Z. Wahrscheinlichkeitstheorie verw. Gebiete
49, 133–153 (1979)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

On Limit Processes for a Class of Additive Functionals of Recurrent Diffusion Processes

Yuji Kasahara and Shin'ichi Kotani
Department of Mathematics, Kyoto University, Kyoto 606, Japan

§ 0. Introduction

Limit theorems for a class of additive functionals, especially occupation times of Markov process, have been studied by many authors, e.g., Darling-Kac [1], Dobrushin [2], Karlin-McGregor [6], C. Stone [12] and Kasahara [7, 8]. So far, these theorems except [12] dealt with the convergence at each fixed time only, but recently Papanicolaou-Stroock-Varadhan [10] discussed the convergence as *stochastic processes:* for example, if $b(t)$ is a 1-dimensional Brownian motion and $l(t,x)$ is its local time (i.e., $2\int_E l(t,x)\,dx=\int_0^t I_E(b(s))\,ds$ for every Borel subset E) and $V(x)$ is a bounded Borel function with compact support, it is easy to see that the processes $t\to\frac{1}{\lambda}\int_0^{\lambda^2} V(b(s))\,ds$ converge, in the sense of probability law on the space of continuous functions, to the process $t\to 2\bar{V}l(t,0)$ as $\lambda\to\infty$, where $\bar{V}=\int_{-\infty}^{\infty} V(x)\,dx$. When $\bar{V}=0$, the limit process is trivial but as they showed in [10] if we change the scaling as

$$t\to\frac{1}{\sqrt{\lambda}}\int_0^{\lambda^2 t} V(b(s))\,ds,$$

then the laws of these processes converge to that of the process $t\to\sqrt{\langle V\rangle}\,\tilde{b}(l(t,0))$ as $\lambda\to\infty$, where $\langle V\rangle$ is the energy of the charge $V(x)\,dx$ and $\tilde{b}(t)$ is another Brownian motion independent of $b(t)$. The purpose of this paper is to study similar problems for 2-dimensional Brownian motion and 1-dimensional recurrent diffusion processes.

Let $B(t)$ be a 2-dimensional Brownian motion and $V(x)$; $x\in\mathbb{R}^2$ be a bounded Borel function with compact support. Kallianpur-Robbins [5] and Kasahara [7], [8] have proved that the distributions of $\frac{1}{u(\lambda)}\int_0^{\lambda} V(B(s))\,ds$ (where $u(\lambda)=\log\lambda$) converge to an exponential distribution as $\lambda\to\infty$, and if $\bar{V}=\int_{\mathbb{R}^2} V(x)\,dx=0$, with $u(\lambda)=\sqrt{\log\lambda}$ the above random variables converge in

0044-3719/79/0049/0133/$04.20

law to a bilateral exponential distribution. As to the convergence of processes Stroock proposed the following problem: What is the limit process, as $\lambda \to \infty$, of the processes

$$A_\lambda(t) = \frac{1}{u(\lambda)} \int_0^{e^{\lambda t}} V(B(s))\, ds,$$

where $u(\lambda) = \lambda$ or $\sqrt{\lambda}$ according as $\overline{V} \neq 0$ or $\overline{V} = 0$? Here, it should be noted that the processes $t \to \frac{1}{\log \lambda} \int_0^{\lambda t} V(B(s))\, ds$ converge to a degenerate one (i.e., independent of time parameter t). For this problem we have obtained the following results: the limit process of $A_\lambda(t)$ is given as

$$2\overline{V} l(M^{-1}(t)), \tag{0.1}$$

where $M(t) = \max_{s \leq t} b(s)$ and $l(t)$ is the local time at 0 of $b(t)$ and, in the case of $\overline{V} = 0$, as

$$\sqrt{\langle V \rangle}\, \tilde{b}(l(M^{-1}(t))), \tag{0.2}$$

where $\tilde{b}(t)$ is another Brownian motion independent of $b(t)$ and

$$\langle V \rangle = -\frac{4}{\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log|x-y|\, V(x)\, V(y)\, dx\, dy.$$

Our method is as follows. First we represent a 2-dimensional Brownian motion as a Brownian motion $X(t) = (b(t), \theta(t))$ on a cylinder $\mathbb{R} \times (\mathbb{R}/2\pi\mathbb{Z})$ run with the clock $S^{-1}(t)$ $\left(\text{where } S(t) = \int_0^t e^{2b(s)}\, ds\right)$. Then, by random time change depending on the parameter λ (see § 3), our problem is reduced to that of finding the limit process of an addidive functional $\frac{1}{u(\lambda)} \int_0^{\lambda^2 t} f(X(s))\, ds$. Since the process $\theta(t)$ is strongly ergodic, only $b(t)$ plays an essential role, which enables us to treat the above functional in a way similar to Papanicolaou-Stroock-Varadhan [10] in the case of 1-dimensional Brownian motion.

We explain the content of this paper. In § 1 and § 2 we give two limit theorems for a class of additive functionals of the Brownian motion $X(t)$ on the cylinder. In § 3 we show that $A_\lambda(t)$ converge in law to the above limit processes (0.1) and (0.2). The process (0.1) is not a Markovian but its inverse is a Markov process (cf. S. Watanabe [13]). Further an interesting fact here is that the process (0.1) increases only with jumps although $A_\lambda(t)$ are continuous. Therefore the convergence is neither in the ordinary weak topology nor in Skorokhod's J-topology. We will see that M_1-convergence introduced by Skorokhod [11] is most suitable. In the final section, we give similar limit theorems for a broad class of 1-dimensional diffusion processes.

Lastly the authors wish to thank D. Stroock for suggesting them this problem and S. Watanabe for his helpful discussions with them.

§ 1. Limit Processes for Additive Functionals of a Brownian Motion on the Cylinder (Positively Charged Case)

As it will be seen later in § 3, the study of limit processes of functionals of a 2-dimensional Brownian motion can be reduced to that of a Brownian motion on the cylinder $G=\mathbb{R}\times(\mathbb{R}/2\pi\mathbb{Z})$. Therefore first we establish limit theorems relating to a Brownian motion on G.

A Brownian motion $X(t)$ on G is defined as the product process of a 1-dimensional Brownian motion $b(t)$ and another 1-dimensional Brownian motion $\theta(t)$ independent of $b(t)$ and viewed in modulo 2π. It has the transition density $p_t(z-z')$ with respect to the Haar measure $d\mu(z)=dx\,d\theta$ $(z=(x,\theta))$, which can be represented in two ways:

$$p_t(z)=\frac{1}{2\pi t}\exp(-|x|^2/2t)\sum_{n=-\infty}^{\infty}\exp[-(\theta+2\pi n)^2/2t] \tag{1.1}$$

$$=\frac{1}{4\pi^2}\int_{G^*}\exp(-|\xi|^2 t/2-i\xi z)\,d\mu^*(\xi), \tag{1.2}$$

where $G^*=\mathbb{R}\times\mathbb{Z}$, $d\mu^*(\xi)=d\lambda\,\delta_{\{n\}}$ for $\xi=(\lambda,n)$, $|\xi|^2=\lambda^2+n^2$ and $\xi z=\lambda x+n\theta$.

Set

$$\Gamma_0(z)=-\frac{1}{2\pi}\log|e^{i\theta}-e^{-|x|}|^2.$$

Then we have

$$\Gamma_1(z)=\frac{\partial\Gamma_0}{\partial x}=-\frac{1}{\pi}\,\frac{(\cos\theta-e^{-|x|})\,e^{-2|x|}}{|e^{i\theta}-e^{-|x|}|^2}\operatorname{sgn}x,$$

$$\Gamma_2(z)=\frac{\partial\Gamma_0}{\partial\theta}=-\frac{1}{\pi}\,\frac{e^{-|x|}\sin\theta}{|e^{i\theta}-e^{-|x|}|^2},$$

where $\operatorname{sgn}x=1$ for $x>0$ and $=-1$ for $x<0$.

Setting $\Gamma_k f(z)=\Gamma_k * f(z)$ $(k=0,1,2)$, for any sufficiently smooth function f we have

$$\frac{1}{2}\Delta\Gamma_0 f(z)=\frac{1}{2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial\theta^2}\right)\Gamma_0 f(z)=-f(z)+\frac{1}{2\pi}\bar{f}(z), \tag{1.3}$$

where $\bar{f}(z)=\int_{-\pi}^{\pi}f(x,\theta)\,d\theta$.

