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How Big are the Increments of a Multi-Parameter Wiener Process? ⋆

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Summary. Let $\beta_T=(2a_T(\log T a_T^{-1}+\log\log T))^{-\frac{1}{2}}$, $0<a_T\leqq T<\infty$ and let R^* be the set of sub-rectangles of the square $[0,T^{\frac{1}{2}}]\times[0,T^{\frac{1}{2}}]$, having an area a_T. This paper studies the almost sure limiting behaviour of $\beta_T \sup_{R\in\mathsf{R}^*}|W(R)|$ as $T\to\infty$, where W is a two-time parameter Wiener process. With $a_T=T$, our results give the well-known law of iterated logarithm and a generalization of the latter is also attained. The multi-time parameter analogues of our two-time parameter Wiener process results are also stated in the text.

1. Introduction

In [2] we have proved the following

Theorem A. *Let $W(t)$ $(0\leqq t<\infty)$ be a standard Wiener process and let a_T be a non-decreasing function of T for which*

(i) $0<a_T\leqq T \quad (T\geqq 0)$,

and

(ii) $T a_T^{-1}$ *is non-decreasing.*

Then

$$(1)\quad \limsup_{T\to\infty}\sup_{0\leqq t\leqq T-a_T}\beta_T|W(t+a_T)-W(t)| =\limsup_{T\to\infty}\sup_{0\leqq t\leqq T-a_T}\sup_{0\leqq s\leqq a_T}\beta_T|W(t+s)-W(t)|=1$$

w.p.1. where $\beta_T=(2a_T(\log T a_T^{-1}+\log\log T))^{-\frac{1}{2}}$.

If we also have

$$\text{(iii)}\quad \lim_{T\to\infty}\frac{\log T a_T^{-1}}{\log\log T}=\infty,$$

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then

$$(2)\quad \lim_{T\to\infty} \sup_{0\le t\le T-a_T} \beta_T |W(t+a_t)-W(t)|$$

$$= \lim_{T\to\infty} \sup_{0\le t\le T-a_T} \sup_{0\le s\le a_T} \beta_T |W(t+s)-W(t)| = 1 \qquad w.p.1.$$

In order to demonstrate what this theorem is all about we mention that the special case $a_T = T$ gives the well-known law of iterated logarithm, while the case $a_T = c\log T$ $(c>0)$ implies the Erdös-Rényi law of large numbers ([3]) when it is applied to a Wiener process.

In this paper we intend to state and prove an analogue of this theorem for a multi-time parameter Wiener process. At first we formulate our results for a two-time parameter Wiener process. The multi-time parameter analogues of these results are given in Section 4 without proofs, for they are entirely similar to the two-time parameter case. The same can be said about the proof of Theorem A.

In order to formulate a possible two-time parameter analogue of Theorem A we introduce the following notations:

Let $\mathsf{R}_T = \mathsf{R}(a_T)$ be the set of rectangles

$$R = [x_1, x_2] \times [y_1, y_2] \qquad (0 \le x_1 < x_2 \le T^{\frac{1}{2}}, 0 \le y_1 < y_2 \le T^{\frac{1}{2}})$$

for which $\lambda(R) = (x_2 - x_1)(y_2 - y_1) \le a_T$. Let $\mathsf{R}^*_T = \mathsf{R}^*(a_T) \subset \mathsf{R}_T$ be the set of those elements R of R_T for which $\lambda(R) = a_T$. For a 2-parameter Wiener process $W(x,y)$ $(0 \le x, y < \infty)$ define the Wiener measure of a rectangle $R = [x_1, x_2] \times [y_1, y_2]$ by

$$W(R) = W(x_2, y_2) - W(x_1, y_2) - W(x_2, y_1) + W(x_1, y_1).$$

Now we state our

Theorem 1. *Let $W(x,y)$ $(0 \le x, y < \infty)$ be a Wiener process and let a_T be a non-decreasing function of T satisfying conditions* (i)–(ii) *of Theorem A. Then*

$$(3)\quad \limsup_{T\to\infty} \sup_{R\in\mathsf{R}_T} \beta_T |W(R)| = \limsup_{T\to\infty} \sup_{R\in\mathsf{R}^*_T} \beta_T |W(R)| = 1$$

w.p.1, where $\beta_T = (2a_T(\log T a_T^{-1} + \log\log T))^{-\frac{1}{2}}$.

