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Random Walks with Internal Degrees of Freedom

I. Local Limit Theorems

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1. Introduction. The Model

Random walks with internal degrees of freedom were introduced by Sinai in 1981 in his Kyoto talk [9]. Let E be a finite or countably infinite set and consider a homogeneous Markov chain ξ_n, $n=0,1,2,\ldots$ on the state space $H=Z^\nu\times E$ $(\nu=1,2,\ldots)$. Suppose that the transition probabilities of $\xi_n=(\eta_n,\varepsilon_n)$ $(\eta_n\in Z^\nu\,\varepsilon_n\in E)$ satisfy the Z^ν-translation invariance condition: for every x_n, $x_{n+1}\in Z^\nu$, $j_n, j_{n+1}\in E$

$$P(\xi_{n+1})=(x_{n+1},j_{n+1})|\xi_n=(x_n,j_n))=p_{x_{n+1}-x_n,j_n,j_{n+1}}. \tag{1.1}$$

The Markov chain ξ_n is called *a random walk with internal degrees of freedom* or briefly with internal states. η_n is the actual random walk component, while ε_n is the internal state.

By inventing this natural generalization of the usual notion of random walks, Sinai's aim was to obtain a tool for the study of the Lorentz process. In fact, (see Sect. 6 for more details), the probability theory of random walks with internal states offers a possibility to use the Markov partition of the Sinai billiard in proving properties of the Lorentz process.

Nevertheless, it is expected that these random walks will find other applications, too, e.g. some models of queueing systems can be described by them.

This part of the paper is organized as follows: in Sect. 2 we give a preliminary version of our main result to make clear the basic line of the proof (see Sect. 3). Section 4 is quite important: it is devoted to the analysis of the arithmetic properties of our random walk. Then in Sect. 5 we can formulate and prove the main result of the first part of the paper. Section 6 contains some remarks.

2. Local Limit Theorems: Pre-formulation

We assume $d=\operatorname{card}E<\infty$.

For convenience we write the usual transition operator $A\colon l_1(H)\to l_1(H)$ attached to our Markov chain ξ_n in the following form

$$(Af)(x) \stackrel{\text{def}}{=} \sum_{y \in Z^\nu} A_y f(x+y),$$

where $x \in Z^\nu$ and, for any $y \in Z^\nu$, A_y: $C^d \to C^d$ is a linear operator defined by

$$A_y \stackrel{\text{def}}{=} (p_{y,j,k})_{j,k=1,\ldots,d},$$

and finally $f \stackrel{\text{def}}{=} \{f(x): x \in Z^\nu\}$ with $f(x) \in C^d$. Let $e_1, \ldots, e_d$ be the unit vectors of C^d and denote $\mathbf{1} \stackrel{\text{def}}{=} (1, \ldots, 1)^* \in C^d$. We will use $*$ for denoting adjoint. Moreover, in general, the indices j and k will denote internal states of the Markov chain ($j, k \in E$).

From (1.1) it immediately follows that $\varepsilon_0, \varepsilon_1, \ldots$ is a homogeneous Markov chain with transition matrix $Q \stackrel{\text{def}}{=} \sum_{y \in Z^\nu} A_y$. Q will throughout be supposed to fulfil ergodicity, i.e. there is a unique $\mu \in C^d$ such that $\sum_j \mu_j = 1$, $\mu_j \geqq 0$ and $Q^* \mu = \mu$.

The spatial translation-invariance (1.1) also suggests the use of Fourier transforms in the spatial coordinates. In fact, for any $f \in l_1(H)$, introduce the Fourier transform $\hat{f}$: $[-\pi, \pi)^\nu \to C^d$ as

$$\hat{f}(t) \stackrel{\text{def}}{=} \sum_{x \in Z^\nu} e^{i(t,x)} f(x). \tag{2.1}$$

Then

$$\widehat{A^n f}(t) = \alpha^n(t) \hat{f}(t)$$

where $\alpha(t) \stackrel{\text{def}}{=} \sum_{y \in Z^\nu} e^{i(t,y)} A_y$ is a linear operator in C^d and $t \in [-\pi, \pi)^\nu$.