From now on we denote by E_z the expectation with respect to the measure P_z of a Brownian motion starting at z and write simply E in place of E_0. We prepare several estimates here.

Lemma 1.1.

$$\left|\int_{-\infty}^{x}\bar{f}(y)\,dy\right|\leq\|f\|_1, \tag{1.4}$$

$$|\Gamma_0 f(z)| \leqq C \|f\|_p \quad (1<p\leqq\infty), \tag{1.5}$$

$$\begin{aligned} &|\Gamma_i f(z)| \leqq C \|f\|_p \quad (2<p\leqq\infty), \\ &\|\Gamma_i f\|_1 \leqq C \|f\|_1 \quad \textit{for } i=1,2, \end{aligned} \tag{1.6}$$

$$|E_z f(X_t)| \leqq p_t(0)^{1/p} \|f\|_p \quad (1\leqq p\leqq\infty), \tag{1.7}$$

where C is a constant independent of f.

Proof. (1.4) is trivial. The others can be deduced from the fact that $\Gamma_0 \in L^q(G)$ $(1\leqq q<\infty)$ and $\Gamma_k \in L^q(G)$ $(0<q<2)$ for $k=1,2$. Using the fact that $p_t(z)\leqq p_t(0)$ and $\int_G p_t(z)\,d\mu(z)=1$, we can prove (1.7) as follows:

$$\begin{aligned} |E_z f(X_t)| &\leqq \int_G p_t(z-z')\,|f(z')|\,d\mu(z') \\ &\leqq \{\int_G p_t(z')^q\,d\mu(z')\}^{1/q}\,\|f\|_p \\ &\leqq p_t(0)^{\frac{q-1}{q}}\,\|f\|_p = p_t(0)^{1/p}\,\|f\|_p. \quad \text{Q.E.D.} \end{aligned}$$

Lemma 1.2. *Suppose $f\in L^1(G)\cap L^p(G)$ for some $1<p\leqq\infty$. Then we have*

$$\lim_{t\to\infty} \frac{1}{t} E\left(\int_0^t f(X_s)\,ds\right)^2 = \frac{1}{4\pi^2}(\bar{\bar{f}})^2,$$

where $\bar{\bar{f}}=\int_G f(z)\,d\mu(z)$.

Proof. The proof is divided into three parts.

1°) $E\left(\int_0^a |f(X_s)|\,ds\right)^2 < \infty$ for any $a<\infty$.

Setting $P_t f(z)=p_t * f(z)$, we have an equality:

$$I=E\left(\int_0^a |f(X_s)|\,ds\right)^2 = 2 \iint_{0\leqq u\leqq s\leqq a} P_u(|f|\,P_{s-u}|f|)(0)\,du\,ds.$$

Applying the inequality (1.7), we see

$$I\leqq 2\|f\|_p^2 \iint_{0\leqq u\leqq s\leqq a} \{p_u(0)\,p_{s-u}(0)\}^{1/p}\,du\,ds.$$

On the other hand, the expression (1.1) implies

$$p_u(0)\leqq \frac{C}{u} \quad \text{for any } u\leqq a.$$

Hence $I<\infty$ follows.

2°) $E\Big(\iint_{\substack{1\leqq u\leqq s\leqq t\\ s-u\leqq 1}} |f(X_u)\,f(X_s)|\,du\,ds\Big)=O(\sqrt{t})$ as $t\to\infty$.

From the inequality (1.7) we have

$$E|f(X_u)f(X_s)|=P_u(|f|P_{s-u}|f|)(0)$$
$$\leqq p_{s-u}(0)^{1/p}p_u(0)\|f\|_1\|f\|_p,$$

which implies that

$$J=E(\iint_{\substack{1\leqq u\leqq s\leqq t\\ s-u\leqq 1}}|f(X_u)f(X_s)|\,du\,ds)$$
$$\leqq\|f\|_p\|f\|_1\iint_{\substack{1\leqq u\leqq s\leqq t\\ s-u\leqq 1}}p_{s-u}(0)^{1/p}p_u(0)\,ds\,du$$
$$\leqq\|f\|_p\|f\|_1\left(\int_1^t p_u(0)\,du\right)\left(\int_0^1 p_s(0)^{1/p}\,ds\right).$$

Here noting $p_s(0)\leqq Cs^{-1}$ for $s\leqq 1$ and $p_u(0)\leqq C\sqrt{u}^{-1}$ for $u\geqq 1$, we easily see that $J=O(\sqrt{t})$ as $t\to\infty$.

3°) $$\lim_{t\to\infty}\frac{1}{t}E(\iint_{\substack{1\leqq u\leqq s\leqq t\\ s-u\geqq 1}}f(X_u)f(X_s)\,du\,ds)=\frac{1}{8\pi^2}(\bar{\bar{f}})^2.$$

Let $\hat{f}(\xi)=\int_G e^{i\xi z}f(z)\,d\mu(z)$. First we note that

$$P_s f(z)=\frac{1}{4\pi^2}\int_{G^*}\exp\left(-i\xi z-\frac{s|\xi|^2}{2}\right)\hat{f}(\xi)\,d\mu^*(\xi)$$

and

$$\widehat{fP_{s-u}f}(\xi_1)=\frac{1}{4\pi^2}\int_{G^*}\hat{f}(\xi_1-\xi_2)\widehat{P_{s-u}f}(\xi_2)\,d\mu^*(\xi_2)$$
$$=\frac{1}{4\pi^2}\int_{G^*}\exp\left(-|\xi_1|^2\frac{u}{2}-|\xi_2|^2\frac{(s-u)}{2}\right)\hat{f}(\xi_2)\hat{f}(\xi_1-\xi_2)\,d\mu^*(\xi_2).$$

Hence

$$Ef(X_u)f(X_s)=P_u(fP_{s-u}f)(0)$$
$$=\frac{1}{16\pi^4}\iint_{G^*\times G^*}\exp\left(-|\xi_1|^2\frac{u}{2}-|\xi_2|^2\frac{(s-u)}{2}\right)\hat{f}(\xi_2)\hat{f}(\xi_1-\xi_2)\,d\mu^*(\xi_1)\,d\mu^*(\xi_2).$$

Changing the variables (u,s) and $(\xi_1=(\lambda_1,n),\ \xi_2=(\lambda_2,m))$ to (ta,tb) and $((\sqrt{t}\lambda_1,n),\ (\sqrt{t}\lambda_2,m))$ respectively, and setting $D_t=\left\{(a,b);\ \frac{1}{t}\leqq a\leqq b\leqq 1,\ b-a\geqq\frac{1}{t}\right\}$, we have

$$E(\iint_{\substack{1\leqq u\leqq s\leqq t\\ s-u\geqq 1}}f(X_u)f(X_s)\,du\,ds)$$
$$=\frac{t}{16\pi^4}\iiiint_{D_t\times G^*\times G^*}\exp\left(-\frac{a}{2}(\lambda_1^2+tn^2)-\frac{(b-a)}{2}(\lambda_2^2+tm^2)\right)$$
$$\cdot\hat{f}\left(\frac{\lambda_2}{\sqrt{t}},m\right)\hat{f}\left(\frac{\lambda_1-\lambda_2}{\sqrt{t}},n-m\right)d\mu^*(\xi_1)\,d\mu^*(\xi_2)\,da\,db.$$

We divide the integral into two parts:

$$I_1(t)=\iiiint_{D_t\times R^2}\exp\left(-\frac{a}{2}\lambda_1^2-\frac{(b-a)}{2}\lambda_2^2\right)\hat{f}\left(\frac{\lambda_2}{\sqrt{t}},0\right)\hat{f}\left(\frac{\lambda_1-\lambda_2}{\sqrt{t}},0\right)d\lambda_1\,d\lambda_2\,da\,db,$$

$$I_2(t)=\iiiint_{\substack{D_t\times G^*\times G^*\\ m^2+n^2\neq 0}}\exp\left(-\frac{a}{2}(\lambda_1^2+tn^2)-\frac{(b-a)}{2}(\lambda_2^2+tm^2)\right)$$

$$\cdot\hat{f}\left(\frac{\lambda_2}{\sqrt{t}},m\right)\hat{f}\left(\frac{\lambda_1-\lambda_2}{\sqrt{t}},n-m\right)d\mu^*(\xi_1)\,d\mu^*(\xi_2)\,da\,db.$$

