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Approximation and the Spectral Multiplicity of Special Automorphisms

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Summary. Conditions have been given [7] under which a special automorphism over an automorphism admitting a simple approximation again admits a simple approximation, and so has simple spectrum.

In this paper using different techniques to those employed in [7], we obtain improved results in the same direction. Specifically, conditions are given for a special automorphism over an automorphism, which admits either a simple approximation, or an approximation with suitable speed, to have bounded spectral multiplicity. Furthermore, we obtain as a corollary a result on primitive automorphisms, which partially generalises a result appearing in [4].

1. Preliminaries

Throughout (X, F, μ) is a measure space isomorphic to the unit interval with Lebesgue measure. A measure-preserving invertible point transformation of X is an *automorphism* of (X, F, μ).

An ordered collection $\xi = \{C_i\colon i=1,2,\dots\}$ of pairwise-disjoint measurable sets in X is called a *partition.* In most cases our partitions will be finite.

If $A \in F$, we write $A \leqq \xi$, if A is a union of members of ξ.

If $A \in F$, we denote by $A(\xi)$ a set such that $A(\xi) \leqq \xi$ and $\mu(A \vartriangle A(\xi))$ is a minimum.

A sequence $\{\xi(n)\}$ of partitions converges to the *unit partition,* written $\xi(n) \to \varepsilon_X$, provided that for each measurable set A, $\mu(A \vartriangle A(\xi(n))) \to 0$ as $n \to \infty$.

We are interested in the following two related, but different types of approximation of automorphisms.

Definition 1. The automorphism T admits an *approximation with speed* $f(n)$ if there exists a sequence of partitions $\{\xi(n)\}$ with $\xi(n) = \{C_i(n)\colon 1 \leqq i \leqq q(n)\}$ such that:

(i) $\xi(n) \to \varepsilon_X$ as $n \to \infty$;

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(ii) $\mu(C_i(n))=\mu(C_{i+1}(n)),\ 1\leqq i\leqq q(n)-1;$

(iii) $\sum_{i=1}^{q(n)-1} \mu(TC_i(n)\triangle C_{i+1}(n))<f(q(n)).$

Definition 2. The automorphism T admits a *simple approximation* if there exists a sequence of partitions $\{\xi(n)\}$ with $\xi(n)=\{C_i(n)\colon 1\leqq i\leqq q(n)\}$ such that:

(i) $\xi(n)\to\varepsilon_X$ as $n\to\infty$;

(ii) $TC_i(n)=C_{i+1}(n),\ 1\leqq i\leqq q(n)-1.$

Stepin [5] has shown that if T admits an approximation with speed θ/n, $\theta<2m/(m+1)$ then T has spectral multiplicity at most m, and Baxter [1] has shown that if T admits a simple approximation then T has simple spectrum.

Definition 3. We say a partition $\eta=\{A_i\colon 1\leqq i\leqq N\}$ is *approximated with speed* $g(n)$ by a sequence of finite partitions $\{\xi(n)\}$, such that $\xi(n)\to\varepsilon$, if there exist sets $A_i(n)\leqq\xi(n)$ such that

$$\mu\left[\bigcup_{i=1}^{N}(A_i\triangle A_i(n))\right]<g(q(n))$$

where $q(n)$ is the number of elements in $\xi(n)$.

Let T be an automorphism of X. Denote by $\mathbb{N}$ the set of non-negative integers, and let $f\colon X\to\mathbb{N}$ be an integrable function. Put $B(k,n)=\{(x,n)\colon x\in X, f(x)=k\}$, and define

$$X(f)=\bigcup_{k\geqq 0}\bigcup_{n=0}^{k}B(k,n).$$

We identify X with the set $\bigcup_{k\geqq 0}B(k,0)$.

We may regard each set $B(k,n)$, $0<n\leqq k$, as a copy of $B(k,0)$. Consequently we may extend μ to $X(f)$ and form a normalised measure μ^f on $X(f)$ in the obvious way.

Definition 4. Let the transformation T_f on $X(f)$ be defined by

$$T_f(x,n)=(x,n+1),\qquad 0\leqq n<f(x),$$
$$T_f(x,f(x))=(Tx,0).$$

T_f is called the *special automorphism* over T built under the function f.

If f is the characteristic function of a set $A\in F$, then the special automorphism T_f is denoted T^A, and called a *primitive automorphism* over T.

Order the sets $B(k,n)$, $k\geqq 0$, $0\leqq n\leqq k$, lexicographically.

