Z. Wahrscheinlichkeitstheorie verw. Gebiete
52, 41–44 (1980)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

A Note on the Random Mean Ergodic Theorem

Alan Saleski

Department of Mathematical Sciences, Loyola University of Chicago, 6525 North Sheridan Road, Chicago, Illinois 60626, USA

Summary. Let $\{S_n\}$ be any non-trivial random walk on the integers. Then, almost surely, for all $p \geqq 1$, for any probability space X, for any mixing automorphism T on X, for all $f \in L_p(X)$,

$$\frac{1}{n}\sum_{i=1}^{n} f \circ T^{S_i} \to E(f) \quad \text{in } L_p(X).$$

Let T be an automorphism (that is, a bimeasurable measure-preserving bijection) of a probability space $(X, \mathscr{X}, m)$. Let $\{S_n\}_0^\infty$ denote a random walk on the integers (that is, $S_0 = 0$, $S_n = \sum_{i=1}^{n} X_i$ where $\{X_i\}$ are independent identically distributed integer-valued random variables) and P be the probability measure associated with the random walk. Assume $P(S_1 = 0) \neq 1$ and $1 \leqq p < \infty$.

As a consequence of the random mean ergodic theorem [4,7], if T is totally ergodic:

$$\text{(A)} \quad \forall f \in L_p(X) \quad P\left(\frac{1}{n}\sum_{i=1}^{n} f \circ T^{S_i} \xrightarrow{L_p} E(f)\right) = 1.$$

If X (and hence $L_p(X)$) is separable, then it follows easily from (A) that:

$$\text{(B)} \quad P\left(\forall p \geqq 1 \; \forall f \in L_p(X) \; \frac{1}{n}\sum_{i=1}^{n} f \circ T^{S_i} \xrightarrow{L_p} E(f)\right) = 1.$$

As a result of the recent work of Reich (Theorem 6.3 of [5]), if the X_i are positive and possess finite mean, then:

$$\text{(C)} \quad P\Big(\forall p \geqq 1 \; \forall \text{ weakly mixing } T \text{ on any probability space } X$$
$$\forall f \in L_p(X) \; \frac{1}{n}\sum_{i=1}^{n} f \circ T^{S_i} \xrightarrow{L_p} E(f)\Big) = 1.$$

0044-3719/80/0052/0041/$01.00

In contradistinction to (A) or (B), (C) asserts that the set of probability one for which convergence takes place does not depend upon p, X, T or f.

The objective of this note is to demonstrate that (C) is valid for arbitrary one-dimensional random walk provided T is mixing. The proof given is similar in spirit to that of Blum and Hanson [1]. For the sake of completeness, a direct proof of (C) for random walk having finite first absolute moment is also provided.

Lemma 1. *Let T be an ergodic automorphism of a non-atomic probability space $(X, \mathscr{X}, m)$, $\varepsilon > 0$, $M \geqq 1$. Then there exists $A \in \mathscr{X}$ such that $0 < m(A) < \varepsilon$ and $m(T^i(A) \cap A) \geqq (1-\varepsilon)\, m(A)$ for $-M \leqq i \leqq M$.*

Proof. Choose n so large that $(n-2M)/n > 1-\varepsilon$. Then, according to Rohlin's theorem, there exists a set $F \in \mathscr{X}$ such that $\{T^{-i}F \mid 1 \leqq i \leqq n\}$ are pairwise disjoint and $n\, m(F) > 1-\varepsilon$. Choose $G \subset F$ such that $0 < m(G) < \varepsilon/n$. Let $A = \bigcup_{i=1}^{n} T^{-i}G$. □

Lemma 2. *Let $\{S_n\}$ be a random walk on the integers and let $M \geqq 1$. Define the random variable $L_n = \frac{1}{n^2} \operatorname{card}\{(i,j) \mid |S_i - S_j| \leqq M,\ 1 \leqq i, j \leqq n\}$. Then $L_n \to 0$ a.s.*