Since $\hat{f}(\xi)$ is a bounded continuous function on G^*, we easily see that

$$\lim_{t\to\infty}I_1(t)=\hat{f}(0)^2\iiiint_{D_\infty\times R^2}\exp\left(-\frac{a}{2}\lambda_1^2-\frac{(b-a)}{2}\lambda_2^2\right)d\lambda_1\,d\lambda_2\,da\,db$$
$$=2\pi^2\hat{f}(0)^2.$$

On the other hand, the boundedness of $\hat{f}$ implies

$$|I_2(t)|\leqq C\iiiint_{D_t\times R^2}\exp\left(-\frac{a}{2}\lambda_1^2-\frac{(b-a)}{2}\lambda_2^2\right)$$
$$\cdot\sum_{m^2+n^2\neq 0}\exp\left(-\frac{a}{2}tn^2-\frac{(b-a)}{2}tm^2\right)d\lambda_1\,d\lambda_2\,da\,db$$
$$=C\iint_{0\leqq a\leqq b\leqq 1}J_t(a,b)\frac{da\,db}{\sqrt{a(b-a)}},$$

where $J_t(a,b)=I_{D_t}(a,b)\sum_{m^2+n^2\neq 0}\exp\left(-\frac{a}{2}tn^2-\frac{(b-a)}{2}tm^2\right)$. ($I_{D_t}$ is the characteristic function of D_t.) Then noting

$$J_t(a,b)\leqq\sum_{m^2+n^2\neq 0}\exp\left(-\frac{n^2+m^2}{2}\right)<\infty$$

and $J_t(a,b)\to 0$ as $t\to\infty$ for each (a,b), by the Lebesgue dominated convergence theorem we see

$$I_2(t)\to 0\quad\text{as}\quad t\to\infty.$$

Consequently we obtain the lemma. Q.E.D.

In null charged case we can get the following convergence in probability:

Lemma 1.3. *Suppose* $f\in L^1(G)\cap L^p(G)$ *for some* $1<p\leqq\infty$, *and* $\int_G f(z)\,d\mu(z)=0$. *Then for each* $T>0$ *and* $\varepsilon>0$,

$$P\left\{\sup_{0\leqq t\leqq T}\left|\frac{1}{\lambda}\int_0^{\lambda^2 t}f(X_s)\,ds\right|>\varepsilon\right\}\to 0\quad as\quad \lambda\to\infty.$$

Proof. 1°) Let $f\in L^1(G)\cap L^p(G)$ for some $1<p\leqq\infty$ and $\int_G f(z)\,d\mu(z)=0$. Define $F(z)$ by

$$F(z) = -\frac{1}{2\pi}\int_0^x dy \int_{-\infty}^{y} \bar{f}(u)\,du + \Gamma_0 f(z). \tag{1.8}$$

Then

$$M_t = F(X_t) - F(0,0) + \int_0^t f(X_s)\,ds$$

becomes a martingale. This is easily seen if f is a smooth function with compact support because by (1.3) we have

$$\tfrac{1}{2}\Delta F = -f.$$

For a general function f, set $f_\varepsilon = P_\varepsilon f$. Then $\int_G f_\varepsilon(z)\,d\mu(z) = 0$ and f_ε converges to f in both $L^1(G)$ and $L^p(G)$. Hence from (1.4), (1.5) and (1.7) it follows that M_t is a martingale even in this case.

2°) For each $\varepsilon > 0$, $P\left\{\sup_{0\leqq t\leqq T} \frac{1}{\lambda}|M_{\lambda^2 t}| > \varepsilon\right\} \to 0 \quad (\lambda\to\infty)$. Since $|M_t|$ is a submartingale, we have

$$P\left\{\sup_{0\leqq t\leqq T} \frac{1}{\lambda}|M_{\lambda^2 t}| > \varepsilon\right\} \leqq \frac{1}{\varepsilon\lambda} E|M_{\lambda^2 T}|.$$

Lemma 1.2 together with $\bar{\bar{f}} = 0$ implies

$$E\left|\int_0^{\lambda^2 T} f(X_s)\,ds\right| = o(\lambda) \quad \text{as} \quad \lambda\to\infty.$$

Let $g(x)$ be the first term in the expression (1.8). Then it is easy to see from $g(x) = o(x)$ that

$$E|g(b_{\lambda^2 T})| = o(\lambda) \quad \text{as} \quad \lambda\to\infty.$$

Noting that the remainder terms of M_t are bounded, we have

$$P\left(\sup_{0\leqq t\leqq T} \frac{1}{\lambda}|M_{\lambda^2 t}| > \varepsilon\right) \to 0 \quad \text{as} \quad \lambda\to\infty.$$

3°) $E\left(\sup_{0\leqq t\leqq T} |g(b_{\lambda^2 t})|\right) = o(\lambda)$ as $\lambda\to\infty$.

Let $m_t^{\pm} = \sup_{0\leqq s\leqq t}(\pm b_s)$ and $h(x) = \sup_{|y|\leqq x}|g(y)|$. Then noting m_t^+ and m_t^- have the same distribution, we have

$$\begin{aligned} E\sup_{0\leqq t\leqq T}|g(b_{\lambda^2 t})| &\leqq E\,h(m_{\lambda^2 T}^+ \vee m_{\lambda^2 T}^-) \\ &\leqq 2E\,h(m_{\lambda^2 T}^+). \end{aligned}$$

Since m_t^+ has the density dominated by $\frac{C}{\sqrt{t}}e^{-x^2/2t}$ and $h(x) = o(x)$, it follows that $E\,h(m_{\lambda^2 T}^+) = o(\lambda)$, which implies the expected estimate. Q.E.D.

Let $l(t)$ be the local time at 0 of a 1-dimensional Brownian motion starting at 0. Denote $\int_G f(z)\,d\mu(z)$ by $\bar{f}$. Then our first theorem is as follows:

Theorem 1.1. *Suppose $f\in L^1(G)\cap L^p(G)$ for some $1<p\leqq\infty$. Then*

$$\left\{\frac{1}{\lambda}\int_0^{\lambda^2 t} f(X_s)\,ds,\ \frac{1}{\lambda}b(\lambda^2 t)\right\}$$

converges weakly in $C([0,\infty)\to\mathbb{R}^2)$ to the process $\{2\bar{f}l(t), b(t)\}$.

Proof. Since we have proved the theorem in case $\bar{f}=0$ in Lemma 1.3, we may assume $\bar{f}\neq 0$. Set

$$\tilde{f}=f-\bar{f}.$$

Then $\tilde{f}\in L^1(G)\cap L^p(G)$ and $\int_G \tilde{f}(z)\,d\mu(z)=0$. Therefore by Lemma 1.3 we see as $\lambda\to\infty$

$$\sup_{0\leqq t\leqq T}\frac{1}{\lambda}\left|\int_0^{\lambda^2 t}\tilde{f}(X_s)\,ds\right|\to 0 \quad \text{in probability.}$$

On the other hand, $\frac{1}{\lambda}\int_0^{\lambda^2 t}\bar{f}(b_s)\,ds=\lambda\int_0^t \bar{f}(b_{\lambda^2 s})\,ds$ holds. Hence noting the equivalence of the two processes $\{b_{\lambda^2 s}\}$ and $\{\lambda b_s\}$, in place of the original process we may consider

$$\lambda\int_0^t \bar{f}(\lambda b_s)\,ds=2\int_{-\infty}^{\infty}\bar{f}(x)\,l\left(t,\frac{x}{\lambda}\right)dx,$$

where $l(t,x)$ is the local time of the Brownian motion at x. Since $\operatorname{supp} l(t,\cdot)=[-m_t^-, m_t^+]$ and $m_t^{\pm}$ are continuous in time-parameter, we easily see that w.p.l.,

$$\sup_{0\leqq t\leqq T} 2\left|\int_{-\infty}^{\infty} l\left(t,\frac{x}{\lambda}\right)\bar{f}(x)\,dx-\int_{-\infty}^{\infty} l(t,0)\bar{f}(x)\,dx\right|\to 0$$

holds as $\lambda\to\infty$ for any $\bar{f}\in L^1(\mathbb{R})$. Q.E.D.