Definition 5. Let ξ be a partition such that every element of ξ is contained in exactly one of the sets $B(k,0)$ for some k. Then ξ^f is the partition in $X(f)$ consisting of the elements $C\in\xi$, together with for each $C\in\xi$, where $C\subset B(k,0)$, a copy of C in each of the sets $B(k,n)$, $0<n\leqq k$.

It is easily seen that if $\xi(n)\to\varepsilon_X$, then $\xi^f(n)\to\varepsilon_{X(f)}$.

Definition 6. If U is a unitary operator on a Hilbert space H and $f \in H$, we put

$$Z(f) = \overline{\{\ldots U^{-1}f, f, Uf, \ldots\}}$$

the closed subspace generated by the vectors $U^n f$, $n = 0, \pm 1, \ldots$.

The principal tool used in the proof of our main theorems, is the following result of Chacon [2].

Theorem 1. *If U is a unitary operator on a separable Hilbert space H, and if the spectral multiplicity of U is at least k, then there exist k orthonormal vectors $u_1, u_2, \ldots, u_k$ in H such that*

$$\sum_{i=1}^{k} d^2(u_i, Z(w)) \geqq k - 1$$

for any $w \in H$. (d is the matrix arising from the norm in H.)

2. Main Results

We first state without proof a simple lemma.

Lemma 1. *Let $\{\xi(n)\}$ be a sequence of partitions, $\xi(n) = \{C_i(n): 1 \leqq i \leqq q(n)\}$ with $\xi(n) \to \varepsilon_X$. Let $\{\eta(n)\}$ be a sequence of partitions, $\eta(n) = \{D_i(n): 1 \leqq i \leqq q(n)\}$, and suppose that*

$$\rho(\eta(n), \xi(n)) = \sum_{i=1}^{q(n)} \mu(D_i(n) \vartriangle C_i(n)) \to 0$$

as $n \to \infty$. Then $\eta(n) \to \varepsilon_X$.

Suppose that $f: X \to \mathbb{N}$ is integrable. Let $\zeta = \{B(k): k \in \mathbb{N}\}$, where $B(k) = B(k, 0) = f^{-1}(k)$.

Suppose further that $\{\xi(n)\}$ is a sequence of partitions, $\xi(n) = \{C_i(n): 1 \leqq i \leqq q(n)\}$, with $\xi(n) \to \varepsilon_X$. Following Definition 3, we shall agree to say that ζ is approximated with speed δ/n, $0 < \delta < m/m+1$, for some fixed $m \in \mathbb{N}$, if there exist sets $F_k(n) \leqq \xi(n)$ such that

$$\mu\left[\bigcup_{k=0}^{\infty} B(k) \vartriangle F_k(n)\right] < \delta/q(n).$$

Note that it is implicit in this definition that all but a finite number of the sets $F_k(n)$ will be empty. Furthermore we can ensure that the sets $F_k(n)$ are pairwise disjoint and $\bigcup_{i=1}^{q(n)} C_i(n) = \bigcup_{k=0}^{\infty} F_k(n)$. That this equality holds is a consequence of the speed of approximation: Let

$$I(n) = \left\{k: C_k(n) \subset \bigcup_{i=1}^{q(n)} C_i(n) \setminus \bigcup_{k=0}^{\infty} F_k(n)\right\}.$$

Then

$$\bigcup_{k\in I(n)} C_k(n) \subset \bigcup_{k=0}^{\infty} (B(k) \triangle F_k(n)).$$

Hence

$$\mu[\bigcup_{k\in I(n)} C_k(n)] < \delta/q(n).$$

However

$$\mu[\bigcup_{k\in I(n)} C_k(n)] = \operatorname{card}(I(n))\, \mu(C_1(n))$$
$$> \operatorname{card}(I(n))\, m/(m+1)\, q(n)$$
$$> \operatorname{card}(I(n))\, \delta/q(n).$$

Hence

$\operatorname{card}(I(n)) = 0.$

We define functions f_n as follows:

$$f_n(x) = k \quad \text{if } x \in F_k(n),$$
$$= 0 \quad \text{if } x \in X \setminus \bigcup_{i=1}^{q(n)} C_i(n).$$

Then

$$\{x : f_n(x) \neq f(x)\} \subset \bigcup_{k=0}^{\infty} (B(k) \triangle F_k(n)).$$

Hence

$$\mu\{x : f_n(x) \neq f(x)\} < \delta/q(n).$$

Following is our main result.