As usual

$$P(\xi_n = (x, \cdot) \mid \xi_0 = (0, j)) = (\delta^*_{0,j} A^n)(x),$$

where $\delta_{0,j} \in l_1(H)$ vanishes for every $x \in Z^\nu$, $x \neq 0$ and $\delta_{0,j}(0) = e_j$. The analogue of the usual Fourier inversion formula (see [4], Theorem 4 in § 3 of Chap. XV) says in our case that

$$(\delta^*_{0,j} A^n)(x) = (2\pi)^{-\nu} \int_{-\pi}^{\pi} \ldots \int_{-\pi}^{\pi} e^{-i(x,t)} e^*_j \alpha^n(t)\, dt. \tag{2.2}$$

Denote

$$M_l \stackrel{\text{def}}{=} \sum_{y \in Z^\nu} y_l A_y, \tag{2.3}$$

$$\Sigma_{l,m} \stackrel{\text{def}}{=} \sum_{y \in Z^\nu} y_l y_m A_y. \tag{2.4}$$

(Here $y = (y_1, \ldots, y_\nu)$). Whenever we use these symbols we implicitely suppose the convergence of the corresponding series.

For simplicity, first we formulate our result for the case $\nu = 1$, when we briefly write $M_1 \stackrel{\text{def}}{=} M$, $\Sigma_{1,1} \stackrel{\text{def}}{=} \Sigma$.

Pre-theorem 2.1. *Suppose that*

(i) *Q is ergodic and aperiodic;*

(ii) $(M\mathbf{1}, \mu) = 0$;

(iii) $\sigma^2 \stackrel{\text{def}}{=} (\Sigma \mathbf{1}, \mu) - 2(M(Q-1)^{-1} M\mathbf{1}, \mu) > 0$;

(iv) $\|\alpha(t)\| < 1$ *unless* $t=0$. ($\| \ \|$ denotes the operator norm.)

Then, for any $j \in E$,

$$\sum_{(x,k)\in H} \left| P(\xi_n = (x,k) \,\middle|\, \xi_0 = (0,j)) - \mu_k \frac{1}{\sqrt{2\pi n}\,\sigma} e^{-\frac{x^2}{2n\sigma^2}} \right| \to 0$$

as $n \to \infty$.

Condition (ii) just says that, with respect to the stationary distribution of internal states, our random walk has no drift. Condition (iii) expresses that the variance of the displacement after n steps grows like const. n. Finally, condition (iv) is responsible for our statement's being called a pre-theorem. Indeed, in Sect. 5 it will be substituted by a more probabilistic assumption and the general case will also be treated.

3. Proof of Pre-theorem 2.1

The existence of M and Σ imply

$$\alpha(t) = Q + itM - \frac{t^2}{2}\Sigma + o(t^2) \qquad (t \to 0).$$

By the Perron-Frobenius theorem ([6]) and condition (i), 1 is a simple eigenvalue of Q ($Q\mathbf{1} = \mathbf{1}$) and all other eigenvalues of Q lie strictly inside the unit circle. Then perturbation theory says that, near 0, the largest eigenvalue $\lambda(t)$ of $\alpha(t)$ is simple and it has a Taylor expansion

$$\lambda(t) = 1 + r_1 t + \frac{r_2}{2} t^2 + o(t^2) \qquad (t \to 0)$$

(cf. [8], Theorem 5.11 of Chap. II). Denote the corresponding eigenvector by $\varphi(t)$ (i.e. $\alpha(t)\varphi(t) = \lambda(t)\varphi(t)$) and accept the normalization $(\mu, \varphi(t)) = 1$.