§2. Limit Processes for Additive Functionals of a Brownian Motion on the Cylinder (Null Charged Case)

As we have seen in Lemma 1.3, in null charged case ($\bar{f}=0$), the limit process of $\frac{1}{\lambda}\int_0^{\lambda^2 t} f(X_s)\,ds$ degenerates to a trivial one. Therefore it is necessary to change the normalization to $\frac{1}{\sqrt{\lambda}}\int_0^{\lambda^2 t} f(X_s)\,ds$. The case where the underlying process X_t is a 1-dimensional Brownian motion has been considered by [10] as a corollary of their general limit theorem. Although we are considering a Brownian motion on G, because of the strong ergodicity of a Brownian motion θ_t on a torus, only a slight modification of their procedure is necessary for the proof of our theorem.

Let $f \in L^1(G) \cap L^p(G)$ for some $2 < p \leqq \infty$, and assume $\bar{\bar{f}} = 0$ and $\int_{-\infty}^{\infty} |\bar{f}(x)|\,|x|\,dx < \infty$. Set

$$\Gamma f(z) = F(z) = -\frac{1}{2\pi} \int_{-\infty}^{x} dy \int_{-\infty}^{y} \bar{f}(u)\,du + \Gamma_0 f(z). \tag{2.1}$$

Lemma 1.1 implies that F belongs to C^1-class and its derivatives are bounded, which allows us to put

$$M_t = \int_0^t \frac{\partial F}{\partial x}(X_s)\,db_s + \int_0^t \frac{\partial F}{\partial \theta}(X_s)\,d\theta_s.$$

As we have remarked in (1.3), for a smooth function f with compact support, F satisfies

$$\tfrac{1}{2}\Delta F = -f.$$

This together with Itô's formula gives an identity

$$\int_0^t f(X_s)\,ds = F(0,0) - F(X_t) + M_t. \tag{2.2}$$

For a non-smooth f, by the method of approximation we see that (2.2) is still valid. Since F is bounded, we can consider $\frac{1}{\sqrt{\lambda}} M_{\lambda^2 t}$ in place of $\frac{1}{\sqrt{\lambda}} \int_0^{\lambda^2 t} f(X_s)\,ds$, which is easier to treat. Set

$$g(z) = \left\{\frac{\partial F}{\partial x}(z)\right\}^2 + \left\{\frac{\partial F}{\partial \theta}(z)\right\}^2.$$

Since Lemma 1.1 implies that $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial \theta}$ are bounded and belong to $L^2(G)$, g is also bounded and belongs to $L^1(G)$. Set

$$y_t = \int_0^t g(X_s)\,ds \quad \text{and} \quad Z_\lambda(t) = \left(\frac{1}{\sqrt{\lambda}} M_t, \frac{1}{\lambda} y_t, \frac{1}{\lambda} b_t\right).$$

Lemma 2.1. *The laws P_λ of the processes $Z_\lambda(\lambda^2 t)$ form a tight family.*

Proof. From the Birkholder-Gundy inequality [3], we have

$$E(M_t - M_s)^6 \leqq C E(y_t - y_s)^3 \quad \text{for } t \geqq s \geqq 0.$$

Hence all we have to do is to show that

$$E(y_t - y_s)^3 \leqq C \sqrt{t-s}^{\,3} \quad \text{for } t \geqq s \geqq 0,$$

where C is a constant independent of t, s. Since $g \in L^1(G) \cap L^\infty(G)$, we easily see

$$0 \leqq p_u g(z) \leqq C \sqrt{u}^{-1} \quad \text{for any } u \geqq 0,$$

which implies

$$E(y_t-y_s)^3=3!\iiint\limits_{s\leqq u\leqq v\leqq w\leqq t} p_u g\, p_{v-u} g\, p_{w-v} g(0)\, du\, dv\, dw$$
$$\leqq C\left(\int_s^t \frac{du}{\sqrt{u}}\right)^3 \leqq C\sqrt{t-s}^{\,3}. \quad \text{Q.E.D.}$$

Let ϕ be a smooth function from $\mathbb{R}^d$ to $\mathbb{R}^1$. We denote $\frac{\partial^2 \phi}{\partial x_j \partial x_k}$ by ϕ_{jk}. The limit process can be characterized by the following martingale problem:

Lemma 2.2. *Let P be a probability measure on $\{\omega; \omega: [0,\infty)\to\mathbb{R}^3$ continuous and $\omega(0)=0\}$ such that for all $\phi\in C_0^\infty(R^3)$*

$$\phi(\omega(t))-\tfrac{1}{2}\int_0^t \phi_{11}(\omega(s))\, d\omega_2(s)-\tfrac{1}{2}\int_0^t \phi_{33}(\omega(s))\, ds$$

becomes a martingale and $\omega_2(t)=\lim_{\varepsilon\to 0}\frac{c}{4\varepsilon}\int_0^t I(-\varepsilon,\varepsilon)(\omega_3(s))\, ds$ holds P a.s., where c is a positive nonrandom constant. Then P coincides with the probability measure induced by the process $(b_2(c\, l_1(t)),\ c\, l_1(t),\ b_1(t))$, where (b_1, b_2) is a 2-dimensional Brownian motion and $l_1(t)$ is the local time at 0 of b_1.

Proof. Under the law P, it is easy to see that ω_3 is a 1-dimensional Brownian motion. Therefore $\omega_2(t)=c\, l_1(t)$. Furthermore restricting P to (ω_1,ω_3), we see that $\omega_1(t)$ and $\omega_3(t)$ are locally square integrable martingales such that

$$\langle\omega_i,\omega_j\rangle_t=0 \qquad (i\neq j)$$
$$\langle\omega_1,\omega_1\rangle_t=\omega_2(t)=c\, l_1(t)$$
$$\langle\omega_3,\omega_3\rangle_t=t.$$

Hence applying the theorem of Knight [9], we arrive at the conclusion of the lemma. Q.E.D.

For a suitable null charged function f, define

$$\langle f\rangle=2\int_G\left\{\left(\frac{\partial F}{\partial x}(z)\right)^2+\left(\frac{\partial F}{\partial \theta}(z)\right)^2\right\} d\mu(z)$$
$$=4\int_G f\Gamma f\, d\mu,$$

where $F=\Gamma f$ (see (2.1)). Then our second theorem is as follows:

Theorem 2.1. *Suppose $f\in L^1(G)\cap L^p(G)$ for some $2<p\leqq\infty$, $\int_G f(z)\, d\mu(z)=0$ and $\int_{-\infty}^{\infty} |y|\, |\bar{f}(y)|\, dy<\infty$. Then the processes $\left\{\frac{1}{\sqrt{\lambda}}\int_0^{\lambda^2 t} f(X_s)\, ds, \frac{1}{\lambda} b(\lambda^2 t)\right\}$ converge in law to the process $\{\sqrt{\langle f\rangle}\, b_2(l_1(t)), b_1(t)\}$, where $l_1(t)$ is the local time at 0 of a 1-dimensional Brownian motion $b_1(t)$ and $b_2(t)$ is another Brownian motion independent of $b_1(t)$.*

Proof. 1°) Let P be an arbitrary limit point of $\{p_\lambda\}$. Then there exists some sequence $\{\lambda_n\}$ diverging to infinity such that P_{λ_n} converge weakly to P. From Itô's formula it follows that under each law P_λ and for any $\phi \in C_0^\infty(\mathbb{R}^3)$

$$\phi(\omega(t)) - \tfrac{1}{2}\int_0^t \phi_{11}(\omega(s))\, d\omega_2(s) - \tfrac{1}{2}\int_0^t \phi_{33}(\omega(s))\, ds$$
$$- \sqrt{\lambda}\int_0^t \phi_{13}(\omega(s)) \frac{\partial F}{\partial x}(\lambda\omega(s))\, ds$$

is a martingale. We first show that

$$I_\lambda = E^{P_\lambda}\left[\sup_{0 \leqq t \leqq T} \sqrt{\lambda}\left|\int_0^t \phi_{13}(\omega(s)) \frac{\partial F}{\partial x}(\lambda\omega(s))\, ds\right|\right] \to 0, \quad \text{as} \quad \lambda \to \infty.$$

Since ϕ_{13} is bounded, I_λ is dominated by

$$C\sqrt{\lambda} E\left[\int_0^T \left|\frac{\partial F}{\partial x}(\lambda b_s, \lambda\theta_s)\right| ds\right] = C\sqrt{\lambda}^{-3} E\left[\int_0^{\lambda^2 T} \left|\frac{\partial F}{\partial x}(b_s, \theta_s)\right| ds\right].$$

On the other hand, noting $\frac{\partial F}{\partial x} \in L^1(G) \cap L^\infty(G)$, we have by Lemma 1.1

$$E\left[\int_0^{\lambda^2 T} \left|\frac{\partial F}{\partial x}(b_s, \theta_s)\right| ds\right] = O(\lambda) \quad \text{as} \quad \lambda \to \infty.$$

Hence $I_\lambda = O\left(\frac{1}{\sqrt{\lambda}}\right)$ as $\lambda \to \infty$.

