk(u)=0.$$

Hence $F(x_2-u)-F(x_2)=0$ for some $u>0$. This is absurd. The proof is complete.

Proof of Theorem 1.3 (i), (ii), (iii). (i) is a well-known result from a general theory (see Feller [2] or Baxter-Shapiro [1]), but it is also a consequence of Theorem 2.1. Namely, if $f(x)>0$ for some $x>\gamma_0$, we have a contradiction with (2.1), noting (2.24). (ii) is shown in Corollary 2.1. Let us prove (iii). By (2.35),

$$(x-\gamma_0)f^*(x)=(\lambda-1)f(x)+\int_{(0,x-\gamma_0)} f(x-u)\,dk(u)$$

for a.e. $x>\gamma_0$. Each term of the right-hand side is nonpositive. By Lemma 4.2, the first term is negative if $F\in \mathrm{I}_1$, and the second term is negative if $F\in \mathrm{I}_2\cup \mathrm{I}_3$.

Proof of Theorem 1.3 (vi). By (2.35),

$$(x-\gamma_0)f^*(x)=\int_{(0,x-\gamma_0)} f(x-u)\,dk(u) \qquad \text{for a.e. } x>\gamma_0.$$

Hence $f^*=0$ a.e. on $(\gamma_0,\gamma_0+\beta)$. If $x>\gamma_0+\beta$, then the right-hand side is negative by Lemma 4.2.

In order to examine $\mathrm{I}_5\cup \mathrm{I}_6$, we need three lemmas.

Lemma 4.3. *Assume* $F\in \mathrm{I}_5\cup \mathrm{I}_6$ *and let*

$$\beta=\sup\{u\colon\ k(u)\geqq 1\}. \tag{4.8}$$

Then, $f'(x)>0$ *on* $(\gamma_0,\gamma_0+\beta]$.

Proof. By Lemmas 2.5 and 2.6, f is continuous on $(-\infty,\infty)$ and C^1 on (γ_0,∞). Suppose $f'(x)\leqq 0$ for some $x\in(\gamma_0,\gamma_0+\beta]$. By Lemma 4.2 and $f(\gamma_0)=0$, we can find in any right neighborhood of γ_0 a point x at which $f'(x)>0$. Hence there is $x_1\in(\gamma_0,\gamma_0+\beta]$ such that $f'(x_1)=0$. Again by Lemma 4.2, $f(x_1)>0$. Choose $x_0\in(\gamma_0,x_1]$ such that $f(x_0)=\max_{x\in[\gamma_0,x_1]} f(x)$ and $f(x)<f(x_0)$ for all $x<x_0$. Since $f'(x_0)=0$, we have

$$f(x_0)=\int_0^\infty (f(x_0-u)-f(x_0))\,dk(u) \tag{4.9}$$

by (2.33). Let $x_0-\gamma_0=\varepsilon$. It follows that

$$(k(\varepsilon-)-1)f(x_0)+\int_{(0,\varepsilon)} (f(x_0-u)-f(x_0))\,dk(u)=0. \tag{4.10}$$

We have $k(\varepsilon-)\geqq 1$ since $\varepsilon\leqq\beta$. Hence both terms in (4.10) vanish. It follows that $k(\varepsilon-)=1$ and $k(\varepsilon-)=k(0+)$, contradicting the assumption $\lambda>1$.

Lemma 4.4. *Let $G_n (n=1,2,\ldots)$ be unimodal distribution functions with mode a_n. If G_n weakly converges to a distribution function G, then G is unimodal and one can find a sequence $n_1<n_2<\cdots$ such that a_{n_p} tends to a mode of G as $p\to\infty$.*

Proof. It is enough to use the fact that the limit of convex functions is convex. See [4], p. 160.

Lemma 4.5. *Let G_n $(n=1,2,\dots)$ be a sequence of distribution functions weakly convergent to a distribution function G. Suppose that, on some interval (c_n, d_n), G_n is absolutely continuous and its density g_n is C^1 and* log-*concave, that $c_n \to c$ and $d_n \to d$, and that G is absolutely continuous on (c,d) and its density g is C^1 and positive on (c,d). Then, g is* log-*concave on (c,d).*

Proof. This is a consequence of Lemma 2 of Yamazato [16].

Proof. of Theorem 1.3 (vii). Lemmas 2.5 and 2.6 show that f is continuous on $(-\infty,\infty)$ and C^1 on (γ_0,∞) for I_5 and that f is C^∞ on $(-\infty,\infty)$ for I_6. Let $F\in I_5\cup I_6$. We will divide our proof into several steps. First, note that, if $x_0>\gamma_0$ and $f'(x_0)=0$, then

$$f(x_0)=\int_0^\infty (f(x_0-u)-f(x_0))\,dk(u). \tag{4.11}$$

Step 1. There exists no pair of points x_1, x_2 such that $\gamma_0<x_1<x_2$, f is increasing on (γ_0, x_1), non-increasing on (x_1, x_2), and $f'(x_1)=f'(x_2)=0$. The proof is as follows. Suppose that such a pair of points exists. Let $x_2-x_1=\varepsilon$. Since (4.11) holds at x_2 and $f(x_2-u)-f(x_2)\geqq 0$ for $u\in[0,\varepsilon]$, we have

$$f(x_2)\leqq \int_{(\varepsilon,\infty)} (f(x_2-u)-f(x_2))\,dk(u).$$

Hence,

$$(k(\varepsilon)-1)f(x_2)\geqq -\int_{(\varepsilon,x_2-\gamma_0)} f(x_2-u)\,dk(u). \tag{4.12}$$

Since $f(x_2)>0$ by Lemma 4.2, it follows that $k(\varepsilon)\geqq 1$. On the other hand,

$$(k(\varepsilon)-1)f(x_1)=-\int_{(0,\varepsilon]} (f(x_1-u)-f(x_1))\,dk(u)-\int_{(\varepsilon,\infty)} f(x_1-u)\,dk(u), \tag{4.13}$$

because (4.11) holds also at x_1. Now note that $f(x_1-u)-f(x_1)<0$ for $u>0$ and that $f(x_1-u)<f(x_2-u)$ for $u\in(\varepsilon, x_2-\gamma_0)$. Moreover, since $\gamma_0+\beta<x_1$ by Lemma 4.3, we have $k((x_2-\gamma_0)-)<1<k(0+)$. It follows from (4.13) that

$$(k(\varepsilon)-1)f(x_1)<-\int_{(\varepsilon,x_2-\gamma_0)} f(x_2-u)\,dk(u). \tag{4.14}$$

This contradicts (4.12), since $(k(\varepsilon)-1)f(x_1)\geqq(k(\varepsilon)-1)f(x_2)$.