If a_T also satisfies condition (iii) *of Theorem A, then*

$$(4)\quad \lim_{T\to\infty} \sup_{R\in\mathsf{R}_T} \beta_T |W(R)| = \lim_{T\to\infty} \sup_{R\in\mathsf{R}^*_T} \beta_T |W(R)| = 1 \qquad w.p.1.$$

It is clear that this Theorem can be considered as an analogue of Theorem A in the 2-parameter case. However it does not imply the law of iterated logarithm for the multi-parameter Wiener process in its full richness. Especially the following result does not follow from our Theorem 1:

Theorem B ([4–7]). *We have*

$$\limsup_{\substack{x\to\infty \\ y\to\infty}} \frac{|W(x,y)|}{\sqrt{4xy\log\log xy}} = 1,$$

that is to say

$$\lim_{T\to\infty} \sup_{\substack{x\geqq T\\ y\geqq T}} \frac{|W(x,y)|}{\sqrt{4xy\log\log xy}}=1 \qquad \textit{w.p.1.}$$

It is somewhat strange that in this Theorem the usual constant 2 of the denominator is replaced by 4. Some explanation of this phenomenon is given in [5] and our Theorem 2 will give a further explanation. We also emphasize that in Theorem B it is assumed that both x and y go to infinity. It is natural to ask what happens if this is not the case. Our next Theorem is somewhat stronger than Theorem B and gives an answer to the latter question.

Theorem B.1. *For any* $\alpha>\frac{1}{2}$ *we have*

$$\text{(5)}\quad \limsup_{T\to\infty} \sup_{(x,y)\in \mathsf{D}_T} \frac{|W(x,y)|}{\sqrt{4T\log\log T}}=\limsup_{T\to\infty} \sup_{(x,y)\in \mathsf{D}_T^*} \frac{|W(x,y)|}{\sqrt{4T\log\log T}}=1$$

w.p.1, where

$$\mathsf{D}_T=\mathsf{D}_T(T^\alpha)=\{(x,y)\colon xy\leqq T,\ 0\leqq x\leqq T^\alpha,\ 0\leqq y\leqq T^\alpha\},$$

$$\mathsf{D}_T^*=\mathsf{D}_T^*(T^\alpha)=\{(x,y)\colon xy=T,\ 0\leqq x\leqq T^\alpha,\ 0\leqq y\leqq T^\alpha\}.$$

Applying this Theorem for $\alpha=1$, it can be seen that it is not necessary to asume in Theorem B that both variables go to infinity. (Cf. Consequence 2.) In our next Theorem we investigate the question of how the function T^α of Theorem B.1 can be replaced by an arbitrary increasing function b_T. We have

Theorem 2. *Let* $b_T\geqq T^{\frac{1}{2}}$ *be a non-decreasing function of* T *and define*

$$\gamma_T=(2T[\log(\log b_T T^{-\frac{1}{2}}+1)+\log\log T])^{-\frac{1}{2}},$$

$$\mathsf{D}_T=\mathsf{D}_T(b_T)=\{(x,y)\colon xy\leqq T,\ 0\leqq x\leqq b_T,\ 0\leqq y\leqq b_T\},$$

$$\mathsf{D}_T^*=\mathsf{D}_T^*(b_T)=\{(x,y)\colon xy=T,\ 0\leqq x\leqq b_T,\ 0\leqq y\leqq b_T\}.$$

Suppose that

(i) γ_T *is a non-increasing function of* T,

(ii) *for any* $\varepsilon>0$ *there exists a* $\theta_0=\theta_0(\varepsilon)>1$ *such that*

$$\limsup_{k\to\infty} \frac{\gamma_{\theta^k}}{\gamma_{\theta^{k+1}}}\leqq 1+\varepsilon$$

if $1<\theta\leqq\theta_0$.

Then

$$\text{(6)}\quad \limsup_{T\to\infty} \sup_{(x,y)\in \mathsf{D}_T} \gamma_T|W(x,y)|=\limsup_{T\to\infty} \sup_{(x,y)\in \mathsf{D}_T^*} \gamma_T|W(x,y)|=1$$

w.p.1.

If we also have

$$\text{(iii)}\quad \lim_{T\to\infty} \frac{\log(\log b_T T^{-\frac{1}{2}}+1)}{\log\log T}=\infty,$$

then

$$(6^*)\quad \lim_{T\to\infty} \sup_{(x,y)\in D_T} \gamma_T |W(x,y)| = \lim_{T\to\infty} \sup_{(x,y)\in D^*_T} \gamma_T |W(x,y)| = 1$$

w.p.1.

We mention some special cases of Theorem 2:

1° if $b_T = T^{\frac{1}{2}}$, we get the simplest form of the law of iterated logarithm (the constant in the denominator is the usual 2);

2° if

$$b_T = T^{\frac{1}{2}} e^{(\log T)^\gamma} \qquad (\gamma \geqq 0);$$

then $\gamma_T \approx (2(\gamma+1) T \log\log T)^{-\frac{1}{2}}$; that is to say for $\gamma = 0$ we get again the law of iterated logarithm with the constant 2 and the constant is increasing as γ is increasing; we get the constant 4 of Theorem B (or Theorem B.1) when $\gamma = 1$;

3° if $b_T = e^T$ then $\gamma_T \approx (2T \log T)^{-\frac{1}{2}}$; that is even the order of magnitude of γ_T has been changed. In this case (iii) of Theorem 2 holds, that is (6*) holds true;

4° if $b_T = e^{e^T}$ then $\gamma_T \approx 2^{-\frac{1}{2}} T^{-1}$.