2°) Let $\Phi(t, \omega)$ be a bounded continuous function. Then

$$E^{P_{\lambda_n}}\left[\int_0^t \Phi(s)\, d\omega_2(s)\right] \to E^P\left[\int_0^t \Phi(s)\, d\omega_2(s)\right] \quad \text{as} \quad \lambda \to \infty.$$

To prove this, first note that $\int_0^t \Phi(s, \omega)\, d\omega_2(s)$ is a continuous function of ω. This is comes from the following facts: if $\omega_n \to \omega$ then $\Phi(\cdot, \omega_n) \to \Phi(\cdot, \omega)$ uniformly on $[0, t]$ and $(\omega_n)_2 \to \omega_2$ uniformly on $[0, t]$ and hence $d(\omega_n)_2 \to d\omega_2$ weakly on $[0, t]$. Let $f_L(x)$ be a smooth function such that $0 \leqq f_L(x) \leqq 1$ and

$$f_L(x) = \begin{cases} 1 & 0 \leqq |x| \leqq L \\ 0 & L+1 \leqq |x| < \infty. \end{cases}$$

Then we have

$$\begin{aligned} E^{P_\lambda}&\left[\int_0^t \Phi(s)\, d\omega_2(s)\right] \\ &= E^{P_\lambda}\left[\int_0^t \Phi(s)\, d\omega_2(s) f_L(\omega_2(t))\right] \\ &\quad + E^{P_\lambda}\left[\int_0^t \Phi(s)\, d\omega_2(s)(1 - f_L(\omega_2(t)))\right]. \end{aligned}$$

As we have seen above, $\int_0^t \Phi(s,\omega)\,d\omega_2(s)$ is a continuous function of ω. Hence $\int_0^t \Phi(s,\omega)\,d\omega_2(s) f_L(\omega_2(t))$ is a bounded continuous function of ω. On the other hand, we have

$$\begin{aligned}&\left| E^{P_\lambda}\left[\int_0^t \Phi(s)\,d\omega_2(s)(1-f_L(\omega_2(t)))\right]\right| \\ &\quad \leqq C E^{P_\lambda}[\omega_2(t) : \omega_2(t) \geqq L] \\ &\quad \leqq \frac{C}{L} E^{P_\lambda}[\omega_2(t)^2] \\ &\quad = \frac{C}{L} E \frac{1}{\lambda^2}\left(\int_0^{\lambda^2 t} g(X_s)\,ds\right)^2 \\ &\quad \leqq Ct/L \quad \text{(see Lemma 1.2).}\end{aligned}$$

Therefore $E^{P_{\lambda_n}}\left[\int_0^t \Phi(s)\,d\omega_2(s)\right] \to E^P\left[\int_0^t \Phi(s)\,d\omega_2(s)\right]$ as $\lambda_n \to \infty$.

3°) P satisfies the conditions of Lemma 2.2.

Under the probability measure P, $\{\omega_3(t)\}$ is a Brownian motion. Further from Theorem 1.1 $\omega_2(t)$ is $2\bar{\bar{g}}$ time the local time at 0 of $\{\omega_3(t)\}$ (note $g \in L^1(G) \cap L^\infty(G)$). This together with the above arguments 1°) and 2°) shows that the measure P satisfies the conditions of Lemma 2.2. Q.E.D.

Remark. In the above two theorems, it is seen that the process $\{\theta_t\}$ has no influence on the results but on the change of the parameters of the limit processes. This comes from the strong ergodicity of $\{\theta_t\}$. Therefore it is natural to expect that any sufficiently ergodic process will lead us to the same conclusions.

§3. Two Limit Theorems for 2-Dimensional Brownian Motions

Let $D = D([0,\infty); \mathbb{R}^n)$ be the set of all $\mathbb{R}^n$-valued right-continuous functions with left-limits. We then define the graph $\Gamma_{x(t)}$ of $x(t) \in D$ as the smallest closed set in $\mathbb{R}^n \times [0,\infty)$ which contains all pairs (x,t) such that x belongs to the segment joining $x(t-)$ and $x(t)$. The pair of functions $(y(s), t(s))$ is called a parametric representation of the graph $\Gamma_{x(t)}$ if those and only those pairs (x,t) belong to it for which an s can be found such that $x = y(s)$, $t = t(s)$, where $y(s)$ is continuous and $t(s)$ is continuous and nondecreasing. A sequence $\{x_n(t)\} \subset D$ is called M_1-convergent to $x_0(t)$ if there exist parametric representations $(y_n(s), t_n(s))$ of $\Gamma_{x_n(t)}$ such that

$$\lim_{n\to\infty} \sup_{0\leqq s\leqq T} (|y_n(s) - y_0(s)| + |t_n(s) - t_0(s)|) = 0$$

for each $T > 0$ (see Skorokhod [11]). Clearly, if $x_n(t)$ and $x_0(t)$ are continuous functions, M_1-convergence implies the convergence in C (i.e. the uniform

convergence on each compact set). We next define the weak M_1-convergence of stochastic processes. Let $\{X_n(t)\}$ be a sequence of D-valued stochastic processes. Then $\{X_n(t)\}$ is said to be weak M_1-convergent to $X_0(t)$ if there exists a sequence $\{\tilde{X}_n(t), n=0,1,\ldots\}$ such that
(i) For each $n \geqq 0$, $\tilde{X}_n(\cdot)$ is equivalent in law to $X_n(\cdot)$.
(ii) $\tilde{X}_n(\cdot)$ is M_1-convergent to $\tilde{X}_0(\cdot)$ a.s.

Theorem 3.1. *Let $B(t)$ be a 2-dimensional Brownian motion and $V(x)$; $x \in \mathbb{R}^2$ be a bounded function such that*

$$\int |V(x)|\,|x|^\alpha\,dx < \infty$$

for some $\alpha > 0$. Then

$$\frac{1}{\lambda}\int_0^{n(\lambda t)} V(B(s))\,ds$$

is weakly M_1-convergent to $2\bar{V} l(M^{-1}(t))$ as $\lambda \to \infty$, where $n(t) = t e^{2t}$, $\bar{V} = \int V(x)\,dx$, $l(t)$ is the local time at 0 of a 1-dimensional Brownian motion $b(t)$ and $M(t) = \max\limits_{0 \leqq s \leqq t} b(s)$.

Theorem 3.2. *Let $B(t)$ be a 2-dimensional Brownian motion and $V(x)$ be a bounded function such that $\int |V(x)|\,|x|^\alpha\,dx$ for some $\alpha > 2$ and $\int V(x)\,dx = 0$. Then*

$$\frac{1}{\sqrt{\lambda}}\int_0^{n(\lambda t)} V(B(s))\,ds$$

is weakly M_1-convergent to $\sqrt{\langle V\rangle}\, b_2(l_1(M^{-1}(t)))$ as $\lambda \to \infty$ where $n(t) = t e^{2t}$, $\langle V\rangle = -\dfrac{4}{\pi}\iint \log|x-y|\,V(x)V(y)\,dx\,dy$, $b_2(t)$ is a 1-dimensional Brownian motion, $l_1(t)$ the local time at 0 of a Brownian motion $b_1(t)$ which is independent of b_2, and $M(t) = \max\limits_{0 \leqq s \leqq t} b_1(s)$.

Remark. Throughout this paper we mean a process starting at 0 whenever we speak of a Brownian motion. However, the assertions of Theorems 3.1 and 3.2 do not depend upon the starting point $x_0 = B(0)$ as far as it is nonrandom since we can replace $V(x)$ by $V(x + x_0)$. So, in the proof of the theorem, we will assume $B(0) = (1, 0)$, for convenience.