Step 2. Let a be the infimum of $x\in(\gamma_0,\infty)$ such that $f'(x)=0$. Then $\gamma_0+\beta<a$ and $f'>0$ on (γ_0,a) by Lemma 4.3. We claim that $f'<0$ on (a,∞). By Step 1, it is enough to show that, if $\varepsilon>0$ is sufficiently small, then $f'(a+\varepsilon)<0$. Fix ε_0 and α such that $0<\varepsilon_0<\beta$ and $0<\alpha<a-\gamma_0-\beta$. Let $0<\varepsilon<\varepsilon_0$. Suppose, for a while, that $F\in I_5$. From (2.33) at $x=a$ and $a+\varepsilon$, we have

$$(a+\varepsilon-\gamma_0)f'(a+\varepsilon)=(\lambda-1)(f(a+\varepsilon)-f(a))+\int_0^\infty (f(a+\varepsilon-u)-f(a-u))\,dk(u). \tag{4.15}$$

The right-hand side is $o(\varepsilon)+A_1+A_2+A_3$, where A_1, A_2, A_3 are the integrals $\int (f(a+\varepsilon-u)-f(a-u))\,dk(u)$ over $(0,\varepsilon]$, $(\varepsilon, a-\gamma_0]$, $(a-\gamma_0, a-\gamma_0+\varepsilon)$, respectively. It is easy to see that $A_1=o(\varepsilon)$ and $A_3 \leqq 0$. We have

$$A_2 \leqq \int_{[\beta,\beta+\alpha]} (f(a+\varepsilon-u)-f(a-u))\,dk(u) \leqq M\varepsilon$$

where

$$M=(k(\beta+\alpha)-k(\beta-))\min_{y\in[a-\beta-\alpha,\, a+\varepsilon_0-\beta]} f'(y).$$

M is negative, since $k(\beta-)\geqq 1>k(\beta+\alpha)$ and $f'>0$ on (γ_0, a). Hence we get $f'(a+\varepsilon)<0$ for small ε, as desired. If $F\in \mathrm{I}_6$, then we need more delicate argument as follows. Instead of (4.15) we have

$$(a+\varepsilon-\gamma_0)f'(a+\varepsilon)=f(a)-f(a-\varepsilon)+\int_0^\infty \tilde{f}(u)\,dk(u)$$

where $\tilde{f}(u)=f(a+\varepsilon-u)-f(a-u)-f(a+\varepsilon)+f(a)$. Using Lemma 2.1, notice that

$$\left|\int_{(0,\varepsilon]} \tilde{f}(u)\,dk(u)\right|=\left|\int_{(0,\varepsilon]} dk(u)\int_a^{a+\varepsilon}(f'(x-u)-f'(x))\,dx\right|\leqq \text{const}\,\varepsilon\left|\int_{(0,\varepsilon]} u\,dk(u)\right|=o(\varepsilon)$$

and that

$$\int_{(\varepsilon,\infty)} (-f(a+\varepsilon)+f(a))\,dk(u)=2^{-1}\varepsilon^2 f''(a+\theta\varepsilon)\,k(\varepsilon)=o(\varepsilon)$$

where $0<\theta<1$. The remaining part of the proof is the same as above.

Step 3. Let $1<\lambda\leqq 2$. Then, $f(x)$ is log-concave on $(\gamma_0, a]$. In fact, just as we proved the last sentence of Corollary 2.1, we can show that f' is absolutely continuous on (γ_0, a) and

$$(x-\gamma_0)(f')^*(x)=(\lambda-2)f'(x)+\int_{(0,x-\gamma_0)} f'(x-u)\,dk(u)$$

a.e. on (γ_0, a). Hence $(f')^*\leqq 0$ a.e. on (γ_0, a). It follows that f' is non-increasing on (γ_0, a) and hence f'/f is non-increasing there.

Step 4. Let $2<\lambda<\infty$. Under the assumption that

$$k(u)=\lambda \qquad \text{on some } (0,\delta), \tag{4.16}$$

we claim that f is log-concave on $(\gamma_0, a]$. We may assume that $k(u)<\lambda$ for $u>\delta$. Let $S(x)=(\log f)''=(f''f-(f')^2)/f^2$. For $x\in(\gamma_0, \gamma_0+\delta)$, (2.34) reduces to $(x-\gamma_0)f'=(\lambda-1)f$, and hence $f(x)=\text{const}(x-\gamma_0)^{\lambda-1}$. Thus $S(x)<0$ on $(\gamma_0, \gamma_0+\delta]$. We have

$$(x-\gamma_0)f''(x)=(\lambda-2)f'(x)+\int_0^\infty f'(x-u)\,dk(u), \qquad x>\gamma_0,$$

from (2.34). It follows from this and (2.34) that

$$(x-\gamma_0)(f''(x)f(x)-f'(x)^2) = -f'(x)f(x) + \int_{(0,x-\gamma_0)} (f'(x-u)f(x)-f(x-u)f'(x))\,dk(u). \tag{4.17}$$

Now, suppose that, for some $x_0 \in (\gamma_0, a]$, $S(x_0)=0$ and $S<0$ on (γ_0, x_0). Then, $x_0 > \gamma_0 + \delta$. Look at (4.17) at $x = x_0$. The left-hand side vanishes, while $f'(x_0)f(x_0) \geqq 0$ and the integral in the right-hand side is

$$\int_{[\delta, x_0-\gamma_0)} f(x_0)f(x_0-u)\left(\frac{f'(x_0-u)}{f(x_0-u)} - \frac{f'(x_0)}{f(x_0)}\right) dk(u),$$

which is negative. This is contradiction. It follows that $S<0$ on $(\gamma_0, a]$.

Step 5. Let $\lambda > 2$. Let us show that f is log-concave on $(\gamma_0, a]$ even if (4.16) is not satisfied or if $\lambda = \infty$. Choose n_0 such that $k(n_0^{-1}) > 2$. For $n \geqq n_0$, let

$$k_n(u) = k(u \vee n^{-1}) \tag{4.18}$$

and let F_n be the distribution with characteristic function

$$\phi_n(t) = \exp\left\{ i\gamma_0 t + \int_0^\infty (e^{itu}-1)\, u^{-1} k_n(u)\, du \right\}. \tag{4.19}$$

Denote the quantities related to F_n by putting subscript n. As $n \to \infty$, $\phi_n(t) \to \phi(t)$ and F_n tends weakly to F. Hence $a_n \to a$ by Lemma 4.4 and by the uniqueness of the mode of F (Step 2). Since f_n is log-concave on $(\gamma_0, a_n]$ by Step 4, f is log-concave on (γ_0, a) by Lemma 4.5 (or, check $f_n \to f$ instead of using Lemma 4.5). By continuity, it is log-concave on $(\gamma_0, a]$. Proof of (vii) is complete.

Proof of the Assertion on I_7 *in Theorem 1.3* (xi), (xii). We divide the proof into two steps. The first step is a slightly weaker version of Step 1 of the preceding proof. Instead of Lemma 4.3, we can now use the assumption $\lambda = \infty$. In the second step, we use the result of (vii).

Step 1. We claim that there exists no pair of points $x_1 < x_2$ such that $f(x)$ is non-decreasing on $(-\infty, x_1)$, $f'(x_1) = f'(x_2) = 0$, $f(x) < f(x_1)$ for all $x \in (-\infty, x_1)$, and $f(x) \geqq f(x_2)$ for all $x \in [x_1, x_2]$. Suppose that such x_1 and x_2 exist. Let $x_2 - x_1 = \varepsilon$. Then we get (4.12) (with ∞ in place of $x_2 - \gamma_0$), (4.13), and $k(\varepsilon) \geqq 1$ in the same way as Step 1 of the proof of (vii). Since $k(\varepsilon) < \infty = k(0+)$, we have

$$-\int_{(0,\varepsilon]} (f(x_1-u) - f(x_1))\, dk(u) < 0.$$

Also, we have $f(x_1-u) \leqq f(x_2-u)$ for $u \in (\varepsilon, \infty)$. Hence we obtain (4.14) with ∞ in place of $x_2 - \gamma_0$. This is a contradiction.