Theorem 2. *Let T admit an approximation with speed θ/n w.r.t. $\{\xi(n)\}$, and suppose that ζ can be approximated with speed δ/n, where $\theta + 2\delta < 2m/m+1$, by $\{\xi(n)\}$. Then T_f has spectral multiplicity at most m.*

Proof. Put

$$D(n) = \bigcap_{i=0}^{q(n)-1} T^{-i}[C_{i+1}(n) \cap \{x : f_n(x) = f(x)\}]$$

then we claim

$$\mu(C_1(n)) - \mu(D(n)) < (\theta + 2\,\delta)/2\,q(n).$$

For, putting

$$K(n) = \bigcap_{i=0}^{q(n)-1} T^{-i}\, C_{i+1}(n)$$

then

$$T^{i-1}\, K(n) \subset C_i(n) \qquad i = 1, \ldots, q(n)$$

and

$$\begin{aligned}
\mu(D(n)) &= \mu\left(K(n)\cap\bigcap_{i=0}^{q(n)-1} T^{-i}\{x: f_n(x)=f(x)\}\right)\\
&= \mu(K(n)) - \mu\left(K(n)\setminus\bigcap_{i=0}^{q(n)-1} T^{-i}\{x: f_n(x)=f(x)\}\right)\\
&= \mu(K(n)) - \mu\left(\bigcup_{i=0}^{q(n)-1} (K(n)\setminus T^{-i}\{x: f_n(x)=f(x)\}\right)\\
&\geqq \mu(K(n)) - \sum_{i=0}^{q(n)-1} \mu[T^i K(n)\setminus\{x: f_n(x)=f(x)\}]\\
&= \mu(K(n)) - \mu\left[\bigcup_{i=0}^{q(n)-1} T^i K(n)\setminus\{x: f_n(x)=f(x)\}\right]\\
&\geqq \mu(K(n)) - \mu\{x: f_n(x)\neq f(x)\}.
\end{aligned}$$

But

$$K(n)\supset C_1(n)\setminus\bigcup_{i=0}^{q(n)} T^{-i}(TC_i(n)\cap(X\setminus C_{i+1}(n)))$$

so

$$\mu(K(n))\geqq\mu(C_1(n))-\theta/2q(n)$$

hence

$$\mu(C_1(n))-\mu(D(n))<(\theta+2\delta)/2q(n).$$

It follows that $\mu(D(n))>0$.

Now define $\eta(n)=\{(C_i(n)\cap\{x: f_n(x)=f(x)\}): 1\leqq i\leqq q(n)\}$. Then,

$$\rho(\eta(n),\xi(n))\to 0$$

as $n\to\infty$ and so $\eta(n)\to\varepsilon_X$ by Lemma 1. Consequently $\eta^f(n)\to\varepsilon_{X(f)}$. Let $\eta^f(n)$ have $q^f(n)$ elements, and $C_k^f(n)$ be the element of $\eta^f(n)$ containing $T_f^{k-1}D(n)$.

Now suppose that U_{T_f} has spectral multiplicity at least $m+1$. Then by Theorem 1 there exist $m+1$ orthonormal vectors $u_1, u_2, \ldots, u_{m+1}$ such that

$$\sum_{i=1}^{m+1} d^2(u_i, Z(w(n)))\geqq m$$

where $w(n)=\chi_{D(n)}$ and $Z(w(n))$ is the cycle generated by $w(n)$.

Since $\eta^f(n)\to\varepsilon_{X(f)}$, the u_i may be arbitrarily closely approximated by simple functions of the form

$$u_j(n)=\sum_{k=1}^{q^f(n)} a_k^j\chi_{C_k^f(n)},\qquad 1\leqq j\leqq m+1.$$

Define

$$h_j(n)\in Z(w(n))$$

by

$$h_j(n)=\sum_{k=1}^{q^f(n)} a_k^j \chi_{T_f^{k-1}D(n)}, \qquad 1\leqq j\leqq m+1.$$

Consider

$$\begin{aligned} d^2(u_j(n), Z(w(n)))&\leqq \|u_j(n)-h_j(n)\|^2 \\ &=\int_{X(f)} \left|\sum_{k=1}^{qf(n)} a_k^j \chi_{C_k^f(n)} - \sum_{k=1}^{q^f(n)} a_k^j \chi_{T_f^{k-1}D(n)}\right|^2 d\mu^f \\ &=\sum_{k=1}^{q^f(n)} |a_k^j|^2 \mu^f(C_k^f(n)\setminus T_f^{k-1} D(n)). \end{aligned}$$