r_1 and r_2 can be calculated by using Schrödinger's implicit method (see [5]) which also applies in the non self-adjoint case. In fact, by introducing the projection Π: $C^d \to C^d$ defined via $\Pi\Psi = (\Psi, \mu)\mathbf{1}$ we have

$$(\alpha(t) - \lambda(t) + c\Pi)\varphi(t) = c(\varphi(t), \mu)\mathbf{1},$$

where $c \neq 0$ is an arbitrary real parameter. Since $c \neq 0$, the operator $\alpha(t) - \lambda(t) + c\Pi$ is invertible and

$$\varphi(t) = c(\varphi(t), \mu)(\alpha(t) - \lambda(t) + c\Pi)^{-1}\mathbf{1},$$

and having taken the inner product of both sides by μ we get

$$1 = c((\alpha(t) - \lambda(t) + c\Pi)^{-1}\mathbf{1}, \mu) \tag{3.1}$$

Observe that for the operator $B = (Q - 1 + c\Pi)^{-1}$ we have $B\mathbf{1} = c^{-1}\mathbf{1}$ and $B^*\mu = c^{-1}\mu$. Also $\Pi^*\Psi = (\Psi, \mathbf{1})\mu$.

To calculate r_1 we only consider Taylor expansions up to linear terms and we transform

$$(\alpha(t)-\lambda(t)+c\Pi)^{-1}=(Q+itM-1-r_1 t+c\Pi+o(t))^{-1}$$
$$=(1+itBM-r_1 tB+o(t))^{-1}B=B-itBMB+r_1 tB^2+o(t).$$

Substituting this expression into (3.1) the comparison of the linear terms on both sides and condition (ii) result in $r_1=0$.

Next take Taylor expansions up to quadratic terms and transform as before

$$(\alpha(t)-\lambda(t)+c\Pi)^{-1}=B-itBMB+\frac{t^2}{2}B\Sigma B+\frac{r_2}{2}t^2B^2-t^2BMBMB+o(t^2).$$

Now by the comparison of the coefficients of the quadratic terms we obtain

$$r_2=-(\Sigma\mathbf{1},\mu)+2(MBM\mathbf{1},\mu).$$

Remember that B depends on the parameter c. Observe, however, that the action of both $(Q-1+c\Pi)$ and B only depends on c in the eigenspace spanned by the vector $\mathbf{1}$. Since the other eigenvectors of $Q-1+c\Pi$ lie in the orthogonal complement to the eigenvector of $(Q-1+c\Pi)^*$ corresponding to the real eigenvalue c $((Q-1+c\Pi)\mathbf{1}=c\mathbf{1})$ and this eigenvector is just μ, we get to the following conclusion: in the orthogonal complement to μ the action of the operators $(Q-1+c\Pi)$ and B as well does not depend on the parameter c. But by condition (ii) $M\mathbf{1}\perp\mu$ and consequently we can take $c\to\infty$ and write $(MBM\mathbf{1},\mu)=(M(Q-1)^{-1}M\mathbf{1},\mu)$ where we should remember that $(Q-1)^{-1}$ only exists in the orthogonal complemnent of μ. Thus we have arrived at

Lemma 3.1. *Under the conditions of pre-Theorem* 2.1

$$\lambda(t)=1-\frac{\sigma^2}{2}t^2+o(t^2)\qquad(t\to 0).$$

Proof of Pre-theorem 2.1. By an elementary argument given, for example, in [7] (Theorem 4.2.2) our statement follows from uniform convergence, i.e. from

$$\lim_{n\to\infty}\sup_{x\in Z}\left\|\sqrt{n}\int_{-\pi}^{\pi}e^{-ixt}e_j^*\alpha^n(t)\,dt-\mu^*(2\pi)^{\frac{1}{2}}\sigma^{-1}\exp\left(-\frac{x^2}{2n\sigma^2}\right)\right\|=0.$$

Via usual transformations (cf. [7], p. 150)

$$\left\|\sqrt{n}\int_{-\pi}^{\pi}e^{-ixt}e_j^*\alpha^n(t)\,dt-\mu^*(2\pi)^{1/2}\sigma^{-1}\exp\left(-\frac{x^2}{2n\sigma^2}\right)\right\|$$
$$\leq\int_{|s|<\delta}\left\|e_j^*\alpha^n\left(\frac{s}{\sqrt{n}}\right)-\mu^*\exp\left(-\frac{\sigma^2s^2}{2}\right)\right\|ds+\|\mu\|\int_{|s|>\delta}\exp\left(-\frac{\sigma^2s^2}{2}\right)ds$$
$$+\int_{\delta<|s|<\gamma\sqrt{n}}\left\|e_j^*\alpha^n\left(\frac{s}{\sqrt{n}}\right)\right\|ds+\int_{\gamma\sqrt{n}<|s|<\pi\sqrt{n}}\left\|e_j^*\alpha^n\left(\frac{s}{\sqrt{n}}\right)\right\|ds=I_1+I_2+I_3+I_4.$$