Step 2. Let n_0 be such that $k(n_0^{-1}) > 2$. For $n \geqq n_0$, define k_n by (4.18), let

$$\phi_n(t) = \exp\left\{ i\gamma t + \int_0^\infty (e^{itu} - 1 - itu(1+u^2)^{-1})\, u^{-1} k_n(u)\, du \right\}, \tag{4.20}$$

and let F_n be the corresponding distribution function. F_n is of type I_5. It converges weakly to F, and $\gamma_{0n} \to -\infty$. It follows that F is unimodal and f is log-concave on $(-\infty, a)$, where a is a mode of F (Lemma 4.4 and 4.5). f' is nonnegative on $(-\infty, a)$, zero at a, and nonpositive on (a, ∞). By Step 1, f is not flat on any interval in $(-\infty, a)$. It follows that $f' > 0$ on $(-\infty, a)$. In fact, if $f' = 0$ at some $x_0 \in (-\infty, a)$, then $f'/f \leqq 0$ on (x_0, a) by log-concavity, and hence $f' = 0$ on (x_0, a), a contradiction. Now using Step 1 again, we see that $f' < 0$ on (a, ∞), completing the proof.

Remark. The above proof indicates that, if $F \in \bigcup_{j=1}^{4} \mathrm{I}_j$, then F satisfies Condition (A) of Section 3, and that, if $F \in \bigcup_{j=5}^{7} \mathrm{I}_j$, then F satisfies Condition (B). Only the fact that f' is bounded on (a, ∞) in the latter case is not yet checked. But, it is a consequence of the Riemann-Lebesgue theorem that $f'(x) \to 0$ as $x \to \infty$. Use (2.20) or (2.22) according as $\lambda > 2$ or $1 < \lambda \leqq 2$.

Proof of Theorem 1.3 (viii) *Except that* $f(\gamma_0 -) = \infty$ *and* $f(\gamma_0 +) = \infty$. Use F_+ and F_- of (4.2) and (4.4). Since $\check{F}_+$ and F_- are of type I_1, Theorem 3.1 (i) applies.

Proof of Theorem 1.3 (ix). By translation, we may assume $\gamma_0 = 0$. As above, f is absolutely continuous on $(-\infty, 0) \cup (0, \infty)$ and f^* is positive a.e. on $(-\infty, 0)$ and negative a.e. on $(0, \infty)$. The proof of Theorem 3.1 shows that

$$f^*(x) = \int_x^\infty f_+^*(z) f_-(x-z)\, dz \qquad \text{a.e. on } (0, \infty).$$

But, by Lemmas 2.5 and 2.6, f is continuous on $(-\infty, \infty)$ and C^1 on $(-\infty, 0) \cup (0, \infty)$. Of course, f' is a version of f^*. Given $x > 0$, choose $x_n \downarrow x$ such that

$$f'(x_n) = \int_{x_n}^\infty f_+^*(z) f_-(x_n - z)\, dz.$$

Using Fatou's lemma, we get

$$f'(x) \leqq \int_0^\infty f_+^*(z) f_-(x-z)\, dz,$$

and hence $f'(x) < 0$. Similarly, $f'(x) > 0$ for $x < 0$.

Proof of Theorem 1.3 (x) *and Theorem 4.1* (ii) *Except that* $f'(\gamma_0 -) = \infty$ *and* $f'(\gamma_0 +) = \infty$. This time, $\check{F}_- \in \mathrm{I}_1$ and $F_+ \in \mathrm{I}_5$. Application of Theorem 3.1 gives the assertion.

Proof of the Assertions on $\mathrm{III}_5, \mathrm{III}_6, \mathrm{III}_7$ *in Theorems 1.3 and 4.1.* f is C^1 on $(-\infty, \infty)$ by Lemma 2.5. Using Theorem 3.1 for F_+ and F_-, we get the assertions for III_5 and III_6. If $F \in \mathrm{III}_7$ and (4.1) holds, then the assertions follow from the decomposition $F = F_- * F_2$. If $F \in \mathrm{III}_7$ and (4.3) holds, use the decomposition $F = F_1 * F_+$. If $F \in \mathrm{III}_7$ and if neither (4.1) nor (4.3) holds, Theorem 3.1 also yields the conclusion in Theorem 1.3 (xi).

Proof of the Assertions on II *in Theorems 1.3 and 4.1.* The conclusion in Theorem 1.3 (xi) and the assertion of Theorem 4.1 (i) are obtained from Theorem 3.1, since F is convolution of a type I distribution and a Gaussian. To see that $f(x)$ is log-concave on $(-\infty, a]$, let

$$k_n(u)=k(u)+n\sigma^2 u^{-1}(1+u^2)\chi_{(0,n^{-1})}(u),$$

where χ is the indicator function, and define $\phi_n(t)$ by (4.20). It is easy to check that ϕ_n is the characteristic function of an L distribution F_n of type I_7 and that F_n weakly converges to F as $n\to\infty$. By Lemma 4.4, a_n tends to a. Since f_n is log-concave on $(-\infty, a_n]$, f is log-concave on $(-\infty, a)$ by Lemma 4.5. By continuity, it is log-concave on $(-\infty, a]$.

Proof of the Assertion on IV *in Theorems 1.3 and 4.1.* We can proceed like the proof of the assertion on type III_7. But, note that we are now using the result on type II in Theorem 1.3 (xii).

Some parts of Theorem 1.3 still remain unproved. But they are automatically proved when we establish Theorems 1.6 and 1.7 in Section 5.

Let us add one result here. We say that f is strictly log-concave if f is positive, C^1, and $(\log f)'$ is decreasing.

Theorem 4.2. *If* $F\in\mathrm{I}_5\cup\mathrm{I}_6$, *then* f *is strictly* log-*concave on* $(\gamma_0, a]$. *If* $F\in\mathrm{I}_7\cup\mathrm{II}$, *then* f *is strictly* log-*concave on* $(-\infty, a]$.

Proof. If $1<\lambda\leqq 2$ and $\sigma^2=0$, then Step 3 of the proof of Theorem 1.3 (vii) actually proves strict unimodality of f. Suppose that $\lambda>2$ or $\sigma^2>0$. Let $\gamma_1=\gamma_0$ if $F\in\mathrm{I}_5\cup\mathrm{I}_6$, and $\gamma_1=-\infty$ if $F\in\mathrm{I}_7\cup\mathrm{II}$. We already know that $(\log f)'$ is non-increasing on $(\gamma_1, a]$. Suppose that $(\log f)'$ is flat on some $[c,d]\subset(\gamma_1, a]$. Then $f(x)=M e^{\alpha x}$ on $[c,d]$ with some constants M, α. If $\lambda>3$ or $\sigma^2>0$, then f is C^3 on (γ_1,∞) and we can differentiate (2.33) under the integral sign.Thus

$$(x-\gamma)f''(x)=-2f'(x)$$
$$+\int_0^\infty (f'(x-u)-f'(x)+f''(x)\arctan u)\,dk(u)-\sigma^2 f'''(x) \tag{4.21}$$

for $x\in(\gamma_1,\infty)$. If $2<\lambda\leqq 3$ and $\sigma^2=0$, then (2.34) can be differentiated under the integral sign. Hence we get (4.21) also in this case. Multiply (2.33) by $f'(x)$ and (4.21) by $f(x)$. Consider the difference. Then we get

$$0=f'(x)f(x)+\int_0^\infty (f(x-u)f'(x)-f'(x-u)f(x))\,dk(u)$$

for $x\in[c,d]$, noting that $f(x)=M e^{\alpha x}$. This is absurd, since the right-hand side must be positive.