Proof. We will prove only Theorem 3.1, because the proof of Theorem 3.2 proceeds similarly. First we reduce the functional $\int_0^t V(B(s))\,ds$ of 2-dimensional Brownian motion to that of Brownian motion on G.

Let $b(t)$ be a 1-dimensional Brownian motion and define

$$S(t) = \int_0^t e^{2b(s)}\,ds.$$

Since, as is well-known, $X(t) = b(S^{-1}(t))$ is a diffusion process with generator

$\frac{1}{2}e^{-2x}\frac{d^2}{dx^2}$, $e^{X(t)}$ turns out to be a Bessel process with exponent 2. Every 2-dimensional Brownian motion starting at (1,0) can be represented by the skew product formula:

$$\exp\left\{X(t)+i\theta\left(\int_0^t e^{-2X(s)}ds\right)\right\},$$

if we take a 1-dimensional Brownian motion $\theta(t)$ which is independent of $b(t)$ (cf. Itô-McKean [4], p. 270). Therefore setting

$$f(u,\theta)=V(e^{u+i\theta})e^{2u}, \qquad (u,\theta)\in\mathbb{R}\times(\mathbb{R}/2\pi\mathbb{Z}),$$

we see that, as stochastic process, $V(B(t))$ is equivalent in law to

$$f(b(S^{-1}(t)),\theta(S^{-1}(t)))\exp\{-2b(S^{-1}(t))\}.$$

This implies that $\int_0^{n(\lambda t)} V(B(s))\,ds$ is equivalent in law to

$$A_\lambda(t)=\int_0^{S^{-1}(n(\lambda t))} f(b(s),\theta(s))\,ds.$$

Let $T_\lambda(t)$ be defined by

$$S^{-1}(n(\lambda T_\lambda(t)))=\lambda^2 t,$$

i.e., $T_\lambda(t)=\frac{1}{\lambda}n^{-1}(S(\lambda^2 t))$. Then our functional $\int_0^{n(\lambda t)} V(B(s))\,ds$ can be reduced to

$$A_\lambda(T_\lambda(t))=\int_0^{\lambda^2 t} f(b(s),\theta(s))\,ds$$

by the time change $t\to T_\lambda(t)$. If we can prove that $\left(\frac{1}{\lambda}A_\lambda(T_\lambda(t)),T_\lambda(t)\right)$ converges in law to $(2\bar{V}l(t),M(t))$, then combining the definition of weak M_1-convergence with Skorohod's theorem ([11])[1] we can finish the proof of our theorem. It is easy to show that $f(u,\theta)\in L^1(G)\cap L^p(G)$ (for some $p>1$) and that

$$\bar{f}=\int_G f(u,\theta)\,d\mu=\int_{\mathbb{R}^2} V(x)\,dx=\bar{V}.$$

Therefore, by Theorem 1.1, $\left(\frac{1}{\lambda}A_\lambda(T_\lambda(t)),\frac{1}{\lambda}b(\lambda^2 t)\right)$ converges in law to $(2\bar{V}l(t),b(t))$ as $\lambda\to\infty$. On the other hand we will prove in Lemma 3.1 below that $\left(\frac{1}{\lambda}b(\lambda^2 t),T_\lambda(t)\right)$ is weakly convergent to $(b(t),M(t))$. Let P_λ be the law induced on the space of continuous functions $C([0,\infty)\to\mathbb{R}^3)$ by the process

[1] The theorem asserts that the convergence in law of random variables on a separable complete metric space can be realized by an almost everywhere convergence without changing the law of each random variable.

$\left(\frac{1}{\lambda}A_\lambda(T_\lambda(t)), \frac{1}{\lambda}b(\lambda^2 t), T_\lambda(t)\right)$. Then the family $\{P_\lambda\}$ is tight because each component converges in law. Further it is easy to see from the above argument that any limit point P^* of $\{P_\lambda\}$ coincides with the law P induced by $(2\sqrt{V}l(t), b(t), M(t))$. Hence, P_λ itself converges to P. Q.E.D.

Lemma 3.1. $\left(\frac{1}{\lambda}b(\lambda^2 t), T_\lambda(t)\right)$ *is weakly convergent to* $(b(t), M(t))$ *as* $\lambda\to\infty$, *where* $M(t)=\max_{0\leqq s\leqq t} b(s)$.

Proof. Set $\tilde{T}_\lambda(t)=\frac{1}{\lambda}n^{-1}\left(\lambda^2\int_0^t e^{2\lambda b(s)}ds\right)$ $(n(t)=te^{2t})$. Then $\left(\frac{1}{\lambda}b(\lambda^2 t), T_\lambda(t)\right)$ is equivalent in law to $(b(t), \tilde{T}_\lambda(t))$, so we have only to show

$$\lim_{\lambda\to\infty}\sup_{0\leqq t\leqq T}|\tilde{T}_\lambda(t)-M(t)|=0 \qquad \text{for every } T>0.$$

Fix $\varepsilon>0$ and $\delta>0$. Then for any $t\in[\delta, T]$ we have

$$\begin{aligned}\lambda^2\int_0^t e^{2\lambda b(s)}ds &\leqq \lambda^2 T e^{2\lambda M(t)}\\ &\leqq \frac{T}{M(\delta)}\lambda e^{-2\lambda\varepsilon}\lambda(M(t)+\varepsilon)e^{2\lambda(M(t)+\varepsilon)}.\end{aligned}$$

If we set $C_\varepsilon(T)=T(2\varepsilon e M(\delta))^{-1}$, then noting the inequality $\lambda e^{-2\lambda\varepsilon}\leqq(2\varepsilon e)^{-1}$, we see

$$\tilde{T}_\lambda(t)\leqq\frac{1}{\lambda}n^{-1}(C_\varepsilon(T)n(\lambda\{M(t)+\varepsilon\})).$$

Since $n^{-1}(\cdot)$ is slowly varying, there exists $\lambda_0\geqq 0$ such that

$$\begin{aligned}n^{-1}(C_\varepsilon(T)n(\lambda\{M(t)+\varepsilon\}))&\leqq(1+\varepsilon)n^{-1}(n(\lambda\{M(t)+\varepsilon\}))\\ &=(1+\varepsilon)\lambda\{M(t)+\varepsilon\}\end{aligned}$$

holds for every $\lambda>\lambda_0$ and $t\in[\delta, T]$, which implies

$$\overline{\lim_{\lambda\to\infty}}\sup_{\delta\leqq t\leqq T}\{\tilde{T}_\lambda(t)-M(t)\}\leqq 0. \tag{3.1}$$

The converse inequality

$$\underline{\lim}_{\lambda\to\infty}\inf_{\delta\leqq t\leqq T}\{\tilde{T}_\lambda(t)-M(t)\}\geqq 0 \tag{3.2}$$

can be proved in the following way: Let $\phi(t)$ be the Lebesgue measure of $\{s\in[0,t]; b(t)\geqq M(t)-\varepsilon\}$. Then w.p. 1, $\phi(t)$ is a positive continuous function on $(0,\infty)$, and hence there exists a positive constant c such that $\phi(t)\geqq c$ for every $t\in[\delta, T]$. This gives an inequality

$$\lambda^2\int_0^t e^{2\lambda b(s)}ds\geqq\lambda^2 c e^{2\lambda(M(t)-\varepsilon)}.$$

Consequently we can prove (3.2) by a similar argument as above. (3.1) and (3.2) imply

$$\lim_{\lambda\to\infty} \sup_{\delta\leqq t\leqq T} |\tilde{T}_\lambda(t)-M(t)|=0.$$

To complete the proof we have only to note that

$$\sup_{0\leqq t\leqq\delta} |\tilde{T}_\lambda(t)-M(t)|\leqq \tilde{T}_\lambda(\delta)+M(\delta)$$
$$\leqq|\tilde{T}_\lambda(\delta)-M(\delta)|+2M(\delta)$$

and that $M(\delta)\to 0$ as $\delta\to 0$.

