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Convergence Rates of the Strong Law for Stationary Mixing Sequences*

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Summary. In this note we improve a theorem of Lai on the convergence rate in the Marcinkiewicz-Zygmund strong law for stationary mixing sequences.

1. Introduction

Let $X_1, X_2, \ldots$ be a sequence of random variables which is strongly stationary, and for $n \in \mathbb{N}$ let $S_n = X_1 + \cdots + X_n$. The convergence rate in the strong law of large numbers can be expressed by

$$(1.1) \quad \sum_1^\infty n^{p\alpha-2} P\{\max_{j \leq n} |S_j| > \varepsilon n^\alpha\} < \infty \qquad \text{for all } \varepsilon > 0.$$

For independent random variables Baum and Katz ((1965), p. 112, Theorem 3) proved that if $\alpha > 1/2$, $p\alpha > 1$, and $EX_1 = 0$ in case $\alpha \leq 1$, then (1.1) is equivalent to

$$(1.2) \quad E|X_1|^p < \infty.$$

Lai ((1977), p. 695, Theorem 1) proved this equivalence for ϕ-mixing and strong mixing sequences of random variables. He needs an additional assumption on bivariate tail probabilities: There exists $\beta > 1$ and a positive integer m such that as $x \to \infty$

$$(1.3) \quad \sup_{i > m} P\{|X_1| > x, |X_i| > x\} = O(P^\beta\{|X_1| > x\}).$$

To prove the equivalence of (1.1) and (1.2) Lai uses an approach completely different from the classical Erdös-Katz approach and, in the ϕ-mixing case, ends up with conditions on ϕ involving β. It is the purpose of this note to show that – in the ϕ-mixing case – the equivalence of (1.1) and (1.2) can be proved using the classical Erdös-Katz methods, leading to weaker conditions on ϕ not involving β. The same methods also work in the case of strong mixing random variables. However, our conditions on the mixing coefficients are weaker than those in Lai (1977) only if p is close to 2. Unfortunately, a lemma of Dvoretzky ((1972), p. 528, Lemma 5.4) stating that strong mixing implies ϕ-mixing, is wrong (see

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Ibragimov and Linnik ((1971), Chapter 17, § 3). The results and proofs can be found in Section 2. Auxiliary lemmas, which might be of independent interest, are defered to Section 3.

2. The Results

Let $X_1, X_2, \ldots$ be a sequence of random variables which is strongly stationary. For $n \in \mathbb{N}$ let $\mathscr{F}_n$ be the σ-field generated by $X_1, \ldots, X_n$ and $\mathscr{F}^n$ the σ-field generated by $X_n, X_{n+1}, \ldots$. Let $\phi\colon \mathbb{N} \to [0,1]$ be nonincreasing and call the sequence $X_1, X_2, \ldots$ ϕ-mixing if:

(2.1) For all $i,j \in \mathbb{N}$, $B_1 \in \mathscr{F}_i$, $B_2 \in \mathscr{F}^{i+j}$

$$|P(B_1 \cap B_2) - P(B_1)\,P(B_2)| \leqq \phi(j)\,P(B_1)$$

(see [8], p. 351, (1.1)).

Let $\alpha\colon \mathbb{N} \to [0, 1/4]$ and call the sequence $X_1, X_2, \ldots$ strong mixing with mixing coefficients $\alpha(j)$ if:

(2.2) For all $i,j \in \mathbb{N}$, $B_1 \in \mathscr{F}_i$, $B_2 \in \mathscr{F}^{i+j}$

$$|P(B_1 \cap B_2) - P(B_1)\,P(B_2)| \leqq \alpha(j)$$

(see [13]).

(2.3) **Theorem.** (i) *Suppose $\alpha > 1/2$, $p > 1/\alpha$, that $EX_1 = 0$ if $\alpha \leqq 1$, and that the sequence $X_1, X_2, \ldots$ satisfies* (1.3) *and* (2.1) *where*

$$(2.4)\quad \sum_1^\infty \phi(n)^{1/\theta} < \infty \qquad \text{for some } \theta > 1.$$

Then $E|X_1|^p < \infty$ implies

$$(2.5)\quad \sum_1^\infty n^{p\alpha-2}\, P\{\max_{j \leqq n} |S_j| > \varepsilon n^\alpha\} < \infty \qquad \text{for all } \varepsilon > 0.$$

(ii) *Suppose $p > 2$, $\alpha > 1/2$, that $EX_1 = 0$ if $\alpha \leqq 1$, and that the sequence $X_1, X_2, \ldots$ satisfies* (1.3) *and* (2.2) *where*

$$(2.6)\quad \sum_1^\infty \alpha^{1/\theta}(n) < \infty \qquad \text{for some } \theta > \max\{\beta/(\beta-1)+2,\ [1+(\alpha p-1)/(\alpha-\gamma)]\, p\gamma/(p\gamma-1)\},$$

where $\gamma = \max(\gamma_1, \gamma_2)$,
$\gamma_1 = \alpha(1+(\beta-1)/\beta)$ *and* $\gamma_2 = (p\alpha+1)/(2p)$.

Then $E|X_1|^p < \infty$ implies (2.5).