Now we can really say that Theorem 2 is a generalization of Theorem B.1 (and, a fortiori, that of Theorem B). However, it is not a generalization of Theorem 1. Now we formulate our main result, which is generalization both Theorems 1 and 2.

Theorem 3. *Let $0 < a_T \leqq T$, $b_T \geqq T^{\frac{1}{2}}$ be non-decreasing functions of T and define*

$$\delta_T = (2 a_T (\log T a_T^{-1} + \log(\log b_T a_T^{-\frac{1}{2}} + 1) + \log\log T))^{-\frac{1}{2}}.$$

Further let $L_T = L_T(a_T, b_T)$ (resp. $L^*_T = L^*_T(a_T, b_T)$) be the set of rectangles $R = [x_1, x_2] \times [y_1, y_2] \subset D_T(b_T)$ for which $\lambda(R) \leqq a_T$ (resp. $\lambda(R) = a_T$).

Suppose that

(i) *δ_T is a non-increasing function of T,*

(ii) *$T a_T^{-1}$ is a non-decreasing function of T,*

(iii) *for any $\varepsilon > 0$ there exists a $\theta_0 = \theta_0(\varepsilon) > 1$ such that*

$$\limsup_{k\to\infty} \frac{\delta_{\theta^k}}{\delta_{\theta^{k+1}}} \leqq 1 + \varepsilon$$

if $1 < \theta \leqq \theta_0$.

Then

$$(7)\quad \limsup_{T\to\infty} \sup_{R\in L_T} \delta_T |W(R)| = \limsup_{T\to\infty} \sup_{R\in L^*_T} \delta_T |W(R)| = 1$$

w.p.1.

If we also have

$$(\text{iv})\quad \lim_{T\to\infty} \frac{\log T a_T^{-1} + \log(\log b_T a_T^{-\frac{1}{2}} + 1)}{\log\log T} = \infty,$$

then

(8) $$\lim_{T\to\infty} \sup_{R\in \mathsf{L}_T} \delta_T |W(R)| = \lim_{T\to\infty} \sup_{R\in \mathsf{L}_T^*} \delta_T |W(R)| = 1$$

w.p.1.

2. An Inequality

The main aim of this section is to prove the following inequality:

Theorem 4. *For any $\varepsilon>0$ there exists a $C=C(\varepsilon)>0$ such that*

(9) $$P\{\sup_{R\in \mathsf{L}_T} |W(R)| \geqq u\, a_T^{\frac{1}{2}}\} \leqq C \frac{T}{a_T}(1+\log T a_T^{-1})(1+\log b_T a_T^{-\frac{1}{2}})\, e^{-\frac{u^2}{2+\varepsilon}} \qquad (u>0),$$

where $\mathsf{L}_T=\mathsf{L}_T(a_T, b_T)$ is the class of rectangles defined in Theorem 3 and a_T and b_T also satisfy the conditions of the latter.

At first we introduce some notations and prove a lemma.
Let $\mu=\mu(T)$ be the smallest integer for which

$$\mu \geqq \log b_T a_T^{-\frac{1}{2}}$$

and, for any integer q, let $Q=Q(q)=2^q$. Define the following sequences of real numbers:

$$z_i = z_i(q) = z_i(q,T) = a_T^{\frac{1}{2}} (b_T a_T^{-\frac{1}{2}})^{i/Q\mu} \qquad (i=0, \pm 1, \pm 2, \dots, \pm Q\mu),$$

$$x_j(i) = x_j(i,T) = j z_i Q^{-1} \qquad (j=0,1,2,\dots),$$

$$y_j(i) = y_j(i,T) = j a_T z_i^{-1} Q^{-1} \qquad (j=0,1,2,\dots),$$

and the following rectangles

$$R_i = R_i(q) = [0, z_i] \times [0, a_T z_i^{-1}],$$

$$R_i(j,l) = R_i(q,j,l) = R_i + (x_j(i), y_l(i)).$$

Let $\mathsf{L}_T^*(q)$ be the set of rectangles $R_i(q,j,l)$ contained in the domain $\mathsf{D}_T(b_T)$. For any $R=[x_1,x_2]\times[y_1,y_2]\in \mathsf{L}_T$ define the rectangle $R(q)\in \mathsf{L}_T^*(q)$ as follows: let $i_0=i_0(R)$ denote the smallest integer for which: $z_{i_0} \geqq x_2 - x_1$ and let $j_0=j_0(R)$, $l_0=l_0(R)$ denote the largest integers for which $x_{j_0}(i_0) \leqq x_1$, $y_{l_0}(i_0) \leqq y_1$ and now let

$$R(q) = (x_{j_0}(i_0), y_{l_0}(i_0)) + [0, z_{i_0}] \times [0, a_T z_{i_0}^{-1}].$$

Lemma 1.