5. Asymptotic Behavior of the Density Function

We will prove Theorem 1.6 appealing to a Tauberian theorem for Laplace transforms. Theorem 1.7 will be proved directly from the inversion formulas of Lemmas 2.5 and 2.6. First, we give a theorem on the derivatives of f for $F\in\mathrm{I}_5\cup\mathrm{I}_6$. We will use this theorem in the proof of Theorem 1.6, but the theorem is interesting by itself.

Theorem 5.1. (i) *Let $F \in \mathrm{I}_5$. Then, there are points*

$$\gamma_0 < a_N < a_{N-1} < \cdots < a_1 < \infty = a_0$$

such that, for $n = 1, \ldots, N$,

$$f^{(n)} \text{ is positive on } (\gamma_0, a_n), \text{ zero at } a_n, \text{ and negative on } (a_n, a_{n-1}). \tag{5.1}$$

On (γ_0, a_N), $f^{(N)}$ is absolutely continuous and $(f^{(N)})^* \leqq 0$ a.e. If $\lambda \neq N+1$, then $(f^{(N)})^* < 0$ a.e. on (γ_0, a_N). Furthermore, for $n = 1, \ldots, N$,

$$\gamma_0 + \beta_n < a_n, \tag{5.2}$$

where

$$\beta_n = \sup\{u\colon k(u) \geqq n\}. \tag{5.3}$$

(ii) *Let $F \in \mathrm{I}_6$. Then, there are points*

$$\gamma_0 < \cdots < a_n < a_{n-1} < \cdots < a_1 < \infty = a_0$$

such that, for each $n \geqq 1$, (5.1) holds. (5.2) is also satisfied for each $n \geqq 1$.

Proof. Let $F \in \mathrm{I}_5$. First, notice that f is C^{N-1} on $(-\infty, \infty)$ and C^N on (γ_0, ∞) (Lemmas 2.5 and 2.6). Let a_1 be the mode of F. For $n = 1$, (5.1) and (5.2) are already proved in Theorem 1.3 and Lemma 4.3. Suppose that $p < N$ and that there are points $\gamma_0 < a_p < \cdots < a_1 < \infty = a_0$ such that, for $n = 1, \ldots, p$, (5.1) and (5.2) are true. We claim that we can find $a_{p+1} \in (\gamma_0, a_p)$ such that (5.1) and (5.2) hold for $n = p+1$.

Step 1. $f^{(p+1)} > 0$ on $(\gamma_0, \gamma_0 + \beta_{p+1}]$.

Step 2. There exists no pair of points x_1, x_2 such that $\gamma_0 < x_1 < x_2 < a_p$, $f^{(p)}$ is increasing on (γ_0, x_1), non-increasing on (x_1, x_2), and $f^{(p+1)}(x_1) = f^{(p+1)}(x_2) = 0$.

Step 3. Since $f^{(p)}(\gamma_0) = f^{(p)}(a_p) = 0$, there is at least one point $x \in (\gamma_0, a_p)$ such that $f^{(p+1)}(x) = 0$. Let a_{p+1} be the infimum of such x. Then, $f^{(p+1)} < 0$ on (a_{p+1}, a_p).

The above three steps are proved exactly in the same way as the proof of Lemma 4.3 and Steps 1 and 2 of the proof of Theorem 1.3 (vii). We have only to notice

$$(x - \gamma_0) f^{(p+1)}(x) = -(p+1) f^{(p)}(x) + \int_0^\infty (f^{(p)}(x-u) - f^{(p)}(x))\, dk(u) \tag{5.4}$$

for $x > \gamma_0$ and to imitate the previous proof with trivial modification. In this way we obtain $\gamma_0 < a_N < \cdots < a_1 < \infty = a_0$ such that (5.1) and (5.2) hold for $n = 1, \ldots, N$. In order to prove the remaining assertion on $f^{(N)}$, we proceed like the proof of the last sentence of Corollary 2.1 and Step 3 of the proof of Theorem 1.3 (vii). On (γ_0, a_N), $f^{(N)}$ is absolutely continuous and

$$(x - \gamma_0)(f^{(N)})^*(x) = (\lambda - N - 1) f^{(N)}(x) + \int_{(0, x-\gamma_0)} f^{(N)}(x-u)\, dk(u) \quad \text{a.e.,}$$

which is nonpositive (negative if $\lambda \neq N+1$). This finishes the proof of (i). The proof of (ii) is given in the same manner.

Proof of Theorem 1.6. Let $F \in \bigcup_{j=1}^{5} \mathrm{I}_j$. By translation, we may assume $\gamma_0 = 0$. Laplace transform of F is

$$\psi(t) = \int_0^\infty e^{-tx}\, dF(x) = \exp \int_0^\infty (e^{-tu} - 1)\, u^{-1} k(u)\, du. \tag{5.5}$$

We claim that

$$\psi(t) \sim c\, t^{-\lambda} K(t^{-1}) \quad \text{as } t \to \infty \tag{5.6}$$

with c of (1.8). In fact,

$$\int_0^{1/t} (e^{-tu} - 1)\, u^{-1} k(u)\, du = \int_0^1 (e^{-u} - 1)\, u^{-1} k(t^{-1} u)\, du \to \lambda \int_0^1 (e^{-u} - 1)\, u^{-1}\, du,$$

$$\int_1^\infty (e^{-tu} - 1)\, u^{-1} k(u)\, du \to -\int_1^\infty u^{-1} k(u)\, du,$$

and

$$\int_{1/t}^1 (e^{-tu} - 1)\, u^{-1} k(u)\, du + \lambda \log t - \log K(t^{-1})$$
$$= \int_{1/t}^1 e^{-tu} u^{-1} k(u)\, du = \int_1^t e^{-u} u^{-1} k(t^{-1} u)\, du \to \lambda \int_1^\infty e^{-u} u^{-1}\, du,$$

proving (5.6). Because $K(x)$ is slowly varying as $x \downarrow 0$, we obtain from (5.6) that

$$F(x) \sim c\, \Gamma(\lambda+1)^{-1} x^\lambda K(x) \quad \text{as } x \downarrow 0 \tag{5.7}$$

by using Karamata's Tauberian theorem (Feller [2], p. 445). Since $f(x)$ is monotone in a right neighborhood of 0, (5.7) leads to

$$f(x) \sim c\, \Gamma(\lambda)^{-1} x^{\lambda-1} K(x) \quad \text{as } x \downarrow 0 \tag{5.8}$$

by the dual version of a theorem of Feller [2], p. 446. f' is monotone in a right neighborhood of 0 by Theorem 5.1, and hence, (5.8) leads to

$$f'(x) \sim c\, \Gamma(\lambda-1)^{-1} x^{\lambda-2} K(x) \quad \text{as } x \downarrow 0.$$

Theorem 5.1 allows us to repeat this procedure to obtain (1.11) for $n = 1, \ldots, N$.

In order to prove Theorem 1.7, we examine behavior of $\phi(x^{-1} s)$ as $x \downarrow 0$.