(2.7) *Remark.* Lemma 5(ii) in [11] yields that (2.5) implies (1.2) in the case considered here. The relations (2.5) and (1.2) are therefore equivalent. For other equivalent formulations of (2.5) including r-quick convergence see [11]. Notice that our assumption (2.4) is strictly weaker than assumption (2.14) in [11]. Other restrictions on p could be dropped in the ϕ-mixing case. Condition (2.6) is weaker than the corresponding assumption in [11] for p close to 2.

As in [11], Theorem 2, we can prove (2.5) without assumption (1.3) if we know that moments exist with order slightly higher than necessary.

(2.8) **Theorem.** (i) *Suppose $\alpha>1/2$, $q>1/\alpha$, that $EX_1=0$ if $\alpha\leqq 1$, and that the sequence $X_1, X_2, \ldots$ satisfies* (2.1) *with*

$$(2.9)\quad \sum_1^\infty \phi(n)^{1/\theta}<\infty \qquad \text{for some } \theta>1.$$

Then $E|X_1|^q<\infty$ implies that (2.5) *holds for all $p\in(1/\alpha, q)$.*

(ii) *Suppose $\alpha>1/2$, $q>2$, $E|X_1|^q<\infty$, that $EX_1=0$ if $\alpha\leqq 1$, that $p\in(1/\alpha, q)$, and the sequence $X_1, X_2, \ldots$ satisfies* (2.2) *with*

$$(2.10)\quad \sum_1^\infty \alpha^{1/\theta}(\mathrm{n})<\infty \qquad \text{for some } 0>[2+q/(q-p)]\,\alpha p/(\alpha p-1).$$

Then (2.5) *holds.*

Proof of the Theorems. (i) For $i\in\{1, \ldots, m+1\}$ and $j\in\mathbb{N}$ let $S_{ji}=\sum X_{i+l(m+1)}$, where the summation extends over all $l\in\mathbb{N}\cup\{0\}$ with $l(m+1)\leqq j-i$. Since $|S_j|>\varepsilon n^\alpha$ implies $|S_{j,i}|>\varepsilon n^\alpha/(m+1)$ for at least one $i\in\{1, \ldots, m+1\}$ it suffices to prove

$$\sum_1^\infty n^{p\alpha-2} P\{\max_{j\leqq n}|S_{j,i}|>\varepsilon n^\alpha\}<\infty \qquad \text{for all } \varepsilon>0 \text{ and } i\in\{1, \ldots, m+1\}.$$

Fix $i\in\{1, \ldots, m+1\}$. For $l\in\mathbb{N}$ and $n\in\mathbb{N}$ with $l(m+1)<n\leqq(l+1)(m+1)$ we have

$$n^{p\alpha-2} P\{\max_{j\leqq n}|S_{j,i}|>\varepsilon n^\alpha\}$$
$$\leqq(m+1)^{p\alpha-2}\max\{l^{p\alpha-2}, (l+1)^{p\alpha-2}\}\, P\{\max_{j\leqq(l+1)(m+1)}|S_{j,i}|>\varepsilon l^\alpha(m+1)^\alpha\}.$$

Since there are at most $m+1$ of these n it suffices to show that for all $\varepsilon>0$

$$(2.11)\quad \sum_1^\infty l^{p\alpha-2} P\{\max_{j\leqq l(m+1)}|S_{j,i}|>\varepsilon l^\alpha\}<\infty.$$

For all $l\in\mathbb{N}$ let $Y_l=X_{i+(l-1)(m+1)}$.

Then

$$\max_{j\leqq l(m+1)}|S_{j,i}|\leqq\max_{j\leqq l}\left|\sum_1^j Y_k\right|.$$

Hence it suffices to prove (2.5) for the sequence $Y_1, Y_2, \ldots$ which is strictly stationary, ϕ-mixing with the same function ϕ and strong mixing with the same function α (because of the monotonicity of ϕ and α), and satisfies (1.3) for $m=0$. After this reduction of the problem we proceed as in [6] and [10]. Let A_n

$=\{|Y_l|>\varepsilon n^\alpha/2$ for at least one $l\in\{1,\dots,n\}\}$, let $\gamma\in(1/2,\alpha)$ which will be specified later, and let

$$B_n=\{\text{there exist } l_1,l_2\in\{1,\dots,n\},\ l_1\neq l_2, \text{ with } |Y_{l_1}|>n^\gamma \text{ and } |Y_{l_2}|>n^\gamma\}$$

and

$$C_n=\{\max_{j\leqq n}|S_j'|>\varepsilon n^\alpha/2\}$$

where $S_j'=\sum_1^{j}{}' Y_l$ and where the summation extends over all

$$l\in\{1,\dots,j\} \quad \text{with } |Y_l|\leqq n^\gamma.$$

Then

$$\left\{\max_{j\leqq n}\left|\sum_1^j Y_l\right|>\varepsilon n^\alpha\right\}\subset A_n\cup B_n\cup C_n$$

and therefore (2.11) follows from

$$(2.12)\quad \sum_1^\infty n^{p\alpha-2}P(A_n)<\infty,$$

$$(2.13)\quad \sum_1^\infty n^{p\alpha-2}P(B_n)<\infty$$

and

$$(2.14)\quad \sum_1^\infty n^{p\alpha-2}P(C_n)<\infty.$$

(ii) The proof of (2.12) is exactly the same as in the independent case. We have

$$n^{p\alpha-2}P(A_n)\leqq n^{p\alpha-1}P\{|Y_1|>\varepsilon n^\alpha/2\}\leqq(2/\varepsilon)^{(p\alpha-1)/\alpha}E|Y_1|^{p-1/\alpha}1_{\{|Y_1|>\varepsilon n^\alpha/2\}}.$$

Now the well known moments lemma (see [12], p. 242) implies (2.12).