(10) $$\operatorname{card} \mathsf{L}_T^*(q) \leqq 48\, Q^3\, T a_T^{-1} (1+\log T a_T^{-1})(1+\log b_T a_T^{-\frac{1}{2}}),$$

(11) *for each* $R \in \mathsf{L}_T^*$ *we have* $\lambda(R \circ R(q)) \leqq 6 a_T Q^{-1}$, *where* λ *is the Lebesgue measure and the operation* $\circ$ *stands for symmetric difference,*

(12) $\lambda(R) = a_T$ *for each* $R \in \mathsf{L}_T^*(q)$.

Proof. At first we evaluate the number of rectangles $R_i(q,j,l)$ belonging to $\mathsf{L}_T^*(q)$ for a fixed i. Clearly if $R_i(q,j,l)$ belongs to the set $\mathsf{L}_T^*(q)$ then its right-upper vertex belongs to the domain

$$A = \{(x,y):\ z_i \leqq x \leqq z_i T a_T^{-1},\ a_T z_i^{-1} \leqq y \leqq T x^{-1}\},$$

and we have $\lambda(A) \leqq T \log T a_T^{-1}$. Since for any fixed i the square $[0, b_T] \times [0, b_T]$ contains at most

$$\mathsf{N} = (b_T Q z_i^{-1} + 1)(z_i Q b_T a_T^{-1} + 1)$$

elements of the sequence $(x_j(i), y_j(i))$, A will contain at most

$$\mathsf{N} b_T^{-2} T \log T a_T^{-1} + 3 Q a_T^{-1}(T - a_T) \leqq 12 Q^2 T a_T^{-1}(\log T a_T^{-1} + 1) = \mathsf{M}_T$$

elements of the sequence $(x_j(i), y_l(i))$. That is for a fixed i the number of rectangles $R_i(q,j,l)$ belonging to $\mathsf{L}_T^*(q)$ is not more than M_T. Since the number of possible values of i is not more than $2Q\mu + 1 \leqq 4Q(\log b_T a_T^{-\frac{1}{2}} + 1)$, we get (10).

Now (11) resp. (12) simply follow from the definition of $R(q)$ resp. that of $\mathsf{L}_T^*(q)$.

In the proof of Theorem 4 the following will also be used:

Lemma A [6]. *Let* $R = [x_1, x_2] \times [y_1, y_2]$ *be any rectangle and let*

$$S = [s_1, s_2] \times [t_1, t_2] \qquad (0 \leqq x_1 \leqq s_1 < s_2 \leqq x_2,\ 0 \leqq y_1 \leqq t_1 < t_2 \leqq y_2).$$

Then we have for any $u > 0$,

$$P\{\sup_{S \subset R} |W(S)| \geqq u\} \leqq 4 P\{|W(R)| \geqq u\}.$$

Now we turn to the

Proof of Theorem 4. For any $R \in \mathsf{L}_T$, the symmetric difference $R(q) \circ R(q+1)$ is the sum of at most 4 rectangles, say $R(q) \circ R(q+1) = R_1(q) + R_2(q) + R_3(q) + R_4(q)$. Denote the class of rectangles $R_i(q)$ $(i = 1, 2, 3, 4)$ by $\tilde{\mathsf{L}}_T^*(q)$. Then we have

$$\text{(13)} \quad \sup_{R \in \mathsf{L}_T} |W(R)| \leqq \sup_{R \in \mathsf{L}_T^*(q)} \sup_{S \subset R} |W(S)| + 4 \sum_{i=0}^{\infty} \sup_{R \in \tilde{\mathsf{L}}_T^*(q+i)} \sup_{S \subset R} |W(S)|,$$

where s is a rectangle with edges parallel to the coordinate axes.