Lemma 5.1. *Let $s \neq 0$. If $\lambda < \infty$ and $\gamma_0 = \sigma^2 = 0$, then*

$$\phi(x^{-1} s)\, x^{-\lambda} K(x)^{-1} \to c\, |s|^{-\lambda} \exp((\operatorname{sgn} s)\, 2^{-1} \pi \mu i) \tag{5.9}$$

as $x \downarrow 0$, where c is (1.8) *and*

$$\mu = k(0+) + k(0-) = \lambda_+ - \lambda_-. \tag{5.10}$$

Proof. Let $l(u)$ be (2.18) and $m(u)=k(u)+k((-u)-)$. We have

$$\phi(x^{-1}s)=\exp\int_{-\infty}^{\infty}(e^{isu/x}-1)\,u^{-1}k(u)\,du$$
$$=\exp\left\{\int_0^{\infty}(\cos x^{-1}su-1)\,u^{-1}l(u)\,du+i\int_0^{\infty}(\sin x^{-1}su)\,u^{-1}m(u)\,du\right\}$$

for $x\neq 0$. Let $s>0$. As $x\downarrow 0$,

$$\int_1^{\infty}(\cos x^{-1}su)\,u^{-1}l(u)\,du\to 0$$

and

$$\int_1^{\infty}(\sin x^{-1}su)\,u^{-1}m(u)\,du\to 0$$

by the Riemann-Lebesgue and

$$\int_0^{x/s}(\cos x^{-1}su-1)\,u^{-1}l(u)\,du\to\lambda\int_0^1(\cos u-1)\,u^{-1}du,$$
$$\int_0^{x/s}(\sin x^{-1}su)\,u^{-1}m(u)\,du\to\mu\int_0^1(\sin u)\,u^{-1}du.$$

Moreover,

$$\int_{x/s}^{1}(\cos x^{-1}su-1)\,u^{-1}l(u)\,du-\log K(x)+\lambda\log x^{-1}s$$
$$=\int_{x/s}^{1}(\cos x^{-1}su)\,u^{-1}l(u)\,du+\int_{x/s}^{x}(\lambda-l(u))\,u^{-1}du$$
$$=l(xs^{-1})\int_1^{s/x}(\cos v)\,v^{-1}dv+\int_{x/s}^{1}dl(u)\int_{su/x}^{s/x}(\cos v)\,v^{-1}dv$$
$$+\int_{1/s}^{1}(\lambda-l(xu))\,u^{-1}du\to\lambda\lim_{A\to\infty}\int_1^{A}(\cos u)\,u^{-1}du$$

and, similarly,

$$\int_{x/s}^{1}(\sin x^{-1}su)\,u^{-1}m(u)\,du\to\mu\lim_{A\to\infty}\int_1^{A}(\sin u)\,u^{-1}du.$$

Noting that

$$\lim_{A\to\infty}\int_0^{A}(\sin u)\,u^{-1}du=2^{-1}\pi\quad\text{and}\quad\lim_{A\to\infty}\int_0^{A}(\cos u-e^{-u})\,u^{-1}du=0,$$

we obtain (5.9) for $s>0$. For $s<0$, it is enough to use $\phi(-t)=\overline{\phi(t)}$.

Proof of Theorem 1.7. Let $F \in \bigcup_{j=1}^{5} \mathrm{III}_j$. By translation, we may assume $\gamma_0 = 0$. There is a constant M_1 such that, for $s \neq 0$ and $0 < |x| \leqq 1$,

$$|\phi(x^{-1} s)| \, |x|^{-\lambda} K(x)^{-1} \leqq M_1 \, |s|^{-\lambda} K(|s|^{-1} \wedge 1). \tag{5.11}$$

To see this, we may assume $x > 0$ and $s > 0$. If $x^{-1} s \leqq 1$, then (5.11) is trivially true with M_1 replaced by 1. If $x^{-1} s > 1$, then

$$|\phi(x^{-1} s)| \, |x|^{-\lambda} \leqq M \, |s|^{-\lambda} K(x s^{-1})$$

by (2.16) of Lemma 2.4, and we have

$$K(x s^{-1})/K(x) = \exp \int_{1/s}^{1} (\lambda - l(x u)) \, u^{-1} \, du \leqq K(s^{-1} \wedge 1).$$

Hence (5.11). Consequently, for each $\alpha < \lambda$, there is a constant M_2 such that, for $|s| \geqq 1$ and $0 < |x| \leqq 1$,

$$|\phi(x^{-1} s)| \, |x|^{-\lambda} K(x)^{-1} \leqq M_2 \, |s|^{-\alpha}. \tag{5.12}$$

Now, by Lemma 2.3,

$$\left| \frac{d}{dx} \left(\frac{1}{x^n} \phi \left(\frac{s}{x} \right) \right) \right| \leqq \frac{n + 2\lambda}{|x|^{n+1}} \left| \phi \left(\frac{s}{x} \right) \right| \tag{5.13}$$

for $s \neq 0$, $x \neq 0$, $n = 0, 1, \ldots$. We have, for $x \neq 0$,

$$f(x) = \frac{\operatorname{sgn} x}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-is} - 1}{-is} \frac{d}{dx} \left(\phi \left(\frac{s}{x} \right) \right) ds, \tag{5.14}$$

$$f^{(n)}(x) = \frac{\operatorname{sgn} x}{2\pi} \int_{-\infty}^{\infty} e^{-is} (-is)^{n-1} \frac{d}{dx} \left(\frac{1}{x^n} \phi \left(\frac{s}{x} \right) \right) ds \tag{5.15}$$

for $n = 1, \ldots, N$. In fact, (5.14) is shown in Lemma 2.6; (5.15) for $n = 1$ is seen from

$$\begin{aligned} f'(x) &= \frac{h'(x) - f(x)}{x} \\ &= \frac{\operatorname{sgn} x}{2\pi} \left(\int_{-\infty}^{\infty} \frac{e^{-is}}{x} \frac{d}{dx} \left(\phi \left(\frac{s}{x} \right) \right) ds - \int_{-\infty}^{\infty} \frac{e^{-is}}{x^2} \phi \left(\frac{s}{x} \right) ds \right) \end{aligned}$$

in Lemmas 2.5 and 2.6; and (5.15) for $n = 2, \ldots, N$ is proved by (2.17), (2.20), and (5.13). We claim that, for any given $\varepsilon > 0$, we can find an A_0 such that, for all $0 < |x| \leqq 1$, $A > A_0$, and $n = 0, \ldots, N$,

$$\left| \frac{f^{(n)}(x)}{|x|^{\lambda - n - 1} K(x)} - \frac{(\operatorname{sgn} x)^n}{2\pi} \int_{-A}^{A} \frac{e^{-is} (-is)^n}{|x|^{\lambda} K(x)} \phi \left(\frac{s}{x} \right) ds \right| < \varepsilon. \tag{5.16}$$