Now

$$\begin{aligned} \mu^f(C_k^f(n)\setminus T_f^{k-1} D(n))/\mu^f(C_k^f(n)) &\leqq \mu^f(C_1(n)\setminus D(n))/\mu^f(C_1(n)) \\ &=\mu(C_1(n)\setminus D(n))/\mu(C_1(n))<(\theta+2\delta)/2q(n)\,\mu(C_1(n)). \end{aligned}$$

Hence

$$\begin{aligned} d^2(u_j(n), Z(w(n)))&<[(\theta+2\delta)/2q(n)\,\mu(C_1(n))] \sum_{i=1}^{q^f(n)} |a_k^j|^2 \mu^f(C_k^f(n)) \\ &=[(\theta+2\delta)/2q(n)\,\mu(C_1(n))]\,\|u_j(n)\|^2. \end{aligned}$$

So

$$\sum_{j=1}^{m+1} d^2(u_j(n), Z(w(n)))<[(\theta+2\delta)/2q(n)\,\mu(C_1(n))] \sum_{j=1}^{m+1} \|u_j(n)\|^2.$$

Thus

$$\lim_{n\to\infty} [(\theta+2\delta)/2q(n)\,\mu(C_1(n))] \sum_{j=1}^{m+1} \|u_j(n)\|^2 \geqq m,$$

which implies $(\theta+2\delta)(m+1)/2\geqq m$; that is

$$\theta+2\delta\geqq 2m/m+1,$$

which contradicts

$$\theta+2\delta<2m/m+1.$$

3. Special Automorphisms over Automorphisms Admitting a Simple Approximation

A similar result to that shown above can be obtained when the automorphism T admits a simple approximation.

Theorem 3. *Let T admit a simple approximation with respect to the sequence $\{\xi(n)\}$, and suppose that ζ can be approximated with speed δ/n, $\delta < m/m+1$, by $\{\xi(n)\}$. Then the spectral multiplicity of T_f is at most m.*

The Corollary below generalises a result appearing in [4].

Corollary 1. *Let T admit a simple approximation with respect to a sequence of partitions $\{\xi(n)\}$, $\xi(n) = \{C_i(n)\colon 1 \leqq i \leqq q(n)\}$, with $\xi(n) \to \varepsilon_X$, and such that*

$$\mu\left(X \setminus \bigcup_{i=1}^{q(n)} C_i(n)\right) < \varepsilon/q(n).$$

Let $A \in F$ be such that there exist sets $A(n) \leqq \xi(n)$ with $\mu(A \,\triangle\, A(n)) < \delta/q(n)$.

If $\varepsilon + \delta < m/m+1$ then the spectral multiplicity of T^A is at most m.

4. Remarks

(i) In [7], it is shown that if T admits a simple approximation with respect to $\xi(n)$, and for each $C_i(n) \in \xi(n)$ there exists $K_i(n)$ such that

$$\mu[C_i(n) \cap B(K_i(n))] > (1 - \delta(n))\,\mu(C_i(n))$$

where $\delta(n) = o\left(\frac{1}{q(n)}\right)$, then T_f admits a simple approximation and so has simple spectrum.

A considerable improvement to this result may be obtained using the techniques of Theorem 2. We can replace $\delta(n)$ by $\delta/q(n)$, $\delta < m/m+1$, and conclude that the spectral multiplicity of T is at most m.

(ii) Using the same method as in [7] it can be shown that if T admits a cyclic approximation (see Katok and Stepin [5]) with speed θ/n^2 and $\delta(n)$ in (i) above is taken to be $\delta(n) = \delta/q(n)^2$, then T_f admits a cyclic approximation with speed α/n, $\alpha < 1$ (provided θ and δ are chosen small enough). It follows that in this case T_f has simple and singular spectrum.

(iii) The relationship between the result of [7] mentioned in (i) above, and Theorem 2 may be clarified if we assume $\mu\left(X \setminus \bigcup_{i=1}^{q(n)} C_i(n)\right) = 0$. Then the statement "for each $C_i(n)$ there exists $K_i(n)$ such that $\mu[C_i(n) \cap B(K_i(n))] > (1 - \delta/q(n))\,\mu(C_i(n))$" implies ζ is approximated with speed $2\delta/n$ by $\{\xi(n)\}$. Conversely the statement "ζ is approximated with speed δ/n by $\{\xi(n)\}$" implies that for each $C_i(n)$, there exists $K_i(n)$ such that $\mu[C_i(n) \cap B(K_i(n))] > (1 - \delta)\,\mu(C_i(n))$.

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