Then by Lemmas 1 and A we have

$$\text{(14)} \quad P\{\sup_{R \in \mathsf{L}_T^*(q)} \sup_{S \subset R} |W(S)| \geqq x a_T^{-\frac{1}{2}}\} \leqq 4 \operatorname{card} \mathsf{L}_T^*(q) e^{-x^2/2}$$

and

$$\text{(15)} \quad P\{\sup_{R \in \tilde{\mathsf{L}}_T^*(q+i)} \sup_{S \subset R} |W(S)| \geqq y_i (6 a_T Q^{-1} 2^{-i})^{\frac{1}{2}}\} \leqq 4 \operatorname{card} \tilde{\mathsf{L}}_T^*(q+i) e^{-y_i^2/2}.$$

Since $\operatorname{card}\tilde{\mathsf{L}}_T^*(q+i) \leqq 4 \operatorname{card}\mathsf{L}_T^*(q+i)$, by (13), (14) and (15) we get

$$(16)\quad P\left\{\sup_{R\in\mathsf{L}_T}|W(R)| \geqq x a_T^{\frac{1}{2}} + 4\sum_{i=0}^{\infty} y_i (6 a_T Q^{-1} 2^{-i})^{\frac{1}{2}}\right\} \leqq 4\operatorname{card}\mathsf{L}_T^*(q)\, e^{-x^2/2} + 16\sum_{i=0}^{\infty}\operatorname{card}\mathsf{L}_T^*(q+i)\, e^{-y_i^2/2}.$$

Choosing $y_i = (6i + x^2)^{\frac{1}{2}}$, we have

$$(17)\quad x a_T^{\frac{1}{2}} + 4\sum_{i=0}^{\infty} y_i (6 a_T Q^{-1} 2^{-i})^{\frac{1}{2}}$$
$$\leqq x a_T^{\frac{1}{2}}\left\{1 + 4(6Q^{-1})^{\frac{1}{2}}\sum_{i=0}^{\infty} 2^{-i/2}\right\} + 24 a_T^{\frac{1}{2}} Q^{-\frac{1}{2}}\sum_{i=0}^{\infty}(i 2^{-i})^{\frac{1}{2}}$$
$$\leqq x a_T^{\frac{1}{2}}(1 + Q^{-\frac{1}{2}} A) + a_T^{\frac{1}{2}} Q^{-\frac{1}{2}} B \leqq (1+\varepsilon)\, x a_T^{\frac{1}{2}},$$

provided that Q is big enough, where $A = 46^{\frac{1}{2}}\sum_{i=0}^{\infty} 2^{-i/2}$ and $B = 24\sum_{i=0}^{\infty}(i 2^{-i})^{\frac{1}{2}}$; further

$$(18)\quad 4\operatorname{card}\mathsf{L}_T^*(q)\, e^{-x^2/2} + 16\sum_{i=0}^{\infty}\operatorname{card}\mathsf{L}_T^*(q+i)\, e^{-y_i^2/2}$$
$$\leqq C T a_T^{-1}(1 + \log T a_T^{-1})(1 + \log b_T a_T^{-\frac{1}{2}})\, e^{-x^2/2}.$$

Choosing $(1+\varepsilon)\, x = u$, (9) follows from (16), (17) and (18).

3. The Proof of Theorem 3

The proof will be given in three steps.

Step 1. *Let*

$$A(T) = \sup_{R\in\mathsf{L}_T} \delta_T\, |W(R)|.$$

Suppose that conditions (i), (ii), (iii) *of Theorem* 3 *are fulfilled. Then*

$$(19)\quad \limsup_{T\to\infty} A(T) \leqq 1 \qquad \textit{w.p.1.}$$

Proof. By Theorem 4 we have

$$P\{A(T) \geqq 1+\varepsilon\} \leqq C\left(\frac{a_T}{T}\right)^{\varepsilon}(1 + \log T a_T^{-1})(1 + \log b_T a_T^{-\frac{1}{2}})^{-\varepsilon}(\log T)^{-1-\varepsilon}.$$

Let $T_k = \theta^k$ $(\theta > 1)$. Then

$$\sum_{k=1}^{\infty} P\{A(T_k) \geqq 1+\varepsilon\} < \infty$$

for every $\varepsilon > 0$, $\theta > 1$, hence, by the Borel-Cantelli lemma,

(20) $$\limsup_{k\to\infty} A(T_k) \leqq 1 \quad \text{w.p.1.}$$

Since

(21) $$\liminf_{k\to\infty} \delta_{T_{k+1}}^{-1} \sup_{T_k \leqq T \leqq T_{k+1}} \delta_T \leqq \limsup_{k\to\infty} \delta_{T_{k+1}}^{-1} \sup_{T_k \leqq T \leqq T_{k+1}} \delta_T \leqq 1+\varepsilon$$

for any $\varepsilon > 0$, if θ is near enough to 1, (19) follows from (20) and (21).