(iii) Here we show that under the assumption of Theorem (2.2)(i) there exists a neighbourhood U of α such that for $\gamma\in U$ with $\gamma<\alpha$ relation (2.13) holds. We have

$$P(B_n)\leqq n\sum_2^n P\{|Y_1|>n^\gamma \text{ and } |Y_l|>n^\gamma\}.$$

Using (1.3) for $m=0$ we obtain with $r=[n^{p\gamma(\beta-1)/2}]$ (where $[x]$ denotes the integral part of $x\in\mathbb{R}$)

$$\sum_2^r P\{|Y_1|>n^\gamma \text{ and } |Y_l|>n^\gamma\}$$

$$\leqq rP^\beta\{|Y_1|>n^\gamma\}\leqq rn^{-p\gamma\beta}(E|Y_l|^p)^\beta=O(n^{-p\gamma-p\gamma(\beta-1)/2}).$$

Using (2.1) we obtain

$$\sum_{r+1}^{n} P\{|Y_1|>n^\gamma \text{ and } |Y_l|>n^\gamma\} \leqq \sum_{r}^{n} (\phi(l) P\{|Y_1|>n^\gamma\} + P^2\{|Y_1|>n^\gamma\})$$
$$\leqq n^{-p\gamma} E|Y_1|^p \sum_{r}^{n} (\phi(l) + n^{-p\gamma} E|Y_1|^p).$$

Assumption (2.4) implies that there exists $\theta>1$ with $\sum^{\infty} \phi(l)^{1/\theta}<\infty$. With a lemma in [1], p. 113 we obtain that $l^\theta \phi(l)$ is bounded. Hence there exists $\tau\in(1,\theta)$ such that

$$\sum_{1}^{\infty} l^{\tau-1}\phi(l) \leqq \sum_{1}^{\infty} l^{-1-(\theta-\tau)} \sup_{l} l^\theta \phi(l)<\infty.$$

For this τ we obtain

$$\sum_{r}^{n} \phi(l)=0(r^{-(\tau-1)})=0(n^{-p\gamma-(\tau-1)p\gamma(\beta-1)/2}).$$

Choose U such that $\gamma\in U$ implies

$$p\alpha<p\gamma+p\gamma(\beta-1)/2, \quad p\alpha<2p\gamma-1, \quad \text{and} \quad p\alpha<p\gamma+(\tau-1)p\gamma(\beta-1)/2.$$

Then (2.13) holds for $\gamma\in U$, $\gamma<\alpha$.

(iv) Now we show that under the assumptions of Theorem (2.8) there exists a neighbourhood U of α such that $\gamma\in U$ and $\gamma<\alpha$ implies (2.13). We have

$$P(B_n) \leqq n\, P\{|Y_1|>n^\gamma\} \leqq n^{1-q\gamma} E|Y_1|^q.$$

If we choose U such that $\gamma\in U$ and $\gamma<\alpha$ implies $p\alpha<q\gamma$ then (2.13) holds.

(v) Now we can show that under the hypothesis of Theorem (2.3)(ii) the relation (2.13) holds for all γ satisfying

(2.15) $\quad \gamma>\alpha(1+\delta)/\beta \quad$ where $\delta=(\beta-1)/\beta$,

and

(2.16) $\quad 2p\gamma-1>p\alpha.$

Let $r=[n^{p\alpha\delta}]$. By (2.6) there exists $\theta>1/\delta+2$ with $\sum_{1}^{\infty} \alpha^{1/\theta}(l)<\infty$. The lemma in [1], p. 113 implies that $l^\theta \alpha(l)$ is bounded. Let $\rho>1$ such that $\theta>\rho/\delta+2$. Then with $q=1-1/\theta$

$$\sum_{1}^{\infty} l^{\rho/\delta} \alpha(l) \leqq \sum_{1}^{\infty} \alpha^{1/\theta}(l) \sup_{l} (l^{\rho/\delta q} \alpha(l))^q$$

and

$$\rho/(\delta q)=(\rho/\delta)/q<(\theta-2)/q=\theta(\theta-2)/(\theta-1)<\theta.$$

Hence $\sum_{1}^{\infty} l^{\rho/\delta}\alpha(l)<\infty$.

Now (1.3) is applied for $m=0$, and (2.2) yields

$$P(B_n)\leqq nr\,P^{\beta}\{|Y_1|>n^{\gamma}\}+n\sum_{r+1}^{n}(\alpha(l)+P^2\{|Y_1|>n^{\gamma}\})$$
$$\leqq nr\,n^{-p\gamma\beta}(E|Y_1|^p)^{\beta}+nr^{-\rho/\delta}\sum^{\infty} l^{\rho/\delta}\alpha(l)+nn^{-2p\gamma}(E|Y_1|^p)^2.$$

Using $p\alpha+p\alpha\delta-p\gamma\beta<0$, $p\alpha-\rho p\alpha<0$, and (2.16), we obtain (2.13).