First, notice that, by (5.12) and (5.13),

$$\frac{1}{2\pi|x|^{\lambda-1}K(x)}\left|\int_{|s|>A}\frac{e^{is}-1}{-is}\frac{d}{dx}\left(\phi\left(\frac{s}{x}\right)\right)ds\right|<\frac{\varepsilon}{2}$$

and

$$\frac{1}{2\pi|x|^{\lambda-n-1}K(x)}\left|\int_{|s|>A}e^{-is}(-is)^{n-1}\frac{d}{dx}\left(\frac{1}{x^n}\left(\frac{s}{x}\right)\right)ds\right|<\frac{\varepsilon}{2},$$

for $n=1,\ldots,N$ uniformly in $0<|x|\leqq 1$ if A is large enough. Next we deal with the integrals over $|s|\leqq A$ in (5.14) and (5.15). Note that

$$\frac{d}{dx}\left(\frac{(-is)^{n-1}}{x^n}\phi\left(\frac{s}{x}\right)\right)=\frac{d}{ds}\left(\frac{(-is)^n}{ix^{n+1}}\phi\left(\frac{s}{x}\right)\right) \tag{5.17}$$

for $s\neq 0$. Make integration by parts. We find, for $n=1,\ldots,N$,

$$\begin{aligned}&\frac{\operatorname{sgn}x}{2\pi|x|^{\lambda-n-1}K(x)}\int_{-A}^{A}e^{-is}(-is)^{n-1}\frac{d}{dx}\left(\frac{1}{x^n}\phi\left(\frac{s}{x}\right)\right)ds\\&=(\operatorname{sgn}x)^n\left(\left[\frac{e^{-is}(-is)^n}{2\pi i|x|^{\lambda}K(x)}\phi\left(\frac{s}{x}\right)\right]_{s=-A}^{A}+\int_{-A}^{A}\frac{e^{-is}(-is)^n}{2\pi|x|^{\lambda}K(x)}\phi\left(\frac{s}{x}\right)ds\right).\end{aligned} \tag{5.18}$$

By (5.12), the absolute value of the integrated term in (5.18) is smaller than $\varepsilon/2$ uniformly in $0<|x|\leqq 1$ for large A. Similar consideration can be made also for $n=0$ by (5.17). Thus we get (5.16).

Now, let us treat (i), (ii), and (iii) separately.

(i) Let $N<\lambda<N+1$ and let $n=N$. By Lemma 5.1, we can find the limit of

$$\frac{(\operatorname{sgn}x)^N}{2\pi}\int_{-A}^{A}\frac{e^{-is}(-is)^N}{|x|^{\lambda}K(x)}\phi\left(\frac{s}{x}\right)ds \tag{5.19}$$

as x tends to 0, because (5.11) guarantees applicability of the dominated convergence theorem. The limit as $x\downarrow 0$ is

$$\begin{aligned}&\frac{i^N c}{2\pi}\left((-1)^N\int_0^A s^{N-\lambda}\exp\left(-is+\frac{i\mu\pi}{2}\right)ds+\int_{-A}^{0}|s|^{N-\lambda}\exp\left(-is-\frac{i\mu\pi}{2}\right)ds\right)\\&=\frac{c}{\pi}\int_0^A s^{N-\lambda}\cos\left(s+\frac{N-\mu}{2}\pi\right)ds.\end{aligned}$$

If $x<0$, (5.19) equals

$$\frac{(-1)^N}{2\pi}\int_{-A}^{A}\frac{e^{is}(is)^N}{|x|^{\lambda}K(x)}\phi\left(\frac{s}{|x|}\right)ds$$

and its limit as $x\uparrow 0$ is, similarly,

$$\frac{(-1)^N c}{\pi}\int_0^A s^{N-\lambda}\cos\left(s+\frac{N+\mu}{2}\pi\right)ds.$$

Hence, noting (5.16) and

$$\lim_{B\to\infty}\int_0^B s^{N-\lambda}\cos\left(s+\frac{N\mp\mu}{2}\pi\right)ds=\Gamma(N+1-\lambda)\sin\frac{\lambda-2N\pm\mu}{2}\pi$$

$$=(-1)^N\Gamma(N+1-\lambda)\sin\lambda_\pm\pi=\pi\Gamma(\lambda-N)^{-1}(\sin\lambda\pi)^{-1}\sin\lambda_\pm\pi,$$

we obtain (1.12) and (1.13).

(ii) Let $\lambda=N+1$. We claim that

$$\lim_{x\downarrow 0}\frac{x}{K(x)}\frac{d}{dx}\left(\frac{f^{(N-1)}(x)-f^{(N-1)}(0)}{x}\right)=-\frac{c}{\pi}\cos\frac{\mu-N}{2}\pi \tag{5.20}$$

with the convention that $f^{(-1)}=F$. Let $x>0$. Let $\varepsilon>0$ be an arbitrary small number. We will show that

$$\frac{x}{K(x)}\frac{d}{dx}\left(\frac{f^{(N-1)}(x)-f^{(N-1)}(0)}{x}\right) \tag{5.21}$$

is within ε of

$$\frac{1}{2\pi K(x)}\int_{-A}^{A}\frac{e^{-is}-1}{s}\frac{d}{ds}\left(\frac{(-is)^{N+1}}{x^{N+1}}\phi\left(\frac{s}{x}\right)\right)ds \tag{5.22}$$

uniformly in $x\in(0,1)$ for large A. If $N\geqq 1$, then, by (2.20) and (5.15),

$$\frac{x}{K(x)}\frac{d}{dx}\left(\frac{f^{(N-1)}(x)-f^{(N-1)}(0)}{x}\right)$$

$$=\frac{1}{K(x)}\left(f^{(N)}(x)-\frac{f^{(N-1)}(x)-f^{(N-1)}(0)}{x}\right)=\frac{1}{2\pi K(x)}\int_{-\infty}^{\infty}\Phi(s,x)\,ds \tag{5.23}$$

where

$$\Phi(s,x)=e^{-is}\frac{d}{dx}\left(\frac{(-is)^{N-1}}{x^N}\phi\left(\frac{s}{x}\right)\right)-(e^{is}-1)\frac{(-is)^{N-1}}{x^{N+1}}\phi\left(\frac{s}{x}\right).$$

By (5.12) and (5.13),

$$\left|\frac{1}{2\pi K(x)}\int_{|s|>A}\Phi(s,x)\,ds\right|<\frac{\varepsilon}{2}$$

uniformly in $x\in(0,1)$ for large A. Rewrite $\Phi(s,x)$ by (5.17). Note that

$$\left|\frac{1}{2\pi K(x)}\int_{-A}^{A}\frac{d}{ds}\left(\frac{(-is)^N}{ix^{N+1}}\phi\left(\frac{s}{x}\right)\right)ds\right|<\frac{\varepsilon}{2}$$

uniformly in $x\in(0,1)$ for large A. Then we see that (5.21) is within ε of

$$\frac{1}{2\pi K(x)}\int_{-A}^{A}(e^{-is}-1)\left\{\frac{d}{ds}\left(\frac{(-is)^N}{ix^{N+1}}\phi\left(\frac{s}{x}\right)\right)-\frac{(-is)^{N-1}}{x^{N+1}}\phi\left(\frac{s}{x}\right)\right\}ds,$$

which equals (5.22). In case $N=0$, we have, by (2.23) and (5.14),

$$\frac{x}{K(x)}\frac{d}{dx}\left(\frac{F(x)-F(0)}{x}\right)=\frac{1}{K(x)}\left(f(x)-\frac{F(x)-F(0)}{x}\right)$$
$$=\frac{1}{2\pi K(x)}\int_{-\infty}^{\infty}\frac{e^{-is}-1}{-is}\left\{\frac{d}{dx}\left(\phi\left(\frac{s}{x}\right)\right)-\frac{1}{x}\phi\left(\frac{s}{x}\right)\right\}ds$$
$$=\frac{1}{2\pi K(x)}\int_{-\infty}^{\infty}\frac{e^{-is}-1}{s}\frac{d}{ds}\left(\frac{-is}{x}\phi\left(\frac{s}{x}\right)\right)ds,$$

which is within ε of (5.22). Now, let us prove (5.20). By integration by parts and again by (5.12), we see that (5.22) is within ε of

$$-\frac{1}{2\pi}\int_{-A}^{A}\left(\frac{e^{-is}-1}{s}\right)'\frac{(-is)^{N+1}}{x^{N+1}K(x)}\phi\left(\frac{s}{x}\right)ds \tag{5.24}$$

uniformly in $x\in(0,1)$ for large A. Let $x\downarrow 0$ and use Lemma 5.1 and (5.11). The limit of (5.24) is

$$-\frac{c}{2\pi}\left(e^{i(\mu-N-1)\pi/2}\int_{0}^{A}\left(\frac{e^{-is}-1}{s}\right)'ds+e^{i(N+1-\mu)\pi/2}\int_{-A}^{0}\left(\frac{e^{-is}-1}{s}\right)'ds\right).$$