Let $\alpha(t)=T_t J_t T_t^{-1}$ be the Jordan form of $\alpha(t)$. If $\mu^*(t)\alpha(t)=\tilde{\lambda}(t)\mu^*(t)$ where $\tilde{\lambda}(0)=1$ and $\mu(0)=\mu$, then $\mu(t)$ is twice differentiable near 0, too. We can suppose that the first diagonal element of J_t is $\lambda(t)$. Then the first column of T_t is $\varphi(t)$ and the first row of T_t^{-1} is $\mu^*(t)$. Since the other eigenvalues of $\alpha(t)$ move in the unit disc bounded away from the boundary for t small and the eigenspaces depend continuously on the perturbation at $t=0$, elementary calculations show that

$$\alpha^n(t)=(\mathbf{1}\mu^*)\left(1-\frac{\sigma^2 t^2}{2}+o(t^2)\right)^n(1+o(1)) \tag{3.2}$$

uniformly for t small. I_2 and I_3 can be made arbitrarily small by choosing δ large and γ small. Since $e_j^*(\mathbf{1}\mu^*)=\mu^*$, (3.2) gives that I_1 is small if n is large. Finally, by the continuity of $\|\alpha(t)\|$ and by (iv) we have $\max\limits_{\gamma\leqq|t|\leqq\pi}\|\alpha(t)\|<1$ and this makes I_4 exponentially small when γ is already fixed. Hence the statement.

4. Arithmetics of Random Walks with Internal States

The aim of this section is to give conditions that imply assumption (iv) of pre-Theorem 2.1 and to analyse the general case when this assumption does not hold. This study generalizes the classical understanding of the arithmetic properties of probability distributions through their characteristic functions.

The domain of definition of the Fourier transform $\alpha(t)$ can, of course, be identified with the ν-dimensional torus T^ν. If $\mathscr{P}=\{p(y): y\in Z^\nu\}$ is a probability distribution on Z^ν, then the same is true for its characteristic function $\varphi(t)=\sum\limits_{y\in Z^\nu} p(y)\,e^{i(y,t)}$.

Lemma 4.1. *If φ is a characteristic function as above, then $|\varphi(s_0)|=1$ implies the existence of an $y_0\in Z^\nu$ such that $p(y_0)>0$ and $\varphi(t)=e^{i(t,y_0)}\tilde{\varphi}(t)$ where $\tilde{\varphi}(t)$ is an s_0-periodic function* (i.e. $\tilde{\varphi}(t+s_0)=\tilde{\varphi}(t)$).

This lemma is a well-known reformulation of the triangle-inequality. Next we reformulate a lemma of Bhattacharya-Rao ([1], Lemma 21.6). If $\mathscr{P}$ is a probability distribution on Z^ν with $p(y_0)>0$, then we can associate with $\mathscr{P}$ the minimal subgroup L of Z^ν generated by $\{x: p(y_0+x)>0\}$.

Lemma 4.2 ([1]). *Suppose that L is the minimal subgroup of Z^ν associated with a probability distribution $\mathscr{P}$. Denote by $\hat{L}$ the set of periods of $|\varphi|$ (φ is the characteristic function of $\mathscr{P}$). Then*

(i) $\hat{L}=\{s: |\varphi(s)|=1\}$

(ii) $\hat{L}=\{s: (s,y)\equiv 0 (\mathrm{mod}\, 2\pi)$ *for all* $y\in L\}$

(iii) $L=\{y: (s,y)\equiv 0 (\mathrm{mod}\, 2\pi)$ *for all* $s\in\hat{L}\}$.