(vi) Now we show that under the hypothesis of Theorem (2.3)(i) and Theorem (2.8)(i) there exists a neighbourhood U of α such that $\gamma\in U$ and $\gamma<\alpha$ implies (2.14). For $n\in\mathbb{N}$ and $l\in\{1,\dots,n\}$ let

$$Y_{nl}=Y_l\,1_{\{|Y_l|\leqq n^{\gamma}\}} \quad\text{and}\quad T_j=\sum_{1}^{j} Y_{nl}.$$

Then

$$P(C_n)=P\{\max_{j\leqq n}|n^{-\gamma}T_j|>\varepsilon n^{\alpha-\gamma}/2\}.$$

Hence (2.14) holds for all $\gamma<\alpha$ for which we can show that

(2.17) for all $k\geqq 3$ there exists $C(k)>0$ such that for all $n\in\mathbb{N}$

$$E\max_{j\leqq n}|n^{-\gamma}T_j|^k\leqq C(k).$$

We first show that there exists a neighbourhood U of α such that for all $\gamma\in U$, $\gamma<\alpha$, the following holds:

(2.18) for all $k\geqq 2$ there exists $C(k)>0$ such that for all $n\in\mathbb{N}$

$$E|n^{-\gamma}T_n|^k\leqq C(k).$$

Choose a neighbourhood U of α such that $\gamma\in U$, $\gamma<\alpha$, implies (3.22). Then Corollary (3.20) implies (2.18). Now, Theorem B in [14] applied for $g(n)=n^{2\gamma}$ and $X_l=Y_{n,l}$, $l=1,2,\dots$ implies (2.17) for all $\gamma\in U$, $\gamma<\alpha$. Serfling states that his constant K in (3.2) on p. 1231 may depend on k, γ, and the joint distribution of the sequence $X_l=Y_{n,l}$ $l=1,2,\dots$, which in the case considered here depends on n. However, looking at the defining equation (3.7) for K on p. 1232 it is easily seen that K can be chosen independent of n.

Now fix some $\gamma<\alpha$ which lies in the intersection of all neighbourhoods we contructed. For this γ the relations (2.13) and (2.14) hold, and with (2.12) this proves the theorems in the ϕ-mixing case.

(vii) Here we show that under the hypothesis of Theorem (2.8)(ii) relation (2.14) holds. We proceed as in (vi) and use Proposition (3.7) instead of Corollary (3.20). If k satisfies

(2.19) $k(\alpha-\alpha\,p/q)>\alpha\,p-1$

then we can find $\gamma<\alpha$ for which $\alpha p<\gamma q$ and

(2.20) $k(\alpha-\gamma)>\alpha p-1$

holds. There exists an even integer $k\leqq(\alpha p-1)/(\alpha-\alpha p/q)+2$ satisfying (2.19). Assumption (2.10) implies that (3.8) holds for this k and all $\gamma\in(1/p, \alpha p/q)$, and for q instead of p. Relation (2.20) and Proposition (3.7) now imply (2.14).

(viii) Finally we show that under the hypothesis of Theorem (2.3)(ii) relation (2.14) holds. We have to find an even integer k for which (2.20) holds, where γ satisfies (2.15) and (2.16), and where $r=(k-1)p\gamma/(p\gamma-1)$ satisfies (3.8). With $\gamma_0=\max(\gamma_1,\gamma_2)$, $\gamma_1=\alpha(2\beta-1)/\beta^2$, $\gamma_2=(\alpha p+1)/(2p)$ we obtain that (2.15) and (2.16) hold for all $\gamma_0<\gamma<\alpha$. There exists an even integer $k\leqq(\alpha p-1)/(\alpha-\gamma_0)+2$ such that for some $\varepsilon>0$ (2.20) holds for $\gamma_0-\varepsilon<\gamma<\gamma_0$. Assumption (2.6) implies that (3.8) holds for at least one of these γ's. Now we proceed as in (vii).

3. Lemmas

We first prove an elementary lemma for strong mixing sequences which is then used to compute upper bounds for moments of sums of truncated strong mixing random variables. To avoid notational difficulties we use a more general framework. Let $(X, \mathscr{A}, P)$ be a probability space, $k\in\mathbb{N}$. $\mathscr{A}_0, \dots \mathscr{A}_k$ sub-σ-fields of $\mathscr{A}$, and for $i=0, \dots, k$ let $\mathscr{P}_i$ be the σ-field generated by $\mathscr{A}_0, \dots, \mathscr{A}_i$, and $\mathscr{F}_i$ the σ-field generated by $\mathscr{A}_i, \mathscr{A}_{i+1}, \dots, \mathscr{A}_k$.

Let $\alpha_1, \dots, \alpha_k$ be real numbers satisfying

(3.1) for $i=1, \dots, k$, $B\in\mathscr{P}_{i-1}$, $C\in\mathscr{F}_i$

$$|P(B\cap C)-P(B)P(C)|\leqq\alpha_i$$

and (w.l.g., see Appendix, (4.2)).