This is within ε of

$$-(2\pi)^{-1}c(i\,e^{i(\mu-N-1)\pi/2}-i\,e^{i(N+1-\mu)\pi/2})$$
$$=-\pi^{-1}c\cos((\mu-N)\pi/2)$$

for large A. Hence we obtain (5.20). Next, we see that

$$\lim_{x\downarrow 0}\frac{f^{(N-1)}(x)-f^{(N-1)}(0)}{xL(x)}=\frac{c}{\pi}\cos\frac{\mu-N}{2}\pi \tag{5.25}$$

by an elementary argument based on

$$\frac{f^{(N-1)}(x)-f^{(N-1)}(0)}{x}=f^{(N-1)}(1)-f^{(N-1)}(0)-\int_{x}^{1}\left(\frac{f^{(N-1)}(y)-f^{(N-1)}(0)}{y}\right)'dy$$

and $\lim_{x\downarrow 0}L(x)=\infty$. Since $\lim_{x\downarrow 0}K(x)/L(x)=0$, the first equality in (5.23) combined with (5.20) and (5.25) proves that $f^{(N)}(x)/L(x)\to\pi^{-1}c\cos((\mu-N)\pi/2)$ as $x\downarrow 0$. Note that we did not use the assumption $\lambda_-\leqq\lambda_+$. Therefore, in order to find the limit as $x\uparrow 0$, it suffices to consider $\check{F}$ and to apply the above result. This completes the proof of (ii).

(iii) Suppose that $\lambda=N+1$. Let $x>0$. By (5.16),

$$K(x)^{-1}(f^{(N)}(x)-f^{(N)}(-x)) \tag{5.26}$$

is within 2ε of

$$\frac{1}{2\pi}\int_{-A}^{A}\frac{e^{-is}(-is)^N}{x^{N+1}K(x)}\left\{\phi\left(\frac{s}{x}\right)-(-1)^N\overline{\phi\left(\frac{s}{x}\right)}\right\}ds \tag{5.27}$$

uniformly in $x\in(0,1)$ for large A. Since (5.26) is real, it is within 2ε of

$$\frac{1}{\pi}\int_{-A}^{A}\frac{s^N\sin s}{x^{N+1}K(x)}\operatorname{Im}\left((-i)^N\phi\left(\frac{s}{x}\right)\right)ds, \tag{5.28}$$

the real part of (5.27). Now use Lemma 5.1. The dominated convergence theorem applies. Thus the limit of (5.28) as $x\downarrow 0$ is

$$\frac{c}{\pi}\int_0^A\frac{\sin s}{s}ds\left(\sin\frac{\mu-N}{2}\pi+(-1)^N\sin\frac{\mu+N}{2}\pi\right).$$

Hence,

$$\lim_{x\downarrow 0}\frac{f^{(N)}(x)-f^{(N)}(-x)}{K(x)}=\frac{c}{2}\left(\sin\frac{\mu-N}{2}\pi+(-1)^N\sin\frac{\mu+N}{2}\pi\right)$$
$$=\frac{c}{2}(\cos\lambda_-\pi+(-1)^{N+1}\cos\lambda_+\pi). \tag{5.29}$$

The proof of Theorem 1.7 is complete.

If F is a one-sided stable distribution with exponent $0<\alpha<1$ and $\gamma_0=0$, then

$$k(u)=ru^{-\alpha}\quad(\text{for } u>0),\qquad 0\quad(\text{for } u<0), \tag{5.30}$$

with $r=\text{const}>0$ and, from the asymptotic expansion of $f(x)$ in Theorem 2.4.6 of [5], we have

$$f(x)\sim c_2x^{-(2-\alpha)/(2-2\alpha)}\exp(-c_1x^{-\alpha/(1-\alpha)})\qquad\text{as } x\downarrow 0 \tag{5.31}$$

with

$$c_1=(1-\alpha)\alpha^{-1}(r\Gamma(1-\alpha))^{1/(1-\alpha)}, \tag{5.32}$$
$$c_2=(2\pi)^{-1/2}(1-\alpha)^{-1/2}(r\Gamma(1-\alpha))^{1/(2-2\alpha)}. \tag{5.33}$$

Let us consider the case where k is close to (5.30).

Theorem 5.2. *If $F\in\mathrm{I}_6$ and if*

$$k(u)\sim ru^{-\alpha}\qquad\text{as } u\downarrow 0 \tag{5.34}$$

for some $0<\alpha<1$ and $r>0$, then

$$\log f(x)\sim -c_1(x-\gamma_0)^{-\alpha/(1-\alpha)}\qquad\text{as } x\downarrow\gamma_0 \tag{5.35}$$

with c_1 of (5.32).

Lemma 5.2. *If $F\in\mathrm{I}_5\cup\mathrm{I}_6$, then $F(x)$ is* log-*concave on* (γ_0,∞).

Proof. Analogous to Steps 4 and 5 of the proof of Theorem 1.3 (vii). Namely, assume (4.16) and let $S(x)=(\log F)''$. We have $S<0$ on $(\gamma_0,\gamma_0+\delta]$ and

$$(x-\gamma_0)(f'(x)F(x)-f(x)^2)$$
$$=-f(x)F(x)+\int_{(0,x-\gamma_0)}(f(x-u)F(x)-F(x-u)f(x))\,dk(u)$$

for $x>\gamma_0$. It follows that $S<0$ on (γ_0,∞). If (4.16) is not satisfied, then define ϕ_n by (4.19). F_n is log-concave on (γ_0,∞). As $n\to\infty$, $F_n(x)\to F(x)$ for all x. Hence F is log-concave on (γ_0,∞).

Proof of Theorem 5.2. Let $\gamma_0=0$ and consider the Laplace transform (5.5). We have

$$\frac{\log\psi(t)}{t^\alpha}=\int_0^t\frac{e^{-u}-1}{u^{1+\alpha}}\frac{u^\alpha}{t^\alpha}k\left(\frac{u}{t}\right)du+\frac{1}{t^\alpha}\int_1^\infty\frac{e^{-tu}-1}{u}k(u)\,du$$

$$\to r\int_0^\infty(e^{-u}-1)u^{-1-\alpha}du=-\alpha^{-1}\Gamma(1-\alpha)r\qquad\text{as } t\to\infty.$$

Applying a Tauberian theorem of exponential type of Minlos-Povzner [9] and Fukushima [3], we get

$$\log F(x)\sim -c_1x^{-\alpha/(1-\alpha)}\qquad\text{as } x\downarrow 0. \tag{5.36}$$

Since $(\log F)'$ is monotone on $(0,\infty)$ by Lemma 5.2, we can aply the method of proof of a theorem of Feller [2], p. 446 and obtain

$$f(x)/F(x)\sim c_1\alpha(1-\alpha)^{-1}x^{-1-\alpha/(1-\alpha)}\qquad\text{as } x\downarrow 0. \tag{5.37}$$

Now (5.35) follows from (5.36) and (5.37).