Our analysis of the set $S=\{s: \|\alpha(s)\|=1\}$ is based on Wielandt's lemma

Lemma 4.3 ([6] §2 of Chapter XIII). *Suppose Q is the transition matrix of an aperiodic, ergodic Markov chain ($Q=(q_{jk})_{1\leqq j,k\leqq d}$, $d<\infty$), and let C be a complex* *matrix* ($C=(c_{jk})_{1\leqq j,k\leqq d}$) *such that for every* j *and* k

$$|c_{jk}|\leqq q_{jk}.$$

Then for the largest eigenvalue γ *of* C *we have*

$$|\gamma|\leqq 1$$

with equality if and only if

1. *for every* j *and* k, $|c_{jk}|=q_{jk}$
2. $C=\gamma DQD^{-1}$

where D *is a unitary diagonal matrix.* $D=(d_{jk})_{1\leqq j,k\leqq d}$ *is uniquely determined if we fix* $d_{11}=1$.

An important content of this lemma is that if once 1 is satisfied then each $c_{jk}=\exp(i\vartheta_{jk})q_{jk}$ where ϑ_{jk} is a real "angle". Also 2. is already independent of the values of the q_{jk} and it only concerns the angles ϑ_{jk}.

Lemma 4.4. *For every* $s\in S$, *there is a complex number* γ_s ($|\gamma_s|=1$) *and a unitary, diagonal matrix* D_s *such that, for every* $t\in T^v$ *and* $s\in S$,

$$\alpha(s+t)=\gamma_s D_s \alpha(t) D_s^{-1}. \tag{4.1}$$

Moreover, γ_s *and* D_s *can be defined to satisfy*

$$\gamma_{s_1+s_2}=\gamma_{s_1}\gamma_{s_2} \quad \textit{and} \quad D_{s_1+s_2}=D_{s_1}D_{s_2} \tag{4.2}$$

($s_1, s_2\in S$).

Proof. According to its definition, $\alpha(t)$ can be written in the form $\alpha(t)=(q_{jk}\varphi_{jk}(t))_{1\leqq j,k\leqq d}$. Here $\varphi_{jk}(t)$ is the characteristic function of the conditional distribution $\mathscr{P}_{jk}$ of one step of the random walk under the condition that the Markov chain of the internal states jumps from j to k (put $\varphi_{jk}(t)\equiv 1$ if $q_{jk}=0$). Fix $s\in S$. By Wielandt's lemma $|\varphi_{jk}(s)|=1$ for every j, k. Then our Lemma 4.1 implies that for suitable $r_{jk}\in Z^v$ $\varphi_{j,k}(t)=e^{i(t,r_{jk})}\tilde{\varphi}_{jk}(t)$ with $\tilde{\varphi}_{jk}$ s-periodic. Consequently,

$$\varphi_{jk}(s+t)=e^{i(s,r_{jk})}\varphi_{j,k}(t). \tag{4.3}$$

By Wielandt's lemma, again,

$$\alpha_{jk}(s)=q_{jk}e^{i(s,r_{jk})}=\gamma_s D_s Q D_s^{-1} \tag{4.4}$$

where $|\gamma_s|=1$ and D_s is a unitary diagonal matrix. Then, in view of (4.3) and (4.4) we obtain (4.1).

(4.2) follows from Wielandt's lemma, too. Q.E.D.

The previous lemma implies straightforward

Lemma 4.5. S *is a closed subgroup of* T^v.

Consequently, $S=S^c\times S^d$ where S^d is a discrete subgroup and S^c is an n-dimensional torus ($0\leqq n\leqq v$). A similar decomposition holds for each set $\hat{L}_{jk}$ corresponding to the distribution $\mathscr{P}_{jk}$ in the sense of Lemma 4.2. (Without restricting the generality we assume each $q_{jk}>0$, because if this is not the case then, first, sufficiently large powers of Q have strictly positive elements and, secondly, by taking large prime powers of Q one can deduce the desired arithmetics for Q itself.) Thus $\hat{L}_{jk}=\hat{L}^c_{jk}\times\hat{L}^d_{jk}$. Each $\hat{L}^d_{jk}$ has $v-n_{jk}$ generators: $\beta^{(1)}_{jk},\dots,\beta^{(v-n_{jk})}_{jk}$ determining a parallelepiped whose interior does not contain any points of $\hat{L}^d_{jk}$. Then, by Lemma 4.1, $\varphi_{jk}(t)=\exp\{i(r_{jk},t)\}\cdot\tilde{\varphi}_{jk}(t)$ and $\tilde{\varphi}_{jk}(t+t')=\tilde{\varphi}_{jk}(t)$ whenever $t\in L^c_{jk}$ or $t=\beta^{(l)}_{jk}\,(l=1,\dots,v-n_{jk})$. Now we are able to formulate the main result of this section.