(3.2) $\alpha_i\leqq 1/4, \quad i=1, \dots, k.$

For $r\in[1,\infty)$ let $\| \ \|_r$ be the norm defined by $\|f\|_r=(E|f|^r)^{1/r}$, and let $\| \ \|_\infty$ be the essential supremum:

$$\|f\|_\infty=\inf\{c>0\colon P\{|f|>c\}=0\}, \quad \inf\emptyset=\infty.$$

Let $f_0, \dots, f_k$ be functions such that for $i=0, \dots, k$

(3.3) f_i is $\mathscr{A}_i$-measurable,

(3.4) $\|f_i\|_\infty\leqq 1$

and

(3.5) $Ef_i=0$

(3.6) **Lemma.** *If* $s>2$ *and* $M=\max\{\|f_i\|_s\colon i=0, \dots, k\}$, *then*

$$\left|E\prod_0^k f_i\right|\leqq 6M^2\prod_1^k(\alpha_i^{(s-2)/(ks)}+6M^2)$$

Proof. For $k=1$ the assertion follows from (4.6). Assume now that $k \geqq 2$ and the assertion holds for $k-1$. Then

$$\begin{aligned}\left|E\prod_0^k f_i\right| &\leqq \min_{1\leqq j\leqq k}\left(\left|E\prod_0^k f_i - E\prod_0^{j-1} f_i\, E\prod_j^k f_i\right| + \left|E\prod_0^{j-1} f_i\, E\prod_j^k f_i\right|\right)\\ &\leqq 6M^2 \min_{1\leqq j\leqq k}(\alpha_j^{(s-2)/s} + 6M^2 \prod_{i\neq j}(\alpha_i^{(s-2)/(ks)} + 6M^2))\\ &\leqq 6M^2 \prod_1^k (\alpha_i^{(s-2)/(ks)} + 6M^2).\end{aligned}$$

With Lemma (3.6) we now compute upper bounds for moments of sums of truncated strong mixing random variables.

(3.7) **Proposition.** *Let $X_1, X_2, \ldots$ be a strongly stationary sequence of random variables which satisfies the strong mixing condition (2.2), where $\alpha\colon \mathbb{N}\to[0, 1/4]$ is nonincreasing. Let $p>2, \gamma\geqq 1/2$, and $k\in\mathbb{N}$ and assume that for $r=(k-1)p\gamma/(p\gamma-1)$*

(3.8) $\sum_1^\infty \alpha_i^{1/r} < \infty.$

Assume that $E|X_1|^p<\infty$, and $EX_1=0$ if $\gamma<1$. For $n\in\mathbb{N}$ and $l\in\{1, \ldots, n\}$ let

$$X_{n,l} = X_l 1_{\{|X_l|\leqq n^\gamma\}} \quad \text{and} \quad S'_n = \sum_1^n X_{nl}.$$

Then there exists $C(k)>0$ such that for all $n\in\mathbb{N}$ $|E(n^{-\gamma}S'_n)^k|\leqq C(k)$.

Proof. Let M be a positive generic constant. Let N be the set of all nonnegative integral n-vectors $v=(v_1, \ldots, v_n)$ with $\sum_1^n v_i = k$. For $R=1, \ldots, k$ let $N(R)$ be the set of all $v\in N$ having exactly R nonzero components. For $R=2, \ldots, k$ and $i_1, \ldots, i_{R-1}\in\mathbb{N}$ let $N(R, i_1, \ldots, i_{R-1})$ be the set of all $v\in N(R)$ with nonzero elements $v_{j_1}, v_{j_2}, \ldots, v_{j_R}$, where $j_1<j_2<\cdots<j_R$, such that for $l=1, \ldots, R-1$

(3.9) $j_{l+1} - j_l = i_l.$

Then $N(1)$ has n elements, and for $R=2, \ldots, p$ and $i_1, \ldots, i_{R-1}\in\mathbb{N}$ the set $N(R, i_1, \ldots, i_{R-1})$ has at most n elements. We have with $W_i = n^{-\gamma}X_{ni}$, $i=1, \ldots, n$,

$$E(n^{-\gamma}S'_n)^k = \sum_{v\in N}\left(k!/\prod_1^n v_i!\right) E\prod_1^n W_i^{v_i} \leqq nEW_1^k + M\sum_{R=2}^k \sum_{v\in N(R)}\left|E\prod_1^n W_i^{v_i}\right|$$

and for $R=2, \ldots, k$

$$\sum_{v\in N(R)}\left|E\prod_1^n W_i^{v_i}\right| = \sum_{i_1, \ldots, i_{R-1}=1}^n \sum \left|E\prod_1^n W_i^{v_i}\right|$$

where the second sum extends over all $v\in N(R, i_1, \ldots, i_{R-1})$.

We first show that for $j\in\mathbb{N}$

(3.10) $|EW_1^j| \leqq M n^{-1}$

where M does not depend on n. With $\|W_1\|_\infty \leqq 1$ it suffices to prove (3.10) for $j=1$ and $j=2$.

If $\gamma<1$, then $EX_1=0$ by assumption, and

$$|EX_{n1}|=|EX_1 1_{\{|X_1|>n^\gamma\}}|\leqq E|X_1|^p n^{-\gamma(p-1)}$$

which yields (3.10) for $j=1$.