§4. Limit Theorems for 1-Dimensional Diffusion Processes

The idea we used in the previous section can be applied to some other diffusion processes. Let $m(x)$ be a right-continuous, nondecreasing function defined on $\mathbb{R}$. Then we can define the Lebesgue-Stieltjes measure $dm(x)$. We exclude the trivial case that $dm(x)$ vanishes identically, so, without loss of generality we can assume that $0\in\operatorname{supp} dm(x)$ and that $m(0)=0$. Let $b(t)$ be a Brownian motion and $l(t,x)$ its local time. It is well known that $X(t)=b(S^{-1}(t))$ becomes a strong Markov process on the support of $dm(x)$ if $S^{-1}(t)$ denotes the right-continuous inverse to $S(t)=S(t,m)=\int l(t,x)\,dm(x)$ (cf. [4, 12]). $X(t)$ is called the (generalized) diffusion process associated with $m(x)$. The local generator of $X(t)$ is given by $\frac{d}{dm}\frac{d}{dx}$. $L_X(t,x)=l(S^{-1}(t),x)$ is called the local time of $X(t)$ because, for every bounded function $f(x)$, we have

$$\int_0^t f(X(s))\,ds=\int_{-\infty}^{\infty} L_X(t,x)f(x)\,dm(x).$$

In this section we study limit theorems for an additive functional $\int_{-\infty}^{\infty} L_X(t,x)\,dF(x)$ where $dF(x)$ is any (signed) finite measure whose support is contained in that of $dm(x)$. Remark that if $dF(x)$ is absolutely continuous with respect to $dm(x)$, then

$$\int_{-\infty}^{\infty} L_X(t,x)\,dF(x)=\int_0^t f(X(s))\,ds$$

where $f(x)=\frac{dF(x)}{dm(x)}$.

First we state two lemmas without proof since we need only a little modification to the proof of Theorems 1.1 and 2.1.

Lemma 4.1. *Let $dF(x)$ be a finite (signed) Borel measure on $\mathbb{R}$. Then, for each $T>0$,*

$$\lim_{\lambda\to\infty}\sup_{0\leqq t\leqq T}\left|\int_{-\infty}^{\infty} l(t,x)\,dF(\lambda x)-\bar{F}\,l(t,0)\right|=0, \qquad a.s.,$$

where $\bar{F}=F(+\infty)-F(-\infty)$.

Lemma 4.2. *Let* $dF(x)$ *be a finite Borel measure with compact support such that* $\bar{F}=\int dF(x)=0$. *Then,* $(\sqrt{\lambda}\int l(t,x)\,dF(\lambda x), b(t))$ *converges in law to* $(c\,b_2(l_1(t)), b_1(t))$ *as* $\lambda\to\infty$, *where* $(b_1(t), b_2(t))$ *is a 2-dimensional Brownian motion,* $l_1(t)$ *the local time at* 0 *of* $b_1(t)$ *and* $c=\left\{2\int_{-\infty}^{\infty}(F(x)-F(-\infty))^2\,dx\right\}^{1/2}$.

Before we state our theorem, we remark some properties of regularly varying functions. A function $r(x)$ defined on $(0,\infty)$ is called a regularly varying function (at ∞) with exponent β $(0\leqq\beta\leqq\infty)$ if and only if

$$\lim_{x\to\infty}\frac{r(\lambda x)}{r(x)}=\lambda^{\beta}, \qquad \lambda>0 \tag{4.1}$$

where $\lambda^{\infty}=\infty$ if $\lambda>1$ and $=0$ if $0<\lambda<1$. A regular varying function with exponent 0 is called a slowly varying function. It is easy to see that in case $\beta<\infty$ a function $r(x)$ varies regularly with exponent β if and only if $r(x)=x^{\beta}L(x)$ for some slowly varying $L(x)$. In case $\beta=\infty$ we often call $r(x)$ a rapidly increasing function. Notice that an increasing, continuous function $r(x)$ varies regularly with exponent β if and only if $r^{-1}(x)$ so does with exponent $1/\beta$ including the case $\beta=0$ or ∞ under the convention $1/0=\infty$ and $1/\infty=0$. For instance, $\log x$ varies slowly and e^x increases rapidly.

From now on we consider generalized diffusion processes associated with speed measures $dm(x)$ satisfying the following condition (C). There exist a continuous, increasing, regularly varying function $n(\lambda)$ with exponent α $(1\leqq\alpha\leqq\infty)$ such that

(i) if $\alpha<\infty$,

$$\lim_{\lambda\to\infty}\frac{\lambda m(\lambda)}{n(\lambda)}=c_1 \qquad (\geqq 0),$$

$$\lim_{\lambda\to-\infty}\frac{\lambda m(\lambda)}{n(\lambda)}=-c_2 \qquad (\leqq 0)$$

$(0<c_1+c_2<\infty)$.

(ii) if $\alpha=\infty$,

$$\lim_{\lambda\to\infty}\frac{\lambda m(\lambda x)}{n(\lambda)}=\begin{cases}\infty, & a_1<x\\ 0, & -a_2<x<a_1\\ -\infty, & x<-a_2\end{cases}$$

$(0<a_1<\infty, 0<a_2\leqq\infty$ or $0<a_1\leqq\infty, 0<a_2<\infty)$.

Set

$$\tilde{T}_\lambda(t)=\frac{1}{\lambda}n^{-1}(\lambda\int l(t,x)\,dm(\lambda x)) \tag{4.2}$$

and

$$T(t)=\begin{cases}(\int l(t,x)\,dm^*(x))^{1/\alpha}, & \alpha<\infty\\ \max\limits_{0\leqq s\leqq t}\left\{\dfrac{b(s)}{a_1}, -\dfrac{b(s)}{a_2}\right\}, & \alpha=\infty\end{cases} \tag{4.3}$$

where

$$m^*(x)=\begin{cases} c_1 x^{\alpha-1}, & x>0\\ -c_2|x|^{\alpha-1}, & x<0.\end{cases}$$

Then we have

Lemma 4.4. *For every $T>0$,*

$$\lim_{\lambda\to\infty}\sup_{0\leqq t\leqq T}|\tilde{T}_\lambda(t)-T(t)|=0 \quad a.s.$$

Proof. Since $l(t,x)$ is continuous in (t,x) and has compact support in $[0,T]\times\mathbb{R}$, it follows from (C) that

$$\lim_{\lambda\to\infty}\frac{\lambda}{n(\lambda)}\int l(t,x)\,dm(\lambda x)=\int l(t,x)\,dm^*(x).$$

Note that $n^{-1}(\lambda)$ varies regularly with exponent $1/\alpha$ because so does $n(\lambda)$ with exponent α. Therefore, if $\alpha<\infty$,

$$\begin{aligned}\lim_{\lambda\to\infty}&\frac{1}{\lambda}n^{-1}(\lambda\int l(t,x)\,dm(\lambda x))\\ &=\lim_{\lambda\to\infty}\frac{1}{\lambda}n^{-1}(n(\lambda)\int l(t,x)\,dm^*(x))\\ &=\lim_{\lambda\to\infty}\frac{1}{\lambda}n^{-1}(n(\lambda))\,T(t)\\ &=T(t),\end{aligned}$$

which proves the assertion because the convergence is clearly uniform in t ($\leqq T$).

In case $\alpha=\infty$, notice that the support of $l(t,\cdot)$ is the interval $[M_-(t), M_+(t)]$ where $M_-(t)=\min\limits_{0\leqq s\leqq t} b(s)$ and $M_+(t)=\max\limits_{0\leqq s\leqq t} b(s)$. On the other hand, by (C), we have

$$\lim_{\lambda\to\infty}\frac{\lambda m(c\lambda)}{n(\lambda)}=\infty, \qquad c>c_1.$$

This, combined with the continuity of the local time, implies

$$\lambda \int_0^\infty l(t,x)\, dm(\lambda x) \geqq \text{const.} \cdot m(\lambda\{M_+(t)-\delta\})$$
$$\geqq n\left(\frac{\lambda}{c}\{M_+(t)-\delta\}\right)$$

for all $c > c_1$ and all sufficiently large λ. Therefore,

$$\begin{aligned}
&\varliminf_{\lambda\to\infty} \tilde{T}_\lambda(t)\\
&\quad \geqq \varliminf_{\lambda\to\infty} \frac{1}{\lambda} n^{-1}\left(\lambda \int_0^\infty l(t,x)\, dm(\lambda x)\right)\\
&\quad \geqq \varliminf_{\lambda\to\infty} \frac{1}{\lambda}\left(\frac{\lambda}{c}(M_+(t)-\delta)\right)\\
&\quad = \frac{1}{c}(M_+(t)-\delta).
\end{aligned}$$

Letting $c \downarrow c_1$ and $\delta \downarrow 0$, we have

$$\varliminf_{\lambda\to\infty} \tilde{T}_\lambda(t) \geqq \frac{1}{c_1} M_+(t).$$

We can also prove

$$\varliminf_{\lambda\to\infty} \tilde{T}_\lambda(t) \geqq -\frac{1}{c_2} M_-(t),$$

and hence we have

$$\begin{aligned}
\varliminf_{\lambda\to\infty} \tilde{T}_\lambda(t) &\geqq \max\left\{\frac{1}{c_1} M_+(t), \frac{-1}{c_2} M_-(t)\right\}\\
&= T(t).
\end{aligned}$$

By a similar (but easier) argument we can obtain

$$\varlimsup_{\lambda\to\infty} \tilde{T}_\lambda(t) \leqq T(t).$$

Thus we have

$$\lim_{\lambda\to\infty} \tilde{T}_\lambda(t) = T(t).$$

It is easy to see that the convergence is uniform on every compact interval in $[0, \infty)$, which implies the assertion (see the proof of Lemma 3.1.).