Theorem 4.6. *S is the maximal subset of T^v with the properties*

(i) $S\subset\bigcap_{j,k}\hat{L}_{jk}$

(ii) *for every $s\in S$ and every j, k*

$$(r_{jk}-r_{11}-r_{j1}+r_{k1},s)\equiv 0 \pmod{2\pi}$$

where r_{jk} is any fixed vector of Z^v such that $p_{r_{jk},j,k}>0$.

This theorem expresses that the arithmetics of random walks with internal states is composed from two factors: (i) the arithmetics of the conditional distributions $\mathscr{P}_{jk}$ and (ii) the structure of the shifts of the $\mathscr{P}_{jk}$'s (each $\mathscr{P}_{jk}$ being concentrated to a shifted minimal subgroup of Z^v).

Proof. From the group property (4.2) it follows that γ_s and the diagonal elements $(d_{jj})_s$ of D_s should satisfy the Cauchy equation on S. Elementary considerations imply the existence of a $\varphi\in Z^v$ and $\rho_j\in Z^v\,(1\leqq j\leqq d)$ such that $\gamma_s=\exp\{i(\varphi,s)\}$ and $(d_{jj})_s=\exp\{i(\rho_j,s)\}$.

Then, by (4.4),

$$(r_{jk},s)\equiv(\varphi,s)+(\rho_j,s)-(\rho_k,s) \pmod{2\pi} \tag{4.5}$$

for every $s\in S$ and, conversely, if φ, ρ_j, $\rho_k\in T^v$ can be given to satisfy (4.5), then (4.4), (4.1) and (4.2) hold for $s\in S$. We note that $d_{11}=1$ involves $\rho_1=0$.

To solve the system of congruences (4.5) put $j=k$. We get $(r_{jj},s)\equiv(\varphi,s)\pmod{2\pi}$ for every j. Next set $k=1$ to obtain $(\rho_j,s)\equiv(r_{j1},s)-(r_{11},s)\pmod{2\pi}$. Thus (4.5) can be solved to be fulfilled for every $s\in S$ if and only if, for every $s\in S$,

$$(r_{jk},s)\equiv(r_{11},s)+(r_{j1},s)-(r_{k1},s) \pmod{2\pi}.$$

Hence the theorem.

For $v=1$ the theorem takes a more transparent form.

Corollary 4.7. *Let $v=1$ and $s_0=\inf\{s:s\in S,s>0\}$. Then*

$$\frac{2\pi}{s_0}=g\cdot c\cdot d\{\pi_{j,k},r_{jk}-r_{11}-r_{j1}+r_{k1},1\leqq j,k\leqq d\}$$

where π_{jk} denotes the span of the distribution $\mathscr{P}_{jk}$ ($\pi_{jk}=0$ of $\mathscr{P}_{jk}$ is degenerate, and $2\pi/s_0=\infty$, if $s_0=0$).

This corollary is a generalization of the classical result saying that the span of an arithmetic distribution is equal to $2\pi/s_0$ where s_0 is the minimal period of the absolute value of its characteristic function.

5. Local Limit Theorem: Final Result

Our final result will be more general than pre-Theorem 2.1 in two respects. First, it will cover the case of multidimensional random walks with internal states and, secondly, we shall drop condition (iv) of the same theorem. We are going to discuss these generalizations separately.