Proof. Let $0 < \varepsilon < \frac{1}{2}$. Since $L_n \geqq 0$, it suffices to show that $P(\limsup L_n \leqq \varepsilon) = 1$. Let T be any totally ergodic automorphism of a non-atomic probability space $(X, \mathscr{X}, m)$. As a consequence of Lemma 1, there exists $A \in \mathscr{X}$ such that $0 < m(A) < \varepsilon$ and $m(T^i A \mid A) \geqq 1-\varepsilon$ for $-M \leqq i \leqq M$. Let f denote the characteristic function of A. Invoking the random mean ergodic theorem and noting that the family $\{T^{X_1}\}$ of transformations is ergodic yields $\left\| \frac{1}{n} \sum_{i=1}^{n} f \circ T^{S_i} - E(f) \right\|_2 \to 0$ a.s. Hence

$$\left\| \frac{1}{n} \sum_{i=1}^{n} f \circ T^{S_i} \right\|_2^2 \to \|E(f)\|_2^2 = (m(A))^2 \quad \text{a.s.}$$

Now, a simple calculation shows:

$$\begin{aligned}\left\| \frac{1}{n} \sum_{i=1}^{n} f \circ T^{S_i} \right\|_2^2 &= \frac{1}{n^2} \sum_{i,j=1}^{n} \langle f, f \circ T^{S_i - S_j} \rangle \\ &\geqq \frac{1}{n^2} \sum \{ \langle f, f \circ T^{S_i - S_j} \rangle \mid 1 \leqq i, j \leqq n, |S_i - S_j| \leqq M \}.\end{aligned}$$

Thus

$$\begin{aligned}(m(A))^2 &\geqq \limsup \frac{1}{n^2} \sum \{ \langle f, f \circ T^{S_i - S_j} \rangle \mid 1 \leqq i, j \leqq n, |S_i - S_j| \leqq M \} \\ &\geqq (1-\varepsilon)\, m(A) \limsup L_n \quad \text{a.s.}\end{aligned}$$

So, $\limsup L_n \leqq m(A)/(1-\varepsilon) < 2\varepsilon$ a.s. □

Theorem 1. *Let $\{S_n\}$ be a one-dimensional random walk. Then, with probability one, for any mixing automorphism T of a probability space X, for all $p \geqq 1$, for all $f \in L_p(X)$,*

$$\frac{1}{n}\sum_{i=1}^{n} f\circ T^{S_i} \xrightarrow{L_p} E(f).$$

Proof. Let E_M denote the event: $\frac{1}{n^2}\operatorname{card}\{(i,j)|1\leqq i,j\leqq n \text{ and } |S_i-S_j|\leqq M\}\to 0$; let $E=\bigcap_{M=1}^{\infty} E_M$. Lemma 2 asserts that each E_M occurs a.s.; hence E occurs a.s. Let $f\in L_2(X)$, $E(f)=0$. Then, if E occurs,

$$\left\|\frac{1}{n}\sum_{i=1}^{n} f\circ T^{S_i}-E(f)\right\|_2=\left(\frac{1}{n^2}\sum_{i,j=1}^{n}\langle f, f\circ T^{S_i-S_j}\rangle\right)^{\frac{1}{2}}\to 0$$

since T is mixing. For arbitrary $g\in L_2(X)$, replace g by $g-E(g)$. To show convergence in $L_p(X)$, $p\neq 2$, a standard approximation argument will succeed. □

Lemma 3. *Assume the random walk $\{S_n\}$ possesses finite first absolute moment. Then there exists a set C of probability one such that, for each $\omega\in C$, for every sequence β of positive integers having zero density,*

$$\frac{1}{n^2}\operatorname{card}\{(i,j)||S_i(\omega)-S_j(\omega)|\in\beta,\ 1\leqq i,j\leqq n\}\to 0.$$