6. Location of the Mode

Let $F\in \mathrm{I}_5\cup\mathrm{I}_6\cup\mathrm{I}_7\cup\mathrm{II}$. From a general theory of infinitely divisible distributions (see Kruglov [7]), it is known that mean m of F exists and $-\infty<m\leqq\infty$, that $m\neq\infty$ if and only if

$$\int_1^\infty k(u)\,du<\infty, \tag{6.1}$$

and that, if (6.1) holds, then

$$m=i^{-1}\phi'(0)=\gamma+\int_0^\infty u^2(1+u^2)^{-1}k(u)\,du. \tag{6.2}$$

Concerning the location of the mode a of F, Wolfe [14] proves that $a\leqq m$ and that, if $m<\infty$, then $a\geqq m-d$ where

$$d=\sup\{u\colon k(u)>0\}. \tag{6.3}$$

He quotes also a result of Johnson-Rogers [6]. We will give strict sense versions of their results and add some other results. Another result in case of type II exists in Theorem 4.1 (i). Information on location of modes in case of types III and IV is obtained, if we combine our results with Theorems 1.3 and 4.1.

Theorem 6.1. *Let* $F\in\mathrm{I}_5\cup\mathrm{I}_6\cup\mathrm{I}_7\cup\mathrm{II}$. *Then the following hold.*

(i) $a<m$.

(ii) *If* $F\in I_5\cup I_6$, *then*

$$(a-\gamma_0)^{-1}\int_0^{a-\gamma_0} k(u)\,du>1. \tag{6.4}$$

This gives an upper bound of a.

(iii) *If* $m<\infty$, *then* $a>m-d$, *where* d *is* (6.3).

(iv) *For every* $\xi>0$,

$$a>\gamma+\int_0^{\xi} u^2(1+u^2)^{-1}k(u)\,du-\int_{\xi}^{\infty}(1+u^2)^{-1}k(u)\,du-\xi k(\xi)-\xi. \tag{6.5}$$

(v) *If* $m<\infty$, *then* $a>m-(3v)^{1/2}$, *where* v *is variance of* F.

(vi) *If* $F\in I_5\cup I_6$, *then* $a>\gamma_0+\beta$, *where* β *is* (4.8).

We need a simple lemma.

Lemma 6.1. *Let* G_1 *be an absolutely continuous distribution function such that, on an interval* $(-\infty, c_1)$, *its density* g_1 *is continuous and non-decreasing. Let* G_2 *be a distribution function supported on* $[b_2,\infty)$, *and let* $G=G_1*G_2$. *Then* G *is absolutely continuous and, on* $(-\infty, c_1+b_2)$, *the density* g *of* G *is continuous and non-decreasing.*

Proof. Notice that $g(x)=\int_{[b_2,\infty)} g_1(x-y)\,dG_2(y)$. It is easy to check the assertion.

Proof of Theorem 6.1. (i) Using the Equation (2.1) at $x=a$, we see that

$$(a-\gamma)f(a)=\int_0^{\infty}(f(a-u)-(1+u^2)^{-1}f(a))\,k(u)\,du. \tag{6.6}$$

Since $f(a-u)<f(a)$ by Theorem 1.3, this and (6.2) prove $a<m$.

(ii) If $F\in I_5\cup I_6$, then $f(x)=0$ on $(-\infty,\gamma_0)$. Hence (6.6) gives

$$(a-\gamma_0)f(a)=\int_0^{a-\gamma_0} f(a-u)\,k(u)\,du.$$

Noting $f(a-u)<f(a)$ again, we get (6.4).

(iii) Suppose that $a\leqq m-d$. By (2.1) and $f'(m)\leqq 0$ we have

$$(m-\gamma)f(m)\geqq\int_0^{d}(f(m-u)-(1+u^2)^{-1}f(m))\,k(u)\,du.$$

But, since $f(m-u)>f(m)$ for $0<u<d$ by Theorem 1.3, the right-hand side is bigger than $(m-\gamma)f(m)$, a contradiction.

(iv) Given $\xi>0$, let

$$\phi_2(t)=\exp\int_0^{\infty}(e^{itu}-1)\,u^{-1}k_2(u)\,du,\qquad k_2(u)=k(u\vee\xi),$$

$$\phi_1(t)=\phi(t)/\phi_2(t).$$

Let F_1, F_2 be the distributions corresponding to $\phi_1(t)$, $\phi_2(t)$, respectively. Then, both F_1 and F_2 are L distributions, and $F=F_1 * F_2$. Let a_1 and m_1 be the mode and the mean of F_1 (let $a_1=\gamma_0$ if $F_1 \in \mathrm{I}_4$). We claim

$$a>m_1-\xi. \tag{6.7}$$

If $F_1 \in \bigcup_{j=5}^{7} \mathrm{I}_j \cup \mathrm{II}$, then (iii) and Lemma 6.1 prove (6.7). If $F_1 \in \bigcup_{j=1}^{4} \mathrm{I}_j$, then use $a>\gamma_0$ and note that

$$m_1=\gamma_0+\int_0^{\xi}(k(u)-k(\xi))\,du \leqq \gamma_0+\xi$$

by (6.2). (6.7) is thus established. Using (6.2) again, we see that $m_1-\xi$ equals the right-hand side of (6.5).

(v) A theorem of Johnson and Rogers [6] shows to us that $|m-a| \leqq (3v)^{1/2}$ for any unimodal distribution. It is easily seen from their proof that the equality holds only if the distribution is degenerate or uniform on an interval. Hence, in our case, $|m-a|<(3v)^{1/2}$.

(vi) is proved in Lemma 4.2.

Example 1. Let F be a one-sided stable distribution with exponent $0<\alpha<1$ and $\gamma_0=0$. We have (5.30) and $F \in \mathrm{I}_6$. Theorem 6.1 (ii) says that $r(1-\alpha)^{-1} a^{-\alpha}>1$, and hence $a<(r/(1-\alpha))^{1/\alpha}$. On the other hand, we have $a>r^{1/\alpha}$ from (vi).

Example 2. Let F be an extremely asymmetric stable distribution with exponent $1 \leqq \alpha<2$ and $\gamma=0$. That is,

$$\phi(t)=\exp\left\{r \int_0^{\infty}(e^{itu}-1-itu(1+u^2)^{-1})\,u^{-1-\alpha}\,du\right\}$$

with $r=\text{const}>0$. Then $F \in \mathrm{I}_7$. Let $\rho(\xi)$ be the right-hand side of (6.5). By elementary calculus, we see that $\rho(\xi)$ is maximum when $\xi=(r\alpha)^{1/\alpha}$. Hence, $a>\rho((r\alpha)^{1/\alpha})$ is the best estimate from below that we have. If $\alpha=r=1$, then the two integrals in the expression of $\rho((r\alpha)^{1/\alpha})$ cancel and we obtain $a>-2$. If $\alpha \neq 1$, then it follows from (i) and (6.2) that $a<m=-2^{-1}\pi r \sec 2^{-1}\pi\alpha$.

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