Extension to the multidimensional case. This extension is straightforward. Though, for multiparameter perturbations, in general, the Taylor expansion of the eigenvalues may not exist, nonetheless simple eigenvalues do have Taylor expansions (cf. [8], § 8 of Chap. II). Thus, analogously to Lemma 3.1, we have

Lemma 5.1. *If for a random walk in Z^v with internal states*

(i) *Q is ergodic and aperiodic;*

(ii) $(\mu, M_l \mathbf{1})=0$ *for* $1 \leqq l \leqq v$;

(iii) *The matrix* $\sigma=(\sigma_{lm})_{1 \leqq l, m \leqq v}$ *whose elements are*

$$\sigma_{lm}=(\mu, \Sigma_{lm}\mathbf{1})-(\mu, M_l(Q-1)^{-1} M_m \mathbf{1})-(\mu, M_m(Q-1)^{-1} M_l \mathbf{1})$$

is positive definite; then, near 0, *the largest eigenvalue* $\lambda(t)=\lambda(t_1, \ldots, t_v)$ *of* $\alpha(t)$ *has the form*

$$\lambda(t)=1-\tfrac{1}{2} t \sigma t+o(|t|^2).$$

Dropping condition (iv) *of pre-Theorem* 2.1.

We allow S to consist of several elements. The approach will be the same as in the proof of pre-Theorem 2.1 but in the limit $n \to \infty$ the neighborhoods of each $s \in S$ give a contribution. Fortunately, Lemma 4.4 greatly simplifies the evaluation of these contributions. Indeed, for $s \in S$ the largest eigenvalue γ_s of $\alpha(s)$ lies on the unit circle. Moreover, (4.1) says that $\alpha(s+t)$, apart from the factor γ_s, is unitary equivalent to $\alpha(t)$. This remark immediately gives the Taylor expansion of the largest eigenvalue $\lambda_s(t)$ of $\alpha(s+t)$ for small t

$$\lambda_s(t)=\gamma_s(1-\tfrac{1}{2} t \sigma t+o(|t|^2)) \qquad (t \to 0).$$

An important consequence of this expansion, and of condition (iii) of Lemma 5.1 is that points of S are isolated, hence S is a discrete group. Moreover, in analogy with (3.2), we obtain that in the neighborhood of $s \in S$

$$\alpha^n(s+t)=(D_s^n \mathbf{1}(D_s^n \mu)^*) \gamma_s^n(1-\tfrac{1}{2} t \sigma t+o(|t|^2))^n \qquad (t \to 0). \tag{5.1}$$

Now $\sqrt{n} P(\xi_n=(x, \cdot) | \xi_0=(0, j)) \stackrel{\text{def}}{=} \Omega$ can be calculated by using (2.2). In fact,

$$\Omega=\frac{\sqrt{n}}{(2\pi)^v} \int_{-\pi}^{\pi} \ldots \int_{-\pi}^{\pi} e^{-i(x, t)} e_j^* \alpha^n(t)\, dt$$

and, as in the proof of pre-Theorem 2.1,

$$\begin{aligned}
\Omega & \sim \frac{\sqrt{n}}{(2\pi)^v} \sum_{s \in S} \int \ldots \int_{|t|<\delta/\sqrt{n}} e^{-i(x, s+t)} e_j^* \alpha^n(s+t)\, dt \\
& =\frac{\sqrt{n}}{(2\pi)^v} \sum_{s \in S} e^{-i(x, s)} \gamma_s^n e_j^*(D_s^n \mathbf{1},(D_s^n \mu)^*) \int_{|t|<\delta/\sqrt{n}} e^{-i(x, t)}(1-\tfrac{1}{2} t \sigma t)^n\, dt+o(1)
\end{aligned}$$

where $\sim$ is to mean that the both sides are equal under $\lim_{\delta \to 0} \lim_{n \to \infty}$. The factor

$$\frac{\sqrt{n}}{(2\pi)^\nu} \int \dots \int_{|t| < \delta/\sqrt{n}} e^{-i(x,t)} (1 - \tfrac{1}{2} t \sigma t)^n \, dt \sim g_\sigma \left(\frac{x}{\sqrt{n}} \right)$$

where $g_\sigma(x)$ denotes the density of the Gaussian distribution with mean 0 and covariance matrix σ. Finally, by the notations of Sect. 4, the k-th component of the vector

$$\vartheta^* = \sum_{s \in S} e^{-i(x,s)} \gamma_s^n e_j^* (D_s^n \mathbf{1} (D_s^n \mu)^*)$$

is

$$\vartheta_k = \mu_k \sum_{s \in S} \exp\{ i(-x + n\varphi + n\rho_j - n\rho_k, s) \}. \tag{5.2}$$