If $\gamma \geqq 1$ then $E|W_1|\leqq n^{-\gamma}E|X_1|\leqq M n^{-1}$. In both cases we have $EW_1^2 \leqq n^{-2\gamma}EX_1^2 \leqq M n^{-1}$. This proves (3.10). To compute an upper bound for

$$\left|E\prod_1^n W_i^{\nu_i}\right|$$

we want to apply Lemma (3.6) for f_i being one of the functions $W_i^{\nu_i}$ with $\nu_i \neq 0$. To do this we must first centralize the functions $W_i^{\nu_i}$ in order to satisfy (3.5). Let $R\in\{2,\dots,k\}$, $i_1,\dots,i_{R-1}\in\mathbb{N}$, $\nu\in N(R,i_1,\dots,i_{R-1})$ with nonzero components ν_{j_1}, $\nu_{j_2},\dots,\nu_{j_R}$, where $j_1<j_2<\dots<j_R$. For $l=1,\dots,R$ let

$$g_{l0}=EW_{j_l}^{\nu_{j_l}}$$

and

$$g_{l1}=W_{j_l}^{\nu_{j_l}}-g_{l0}.$$

Then

$$E\prod_1^n W_i^{\nu_i}=\sum\prod_1^R g_{l\delta l}$$

where the summation extends over all $\delta=(\delta_1,\dots,\delta_R)$ in $\{0,1\}^R$, and $Eg_{l1}=0$ for $l=1,\dots,R$.

Fix $\delta\in\{0,1\}^R$, let $A=\{l\in\{1,\dots,R\}:\delta_l=1\}$ and $B=\{1,\dots,R\}-A$. Then

$$E\prod_1^R g_{l\delta_l}=E\prod_{l\in A}g_{l1}\prod_{l\in B}g_{l0}$$

and from (3.9)

$$(3.11)\qquad \Big|\prod_{l\in B}g_{l0}\Big|\leqq M n^{-b}$$

where b is the number of elements in B. We now apply Lemma (3.6) to obtain upper bounds for

$$\Big|E\prod_{l\in A}g_{l1}\Big| \qquad \text{if } A\neq\emptyset.$$

Let $k'=R-b-1$ and $f_0,\dots,f_{k'}$ the functions g_{l1} with $l\in A$: if $l_1<l_2<\dots<l_{k'+1}$ are the elements of A then for $i=0,\dots,k'$

$$f_i=g_{l_{i+1},1}.$$

For $i=0,\dots,k'$ let $\mathscr{A}_i$ be the σ-field generated by $X_{j_{l_i}}$. Since α is nonincreasing and, from (3.9),

$$j_{l_{i+2}}-j_{l_{i+1}}\geqq j_{l_{i+1}+1}-j_{l_{i+1}}=i_{l_{i+1}}$$

(3.1) is satisfied with $\alpha_i=\alpha(i_{l_i})$, $i=1,\dots,k'$.

The assumptions (3.3), (3.4) and (3.5) are now satisfied.

For $s=2p\gamma$ we have $\|f_i\|_s\leqq Mn^{-1/2}$, $i=0,\dots,k'$. Lemma (3.6) implies

$$\left|E\prod_0^{k'} f_i\right|\leqq Mn^{-1}\prod_1^{k'}(\alpha_i^{(s-2)/(k's)}+Mn^{-1}).$$

Therefore with $\rho=(s-2)/((k-1)s)$

$$\left|E\prod_1^{R} g_{l\delta_l}\right|\leqq Mn^{-1}\prod_1^{R-1}(\alpha(i_l)^\rho+Mn^{-1})$$

and hence

$$\left|E\prod_1^{n} W_i^{\nu_i}\right|\leqq Mn^{-1}\prod_1^{R-1}(\alpha(i_l)^\rho+Mn^{-1})$$

for $\nu\in N(R,i_1,\dots,i_{R-1})$. Summing up over all—at most n—$\nu\in N(R,i_1,\dots,i_{R-1})$ and then over all $i_1,\dots,i_{R-1}$ in $\{1,\dots,n\}$ gives

$$\sum_{\nu\in N(R)}\left|E\prod_1^{n} W_i^{\nu_i}\right|\leqq M\left(\sum_1^n\alpha(l)^\rho+M\right)^{R-1}.$$

Summing up over $R\in\{2,\dots,k\}$ and using (3.10) for $\nu\in N(1)$ yields the result.

(3.12) *Remark.* If we apply Proposition (3.7) for $p>2$ and $\gamma=1/2$, then we obtain that for all $k\in\mathbb{N}$ there exists $C(k)>0$ such that for all $n\in\mathbb{N}$

$$E\left|n^{-1/2}\sum_1^n X_{nl}\right|^k\leqq C(k)$$

provided $\sum\alpha^\vartheta(n)<\infty$ for all $\vartheta>0$. Notice that the proposition is true also for non stationary sequences if instead of $E|X_1|^p<\infty$ we assume $\sup\{E|X_i|^p: i\in\mathbb{N}\}<\infty$.

The following result improves upon a proposition of G. Jogesh Babu (1978).