Step 2. *Suppose that conditions* (i), (ii), (iii) *of Theorem* 3 *are satisfied. Then for any* $\varepsilon > 0$

(22) $$\limsup_{T\to\infty} \sup_{R \in \mathsf{L}_T^*} \delta_T |W(R)| \geqq 1-\varepsilon \quad \textit{w.p.1.}$$

Proof. Let $\theta > 1$ and $M > 1$ to be specified later on, and set

$$\lim_{T\to\infty} a_T T^{-1} = \rho$$

$$T_k = \theta^k,$$

$$L = L(T) = \begin{cases} \text{the largest integer for which } \left(\dfrac{T-a_T}{T}\right)^L b_T \geqq a_T^{\frac{1}{2}} & \text{if } \rho < 1, \\ \text{the largest integer for which } a_T^{\frac{1}{2}} M^L \leqq b_T & \text{if } \rho = 1, \end{cases}$$

$$L' = \max(L, 1), \qquad L_k = L(T_k),$$

$$R_i(T_k) = R_i =$$

$$\begin{cases} \left[\left(\dfrac{T_k - a_{T_k}}{T_k}\right)^{i+1} b_{T_k}, \left(\dfrac{T_k - a_{T_k}}{T_k}\right)^{i} b_{T_k}\right] \\ \times \left[\dfrac{T_{k-1} \cdot T_k^{i+1}}{(T_k - a_{T_k})^{i+1} b_{T_k}}, \dfrac{T_k^{i+1}}{(T_k - a_{T_k})^{i} b_{T_k}}\right] & \text{if } \rho < 1 \text{ and } L_k \geqq 1, \\ [T_k^{\frac{1}{2}} M^i, T_k^{\frac{1}{2}} M^{i+1}] \times [T_{k-1} T_k^{-\frac{1}{2}} M^{-i}, T_k^{\frac{1}{2}} M^{-i-1}] & \text{if } \rho = 1 \text{ and } L_k \geqq 1, \\ [T_{k-1}^{\frac{1}{2}}, T_k^{\frac{1}{2}}] \times [T_{k-1}^{\frac{1}{2}}, T_k^{\frac{1}{2}}] & \text{if } L_k = 0 \end{cases}$$

$(i = 0, 1, 2, \ldots, L' - 1, k = 1, 2, \ldots)$.

Now we have

$$R_i(k) \subset \mathsf{D}_{T_k} - \mathsf{D}_{T_{k-1}}$$

and

$$L' \geqq C T a_T^{-1} (\log b_T a_T^{-\frac{1}{2}} + 1)$$

if T is large enough, where C is a positive constant depending only on ρ resp. M. We also have

$$a_{T_k}(1-\varepsilon_1) \leqq \lambda(R_i(k)) \leqq a_{T_k}$$

where ε_1 can be an arbitrarily small positive number if θ and M are sufficiently large.

Since the sets R_i $(i=0,1,2,\ldots,L'-1)$ are disjoint we get

$$P\{\max_{0\leq i\leq L'-1} \delta_{T_k}|W(R_i(k))|\geqq 1-\varepsilon\}$$
$$\geqq 1-(1-\Phi((1-\varepsilon)[2(\log T_k a_{T_k}^{-1}+\log(\log b_T a_T^{-\frac{1}{2}}+1)+\log\log T_k]^{\frac{1}{2}}))^{L'}$$
$$\geqq(\log T_k)^{-1+\varepsilon^*}$$

if T is big enough, where $\varepsilon^*>0$ provided that θ and M are sufficiently large.

Then we get (22) applying the Borel-Cantelli lemma and the result of Step 1.

Step 3. *Suppose that conditions* (i)–(iv) *of Theorem* 3 *are satisfied. Then for any* $\varepsilon>0$ *we have*

(23) $$\liminf_{T\to\infty}\sup_{R\in \mathsf{L}_T^*}\delta_T|W(R)|\geqq 1-\varepsilon \qquad \textit{w.p.1.}$$

Proof. Define ρ and L just like in Step 2. and set

$$\tilde{R}_i=\tilde{R}_i(T)=\begin{cases}\left[\left(\frac{T-a_T}{T}\right)^{i+1}b_T,\left(\frac{T-a_T}{T}\right)^{i}b_T\right]\times\left[0,\frac{T^{i+1}}{(T-a_T)^i b_T}\right] & \text{if } \rho<1\\ [a_T^{\frac{1}{2}}M^i, a_T^{\frac{1}{2}}M^{i+1}]\times[0,a_T^{\frac{1}{2}}M^{-i-1}] & \text{if } \rho=1.\end{cases}$$

Since the sets $\tilde{R}_i$ $(i=0,1,2,\ldots,L-1)$ are disjoint, we have

$$P\{\max_{0\leq i\leq L-1}\delta_T|W(\tilde{R}_i(T))|\leqq 1-\varepsilon\}$$
$$\leqq(1-\Phi((1-\varepsilon)[2(\log T a_T^{-1}+\log(\log b_T a_T^{-\frac{1}{2}}+1)+\log\log T)]^{\frac{1}{2}}))$$
$$\leqq\exp(-(T a_T^{-1})^{\varepsilon'}(\log b_T a_T^{-\frac{1}{2}}+1)^{\varepsilon'}(\log T)^{-1+\varepsilon'}$$

for a suitable $\varepsilon'>0$ if T is large enough. Then (23) follows from condition (iv) of Theorem 3. and the Borel-Cantelli lemma.