Proof. Let $0<\varepsilon<1$. Applying the strong law of large numbers and Lemma 2, there exists a set C of probability one such that for any $\omega\in C$ there exist integers K and L satisfying $|S_n(\omega)|<Kn$ for all $n\geqq 1$ and $\frac{1}{n^2}\operatorname{card}\{(i,j)||S_i(\omega)-S_j(\omega)|\geqq 1, 1\leqq i,j\leqq n\}>1-\varepsilon$ for all $n\geqq L$. Hence, for any $\omega\in C$ and for each $n\geqq L$, there exists a subset G_n of $\{1,2,\dots,n\}$ such that $\operatorname{card} G_n\geqq(1-\varepsilon)n$ and $S_i(\omega)\neq S_j(\omega)$ whenever i and j are distinct elements of G_n. Let $A_{ij}(\omega)=|S_i(\omega)-S_j(\omega)|$; note that $A_{ij}(\omega)\leqq 2Kn$ if $i,j\leqq n$. A simple counting argument shows that, for any integer p, $\operatorname{card}\{(i,j)|i,j\in G_n,\ A_{ij}(\omega)=p\}\leqq 2n$. Thus, for each $\omega\in C$ and any sequence $\beta=\{\beta_i\}$ of zero density,

$$\begin{aligned}&\frac{1}{n^2}\operatorname{card}\{(i,j)|A_{ij}(\omega)\in\beta, 1\leqq i,j\leqq n\}\\&\leqq\frac{1}{n^2}((2\varepsilon-\varepsilon^2)n^2+2n\operatorname{card}\{i|\beta_i\leqq 2Kn\})\\&=2\varepsilon-\varepsilon^2+4K\left(\frac{1}{2Kn}\operatorname{card}\{i|\beta_i\leqq 2Kn\}\right)\to 2\varepsilon-\varepsilon^2.\end{aligned}$$ □

Theorem 2. *Let $\{S_n\}$ be a one-dimensional random walk on the integers having finite first absolute moment. Then, with probability one, for any weakly mixing automorphism T of a probability space X, for all $p\geqq 1$, for all $f\in L_p(X)$, $\frac{1}{n}\sum_{i=1}^{n} f\circ T^{S_i}\xrightarrow{L_p} E(f)$.*

Proof. Choose C to be the set of probability one defined in Lemma 3. Let (T, X) be a weakly mixing dynamical system and let f be any simple function on X. Then there exists a sequence β of positive integers having zero density such that, if $|a_n| \notin \beta$ and $|a_n| \to \infty$, then $\lim_{n \to \infty} \langle f, f \circ T^{a_n} \rangle = 0$. If $\omega \in C$ then, arguing as in the proof of Theorem 1, $\frac{1}{n} \sum_{i=1}^{n} f \circ T^{S_i(\omega)} \to E(f)$ in L_2. A routine approximation argument completes the proof of this theorem. □

References

1. Blum, J.R., Hanson, D.L.: On the mean ergodic theorem for subsequences. Bull. Amer. Math. Soc. **66**, 308–311 (1960)
2. Friedman, N.A.: Introduction to Ergodic Theory. New York-Cincinnati-Toronto-London-Melbourne: Van Nostrand Reinhold 1970
3. Halmos, P.R.: Lectures on Ergodic Theory. New York: Chelsea 1956
4. Kakutani, S.: Random ergodic theorems and Markoff processes with a stable distribution. In: Proc. 2nd Berkeley Sympos. Math. Statist. and Probab. pp. 247–261. Berkeley-Los Angeles: Univ. Calif. Press 1951
5. Reich, J.I.: On the individual ergodic theorem for subsequences. Ann. Probability **5**, 1039–1046 (1977)
6. Spitzer, F.: Principles of Random Walk (2nd Edition). New York-Heidelberg-Berlin: Springer 1976
7. Ulam, S.M., von Neumann, J.: Random ergodic theorem (abstract). Bull. Amer. Math. Soc. **51**, 660 (1945)

Received December 28, 1978