(3.13) **Proposition.** *Let $X_1,X_2,\dots$ be a sequence of random variables which is φ-mixing, where $\varphi\colon\mathbb{N}\to[0,1/4]$ is nonincreasing and satisfies*

(3.14) $\sum\varphi^\delta(n)<\infty$

for some $\delta>0$. Let $p>0$ such that

(3.15) $\sup_n E|X_n|^p<\infty.$

In case $p>1$ we assume $EX_n=0$ for all $n\in\mathbb{N}$. For $d>1$ and $n\in\mathbb{N}$ let

$$Y_n=X_n 1_{\{|X_n|\leqq d\}}\quad\text{and}\quad S_{n,h}=\sum_{i=1}^n Y_{h+i}.$$

Then for each $k \geqq 2$ *and* $0<q<p$ *there exists a constant* K *not depending on* d *such that for all* $1 \leqq n \leqq d^q$

(3.16) $\sup_n E|S_{n,h}|^k \leqq K(n^{k/2} + n\, d^{k-p})$.

Proof. G. Jogesh Babu proved (3.16) assuming (3.14) for $\delta \leqq 1$ if $p \leqq 1$, and $\delta \leqq \max(1/p, 1-1/p)$ if $p>1$. His proof is done by induction, and the stronger condition on φ is used only for $k=2$. Here we prove (3.16) for $k=2$ under the assumption that (3.14) holds for some $\delta>0$. We have $\sup\{|EY_n|: n \in \mathbb{N}\} \leqq M d^{1-p}$, where $M = \sup\{E|Y_n|^p: n \in \mathbb{N}\}$. Hence

(3.17) $\sup\{|ES_{n,h}|: h \in \mathbb{N}\} \leqq M\, n\, d^{1-p}$.

For $n \in \mathbb{N}$ let $D(n) = \sup\{ES^2_{n,h}: h \in \mathbb{N}\}$. Fix $h \in \mathbb{N}$, and for $t \in \mathbb{N}$ define $Z_n = Y_{h+1} + \cdots + Y_{h+n}$, $Z_{n,t} = Z_{2n+t} - Z_{n+t}$, and $S_{n,t} = Z_{n+h+t} - Z_{n+h}$. Then

(3.18) $E(Z_n + Z_{n,t})^2 \leqq EZ_n^2 + EZ^2_{n,t} + |EZ_n EZ_{n,t}| + 2\varphi^{1/2}(t)(EZ_n^2 EZ^2_{n,t})^{1/2}$.

With (3.17) we obtain

(3.19) $E(Z_n + Z_{n,t})^2 \leqq 2D(n)(1+\varphi^{1/2}(t)) + M^2 n^\alpha d^{2-p}$

where $\alpha \in (0,1)$ depends on p and q only. Now the proof can be done as in [15] showing that (3.16) holds for $k=2$ and $n=2^j$, $j \in \mathbb{N}$, and finally using a binary decomposition of n for arbitrary positive integer $n \leqq d^q$. See also [7], p. 226–227.

An immediate consequence of Proposition (3.13) is the following

(3.20) **Corollary.** *Let* $X_1, X_2, \ldots$ *be a sequence of random variables which is strongly stationary and* ϕ*-mixing, where* $\phi: \mathbb{N} \to [0,1]$ *is nonincreasing and satisfies* $\sum \varphi^\delta(n) < \infty$ *for some* $\delta>0$. *Let* $p>0$ *and* $\gamma \geqq 1/2$ *satisfy* $p\gamma>1$.

Assume that $E|X_1|^p < \infty$, *and* $EX_1 = 0$ *if* $\gamma<1$. *For* $n \in \mathbb{N}$ *and* $l \in \{1, \ldots, n\}$ *let*

$$X_{n,l} = X_l 1_{\{|X_l| \leqq n^\gamma\}}$$

and

$$S'_n = \sum_1^n X_{nl}.$$

Then for all $k \in \mathbb{N}$ *there exists* $C(k)>0$ *such that for all* $n \in \mathbb{N}$

$$E|n^{-\gamma} S'_n|^k \leqq C(k).$$

4. Appendix

Here we collect some general results on mixing conditions. Let $(X, \mathscr{A}, P)$ be a probability space, and $\mathscr{P}$ and $\mathscr{F}$ two sub-σ-fields of $\mathscr{A}$. The strong mixing coefficient between $\mathscr{P}$ and $\mathscr{F}$ is defined as $\alpha = \sup\{|P(A \cap B) - P(A)P(B)|: A \in \mathscr{P},$

$B\in\mathscr{F}\}$, and the ϕ-mixing coefficient between $\mathscr{P}$ and $\mathscr{F}$, regarding $\mathscr{P}$ as "past" and $\mathscr{F}$ as "future" is defined as $\phi=\sup\{\|E(B\mid\mathscr{P})-P(B)\|_\infty\colon B\in\mathscr{F}\}$. We then have

(4.1) $\phi\leqq 1$

and

(4.2) $\alpha\leqq 1/4$

and these bounds are attained in many cases when $\mathscr{P}=\mathscr{F}$. Relation (4.2) follows from Hölder's inequality: Let $f=1_A-1/2$, $g=1_B-1/2$; then $\|f\|_\infty\leqq 1/2$, $\|g\|_\infty\leqq 1/2$, and $|P(A\cap B)-P(A)\,P(B)|=|Efg-EfEg|=|Ef(g-Eg)|\leqq\|f\|_2\ \|g-Eg\|_2\leqq\|f\|_2\ \|g\|_2\leqq 1/4$.