4. The Multi-Parameter Case

The multi-parameter analogue of Theorem 3 is straightforward, hence only the statement will be formulated.

Theorem 3*. *Let* $b_T\geqq T^{1/d}$, $0<a_T\leqq T^{1/d}$ *be non-decreasing functions of* T *and define*

$$\delta_T=(2a_T(\log T a_T^{-1}+\log(\log b_T a_T^{-1/d}+1)^{d-1}+\log\log T))^{-\frac{1}{2}}.$$

Further let $\mathsf{L}_T=\mathsf{L}_T(a_T,b_T)$ (resp. $\mathsf{L}_T^*=\mathsf{L}_T^*(a_T,b_T)$) *be the set of rectangles* $\mathsf{R}=\prod_{i=1}^{d}[x_1^{(i)},x_2^{(i)}]\subset\mathsf{D}_T(b_T)$ *for which* $\lambda(R)\leqq a_T$ (resp. $\lambda(R)=a_T$) *where*

$$\mathsf{D}_T=\mathsf{D}_T(b_T)=\{(x^{(1)},x^{(2)},\ldots,x^{(d)}): x^{(1)}x^{(2)}\ldots x^{(d)}\leqq T, 0\leqq x^{(i)}\leqq b_T\}.$$

Suppose that conditions (i), (ii), (iii) *of Theorem* 3 *holds. Then* (7) *will hold true. If we also have* (iv) *of Theorem* 3. *then* (8) *also holds true.*

5. Applications and a Generalization

Theorem 2 easily implies:

Consequence 1. *Let b_T be a function of T satisfying the conditions of Theorem 2 and define*

$$\bar{\mathsf{D}}_T = \bar{\mathsf{D}}_T(b_T) = \bigcup_{S \geqq T} \mathsf{D}_S^*.$$

Then

$$\lim_{T\to\infty} \sup_{(x,y)\in\bar{\mathsf{D}}_T} \gamma_{xy} |W(x,y)| = 1 \qquad w.p.1.$$

The domain $\bar{\mathsf{D}}_T$ seems to be a rather artificial one. However using this Consequence, one can get similar results for many concrete domains. As an example we give:

Consequence 2. *Let*

$$\mathsf{E}_U = \{(x,y)\colon x \geqq 1,\ y \geqq U\}.$$

Then

$$\lim_{U\to\infty} \sup_{(x,y)\in\mathsf{E}_U} \frac{|W(x,y)|}{\sqrt{4xy\log\log xy}} = 1 \qquad w.p.1. \tag{24}$$

or equivalently

$$\lim_{y\to\infty} \sup_{x\geqq 1} \frac{|W(x,y)|}{\sqrt{4xy\log\log xy}} = 1 \qquad w.p.1. \tag{24*}$$

Proof. (24) follows from Consequence 1 and from the trivial relation

$$\bar{\mathsf{D}}_T(T^{\frac{3}{4}}) \subset \mathsf{E}_U \subset \bar{\mathsf{D}}_U(U)$$

if T is big enough, (for example if $T \geqq U^4$).

Formula (24*) is very suitable to get a strong law for the Kiefer Process. We recall that a Kiefer Process $K(x,y)$ $(0 \leqq x \leqq 1,\ 0 \leqq y < \infty)$ can be defined by the following transformation of a two-time parameter Wiener Process.

$$K(x,y) = (1-x)\, W\!\left(\frac{x}{1-x}, y\right) \qquad (0 \leqq x \leqq 1,\ 0 \leqq y < \infty). \tag{25}$$

Then (24*) and (25) imply

Consequence 3.

$$\lim_{y\to\infty} \sup_{0<x<1} \frac{|K(x,y)|}{\sqrt{4x(1-x)y\log\log y/x(1-x)}} = 1 \qquad w.p.1. \tag{26}$$

At the same time, it is well-known that

$$(27)\quad \lim_{y\to\infty} \sup_{\varepsilon<x<1-\varepsilon} \frac{|K(x,y)|}{\sqrt{2x(1-x)\,y\log\log y}}=1 \qquad \text{w.p.1.}$$

for any $0<\varepsilon<\frac{1}{2}$. (In fact this result is also a special case of our Consequence 5.)

Comparing (26) and (27), it is natural to ask: what is the behaviour of a Kiefer Process $K(x,y)$ in an interval $\varepsilon_y<x<1-\varepsilon_y$ when ε_y is a non-increasing positive function. In order to give an answer to this question we have to generalize Theorem 2. Namely, the domains D_T and D_T^* of Theorem 2. resp. $\bar{\mathsf{D}}_T$ of Consequence 1 were symmetric to the line $y=x$. In the next Theorem the non-symmetric case is studied.