Theorem 4.1. *Assume* (C) *and let* $dF(x)$ *be a finite measure whose support is compact and contained in that of* $dm(x)$. *Then,*

(i) $\frac{1}{\lambda}\int L_X(n(\lambda t), x)\, dF(x)$ *is weakly* M_1*-convergent to* $\bar{F}\, l(T^{-1}(t), 0)$ *as* $\lambda \to \infty$ *where* $\bar{F} = F(+\infty) - F(-\infty)$ *and* $T(t)$ *is the process defined in* (4.3).

(ii) *In case* $\bar{F} = 0$,

$\frac{1}{\sqrt{\lambda}}\int L_X(n(\lambda t)), x)\, dF(x)$ *is weakly* M_1*-convergent to* $\sqrt{\langle F\rangle}\,\tilde{b}(l(T^{-1}(t), 0))$ *as* $\lambda \to \infty$ *where* $\langle F\rangle = 2\int_{-\infty}^{\infty} (F(x) - F(-\infty))^2\, dx$ *and* $\tilde{b}(t)$ *is a Brownian motion which is independent of* $b(t)$.

Proof. Define $\bar{T}_\lambda(t) = \frac{1}{\lambda} n^{-1}(\int l(\lambda^2 t, x)\, dm(x))$. We will prove that the joint distributions of

$$\left(\frac{1}{\lambda}\int L_X(n(\lambda \bar{T}_\lambda(t), x)), x)\, dF(x), \bar{T}_\lambda(t)\right)$$

converge to that of $(\bar{F}\, l(t, 0), T(t))$. In fact it is easy to see that this implies (i) if we take into account of the Skorokhod theorem and the definition of weak M_1-convergence. First note, for $x \in \operatorname{supp} dm$, $l(S^{-1}(S(t)), x) = l(t, x)$ $(S(t) = \int l(t, x)\, dF(x))$ and hence

$$\begin{aligned}
&\frac{1}{\lambda}\int L_X(n(\lambda \bar{T}_\lambda(t)), x)\, dF(x)\\
&\quad = \frac{1}{\lambda}\int L_X(S(\lambda^2 t), x)\, dF(x)\\
&\quad = \frac{1}{\lambda}\int l(S^{-1}(S(\lambda^2 t)), x)\, dF(x)\\
&\quad = \frac{1}{\lambda}\int l(\lambda^2 t, x)\, dF(x).
\end{aligned}$$

Furthermore, since $\frac{1}{\lambda} l(\lambda^2 t, \lambda x)$ is the local time of $\frac{1}{\lambda} b(\lambda^2 t)$, we easily see that $\left(\frac{1}{\lambda}\int l(\lambda^2 t, x)\, dF(x), \frac{1}{\lambda} b(\lambda^2 t), \bar{T}_\lambda(t)\right)$ is equivalent in law to

$$\left(\int l\left(t, \frac{x}{\lambda}\right) dF(x), b(t), \tilde{T}_\lambda(t)\right), \tag{4.4}$$

where $\tilde{T}_\lambda(t)$ is defined in (4.2). Let $\{P_\lambda\}$ denote the law induced in $C([0, \infty) \to \mathbb{R}^3)$ by the process (4.4). Then $\{P_\lambda\}$ is precompact because $\left(\int l\left(t, \frac{x}{\lambda}\right) dF(x), b(t)\right)$ and $(b(t), \tilde{T}_\lambda(t))$ are weakly convergent to $(\bar{F}\, l(t, 0), b(t))$ and $(b(t), T(t))$, respectively as we have seen in Lemmas 4.1 and 4.4. Let P^* be any limit point of $\{P_\lambda\}$ and denote by $(x(t), y(t), z(t))$ the element of $C([0, \infty) \to \mathbb{R}^3)$. Then, using Lemmas 4.1 and 4.4 again, we see that the marginal distribution of $(x(t), y(t))$ is equal to that of $(\bar{F}\, l(t, 0), b(t))$ and that the marginal distribution of $(y(t), z(t))$ equals that of

$(b(t), T(t))$. Therefore, noting that $T(t)$ and $l(t,x)$ are functionals of $b(t)$, we see that P^* equals the law of $(\bar{F}\,l(t,0), b(t), T(t))$, which implies that $\left(\frac{1}{\lambda}\int l(\lambda^2 t, x)\, dF(x), \frac{1}{\lambda} b(\lambda^2 t), \tilde{T}_\lambda(t)\right)$ is weakly convergent to $(\bar{F}\,l(t,0), b(t), T(t))$. This, as we have mentioned, proves (i). Similarly we can prove (ii), but we omit the details. Q.E.D.

Remarks. (i) In case $\alpha < \infty$, we can obtain similar limit theorems for $\int L_X(n(\lambda)t, x)\, dF(x)$ (see C. Stone [12]). But since $n(\lambda t) \sim t^\alpha n(\lambda)$, there is no crucial difference from the type we treated. So we omit the details.

(ii) Weak M_1-convergence, in general, does not necessarily imply the convergence of finite-dimensional marginals. However, in all our cases, the first implies the latter since the limit processes have no fixed-discontinuities.

References

1. Darling, D.A., Kac, M.: On occupation times for Markov processes. Trans. Amer. Math. Soc., **84**, 444–458 (1957)
2. Doburusin, R.L.: Two limit theorems for the simplest random walk on a line. Uspehi Mat. Nauk. **10**, 139–146 (1955) (English translation)
3. Getoor, R.K., Sharpe, M.J.: Conformal martingales. Invent. Math. **16**, 271–308 (1972)
4. Itô, K., McKean Jr., H.P.: Diffusion processes and their sample paths. Berlin-Heidelberg-New York: Springer 1965
5. Kallianpur, G., Robbins, H.: Ergodic property of the Brownian motion process. Proc. Nat. Acad. **39**, 525–533 (1953)
6. Karlin, S., McGregor, J.L.: Occupation time laws for birth and death processes. Proc. 4th Berkeley Sympos. Math. Statist. Probab. Univ. Calif. vol. **2**, 249–272 (1960)
7. Kasahara, Y.: Spectral theory of generalized second order differential operators and its applications to Markov processes. Japan. J. Math. **1**, 67–84 (1975)
8. Kasahara, Y.: Limit theorems of occupation times for Markov processes. Publ. RIMS, Kyoto Univ. **12**, 801–818 (1977)
9. Knight, F.B.: A reduction of continuous square-integrable martingale to Brownian motion. Lecture notes in Math. **190**, 19–31. Berlin-Heidelberg-New York: Springer 1971
10. Papanicolaou, G.C., Stroock, D., Varadhan, S.R.S.: Martingale approach to some limit theorems. Duke Univ. Mathematics series III, Statistical Mechanics and Dynamical systems, 1977
11. Skorokhod, A.V.: Limit theorems for stochastic processes. (English transl.) Theor. Probability Appl. **1**, 261–290 (1956)
12. Stone, C.: Limit theorems for random walks, birth and death processes, and diffusion processes. Illinois J. Math. **7**, 638–660 (1963)
13. Watanabe, S.: A limit theorem for sums of non-negative i.i.d. random variables with slowly varying tail probabilities. (To appear in Proc. 5th International Multivariate Analysis.)

Received May 15, 1978