(4.3) $\phi=\inf\{\psi\colon\forall\, A\in\mathscr{P},\, B\in\mathscr{F}\colon |P(A\cap B)-P(A)\,P(B)|\leqq\psi P(B)\}$,

(4.4) $\phi=\inf\{\psi\colon\forall f\ \mathscr{P}$-measurable, $g\ \mathscr{F}$-measurable, $r, s>0$ with $1/r+1/s=1$ $|Efg-EfEg|\leqq 2\psi^{1/r}\|f\|_r\|g\|_s\}$.

Relation (4.3) is due to Ibragimov [8]. Relation (4.4) follows from Lemma (1.1) in [8] and $r\to\infty$, $s\to 1$.

Let $r, s>0$ satisfy $1/r+1/s=1$. Then for all $\mathscr{P}$-measurable f and $\mathscr{F}$-measurable g

(4.5) $|Efg-EfEg|\leqq 4\alpha^{1/r}\|g\|_\infty\inf\{\|f-\mu\|_s\colon \mu\in\mathbb{R}\}$.

If $r, s, t>0$ satisfy $1/r+1/s+1/t=1$ then for all $\mathscr{P}$-measurable f and $\mathscr{F}$-measurable g

(4.6) $|Efg-EfEg|\leqq 6\alpha^{1/r}\|f\|_s\|g\|_t$.

With 6 replaced by 10 this is shown in [4], p. 871. See also [3].

Proof of (4.5). We may restrict the considerations on $\mathscr{P}$-measurable functions $f=\sum_1^m\beta_i 1_{B_i}$, where $m\in\mathbb{N}$, $\beta_1,\dots,\beta_m\in\mathbb{R}$ and $B_1,\dots,B_m\in\mathscr{P}$ are disjoint. We first treat the case $g=1_A$ with $A\in\mathscr{F}$. Then

$$\begin{aligned}|Efg-EfEg|&\leqq\sum_1^m|\beta_i|\,|P(A\cap B_i)-P(A)\,P(B_i)|\\&\leqq\left(\sum_1^m|\beta_i|^s\,|P(A\cap B_i)-P(A)\,P(B_i)|\right)^{1/s}\left(\sum_1^m|P(A\cap B_i)-P(A)\,P(B_i)|\right)^{1/r}.\end{aligned}$$

Note first that for $i=1,\dots,m$ $|P(A\cap B_i)-P(A)\,P(B_i)|\leqq 2P(B_i)$ and with

$$B=\bigcup\{B_i\colon P(A\cap B_i)\geqq P(A)\,P(B_i)\}$$

and

$$B' = \bigcup \{B_i : P(A \cap B_i) < P(A) P(B_i)\}$$

we obtain

$$\sum_1^m |P(A \cap B_i) - P(A) P(B_i)| = P(A \cap B) - P(A) P(B) + P(A) P(B') - P(A \cap B') \leqq 2\alpha.$$

Hence $|Efg - Ef\,Eg| \leqq 2\alpha^{1/r} \|f\|_s$. Keep f fixed and let G be the set of all $\mathscr{F}$-measurable g with $0 \leqq g \leqq 1$ and $|Efg - Ef\,Eg| \leqq 2\alpha^{1/r} \|f\|_s$.

Then $1_A \in G$ for $A \in \mathscr{F}$. Since G is convex and $\|\ \|_\infty$-closed we obtain that for all $\mathscr{F}$-measurable g with $0 \leqq g \leqq 1$ we have $g \in G$. This proves that for all $\mathscr{P}$-measurable f and $\mathscr{F}$-measurable g we have $|Efg - Ef\,Eg| \leqq 4\alpha^{1/r} \|f\|_s \|g\|_\infty$. Since the left hand side of this inequality remains unchanged if we substitute f by $f - \mu$ with $\mu \in \mathbb{R}$ (4.5) follows.

Proof of (4.6). Assume w.l.g. that $\|g\|_t = 1$. Let $s' > 0$ be defined by $1/s + 1/s' = 1$. For $N = \alpha^{-1/t}$, $g^{(N)} = g 1_{\{|g| \leqq N\}}$, $g_{(N)} = g - g^{(N)}$, we obtain with (4.5) and Hölders inequality

$$\begin{aligned} |Efg - Ef\,Eg| &= |E(f - Ef)\, g^{(N)} + E(f - Ef)\, g_{(N)}| \\ &\leqq 4N\alpha^{1/s'} \inf\{\|f - \mu\|_s : \mu \in \mathbb{R}\} + 2\|f\|_s \|g_{(N)}\|_{s'} \end{aligned}$$

Markov's inequality yields

$$\begin{aligned} \|g_{(N)}\|_{s'} &= (E|g|^{s'} 1_{\{|g| > N\}})^{1/s'} \\ &\leqq N^{-(t-s')/s'} (E|g|^t)^{1/s'} = N^{-(t-s')/s'} = \alpha^{1/s' - 1/t} = \alpha^{1/r}. \end{aligned}$$

With $N\alpha^{1/s'} = \alpha^{1/s' - 1/t} = \alpha^{1/r}$ we obtain $|Efg - Ef\,Eg| \leqq 6\alpha^{1/r} \|f\|_s$ which proves the assertion.

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