Theorem 2*. *Let $b_1=b_1(T)$ and $b_2=b_2(T)$ be two non-decreasing functions of T for which $b_1 b_2\geqq T$ and define*

$$\gamma_T=(2T[\log(\log b_1 b_2 T^{-1}+1)+\log\log T])^{-\frac{1}{2}},$$

$$\mathsf{D}_T=\{(x,y):\ xy\leqq T,\ 0\leqq x\leqq b_1(T),\ 0\leqq y\leqq b_2(T)\}$$

$$\mathsf{D}_T^*=\{(x,y):\ xy=T,\ 0\leqq x\leqq b_1(T),\ 0\leqq y\leqq b_2(T)\}.$$

Suppose that γ_T satisfies conditions (i), (ii) *of Theorem* 2. *Then* (6) *holds true. If we also have* (iii) *of Theorem* 2. *then* (6*) *will also hold true.*

Applying the above Theorem the following analogue of Consequence 2. can be obtained:

Consequence 4. *Let $f(y)$ be a non-decreasing function of y tending to $+\infty$ and define $g(x)=xf^{-1}(x)$ and*

$$(28)\quad \gamma_{xy}=(2xy[\log(\log g^{-1}(xy)+1)+\log\log xy)])^{-\frac{1}{2}}.$$

Suppose that γ_{xy} satisfies the conditions (i), (ii) *of Theorem* 2.
Then

$$(29)\quad \limsup_{y\to\infty} \sup_{1\leqq x\leqq f(y)} \gamma_{xy}|W(x,y)|=1 \qquad \textit{w.p.1.}$$

Proof. Applying Theorem 2* for $b_1(y)=g^{-1}(y)$ and $b_2(y)=y$ one gets (29).

Consequence 5. *Let $0<\varepsilon_y<\frac{1}{2}$ be a non-increasing function of y and define $f(y)=1/\varepsilon_y$ and $g(x)=xf^{-1}(x)$. Then*

$$(30)\quad \limsup_{y\to\infty} \sup_{\varepsilon_y\leqq x\leqq 1-\varepsilon_y} \left(2yx(1-x)\left[\log\left(\log g^{-1}\left(\frac{y}{x(1-x)}\right)+1\right)\right.\right.$$
$$\left.\left.+\log\log\frac{y}{x(1-x)}\right]\right)^{-\frac{1}{2}}|K(x,y)|=1 \qquad \textit{w.p.1.}$$

Especially if $\varepsilon_y=e^{-(\log y)^\gamma}$ $(0<\gamma<1)$, then

$$(31)\quad \limsup_{y\to\infty} \sup_{\varepsilon_y\leqq x\leqq 1-\varepsilon_y} \left(2yx(1-x)(\gamma+1)\log\log\frac{y}{x(1-x)}\right)^{-\frac{1}{2}}|K(x,y)|=1 \qquad \textit{w.p.1.}$$

Let $X_1, X_2, \ldots$ be a sequence of independent r.v.'s uniformly distributed over $[0,1]$ and let $F_n(x)$ be the empirical distribution function based on the sample $X_1, X_2, \ldots, X_n$. The invariance principle ([8]) says that the empirical process $n^{\frac{1}{2}}(F_n(x)-x)=e_n(x)$ can be approximated by a Kiefer Process such that

$$\sup_x |K(x,n)-n^{\frac{1}{2}} e_n(x)| = O(\log^2 n) \qquad \text{w.p.1.}$$

This result and our Consequence 5. imply

Consequence 6. *Let* $n^{-1}\log^4 n < \varepsilon_n < \frac{1}{2}$ *be a non-increasing sequence of positive numbers satisfying the conditions of Consequence* 5. *Then*

$$(32) \quad \limsup_{n\to\infty} \sup_{\varepsilon_n \leqq x \leqq 1-\varepsilon_n} \left(2x(1-x)\log\left(\log g^{-1}\left(\frac{n}{x(1-x)}\right)+1\right) + \log\log\frac{n}{x(1-x)}\right)^{-\frac{1}{2}} |e_n(x)| = 1 \qquad \textit{w.p.1.}$$

This result was proved at first by Csáki ([1]), who also evaluated the lim sup of the sup of e_n taking the sup over more general intervals $(\varepsilon_n, 1-\varepsilon_n)$. Comparing Consequence 5. and the result of Csáki one can see that the behaviour of the Kiefer Process is the same as that of the empirical process if $\varepsilon_n \geqq d_0 n^{-1}\log\log n$ where $d_0 = 0{,}236\ldots$ and different if $\varepsilon_n \leqq d n^{-1}\log\log n$ where $d<d_0$. However the agreement of the two processes does not follow from the invariance principle when $\varepsilon_n = n^{-1}(\log n)^\alpha$ $(0<\alpha<3)$ or less.

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