But, since $\exp\{i(z, s)\}$ ($z \in Z^\nu$) is a character of the group S, the sum in (5.2) equals either 0 or card S. More exactly, the group S determines a dual subgroup L of Z^ν in the same way as $\hat{L}$ determines L in statement (iii) of Lemma 4.2. Then the sum in (5.2) equals card S if and only if $-x + n(\varphi + \rho_j - \rho_k) \in L$, otherwise its value is 0.

Finally, we show that in bounding the terms corresponding to I_4 of the proof of pre-Theorem 2.1 it is sufficient to know that, in compact subsets of the torus, disjoint of S, the spectral radius $R(\alpha(t))$ of $\alpha(t)$ is uniformly less than some $C < 1$. In fact, by $R(\alpha(t)) = \lim_{n \to \infty} \|\alpha^n(t)\|^{1/n}$ and the continuity of the norm, for every t such that $R(\alpha(t)) < C$ there exists an open neighbourhood U of t and a natural number n_0 such that $\|\alpha^{n_0}(s)\|^{1/n_0} < C$ whenever $s \in U$. Moreover, if $n = kn_0 + l$ ($0 \leqq l < n_0$), then

$$\int_U \|\alpha^n(s)\| \, ds \leqq C^{kn_0} \int_U \|\alpha(s)\|^l \, ds.$$

This inequality complemented by compactness arguments yields the exponential convergence to 0 of I_4.

Denote card $S = m$. We can now formulate our final result

Theorem 5.2. *Under the conditions of Lemma* 5.1

$$\sum_{(x,k) \in H} \left| (\xi_n = (x, k) \mid \xi_0 = (0, j)) - \frac{1}{\sqrt{n}} m \mu_k g_\sigma \left(\frac{x}{\sqrt{n}} \right) \chi_L(-x + n(\varphi + \rho_j - \rho_k)) \right| \to 0$$

as $n \to \infty$.

Here $\chi_L(x)$ is the indicator of the set L.

6. Remarks

1. The first local limit theorem for random walks with internal states was proved by Sinai in [9]. He proved a similar result as ours for simple, self-adjoint random walks, i.e. he assumed that $p_{y,j,k} = 0$ unless $|y| = 1$ and $p_{y,j,k} = p_{-y,k,j}$ (self-adjointness!).

2. Some conditions of Theorem 5.2 (and of Lemma 5.1) can be weakened without any difficulty. If Q is not aperiodic, then it has several eigenvalues on the unit circle and a Taylor expansion, near 0, should be calculated for each of them. In the non-ergodic case, the limit distribution will be a mixture.

If $(\mu, M_l \mathbf{1})$ is not equal to 0, then our result remains true by using a suitable centering.

Without condition (iii) the limit distribution may be degenerate. We remark that if ε_0 has the stationary distribution μ of Q, then $\mu_0, \mu_1, \ldots$ is a stationary sequence and σ_{jk} coincides with the asymptotic coefficient for covariance usually used at stationary sequences (cf. [7]).

3. Consider a Sinai billiard on the ν-dimensional torus T^ν and continue it periodically to the whole space R^ν (this can be done by the decomposition $R^\nu = \bigcup_{Z^\nu}$ {elementary lattice cubes} and by identifying T^ν with any of the elementary lattice cubes). This periodical continuation gives rise to the Lorentz process: a particle starts according to some initial distribution and moves along trajectories determined by specular reflection at the scatterers and uniform motion between them. The Markov partition of the Sinai billiard constructed for $\nu=2$ by Buninovich and Sinai [2] is a countable partition of the phase space of the Sinai billiard and, to treat properties of the Lorentz process, it can also be continued periodically. E.g. the Wiener approximation of the Lorentz process given by Bunimovich and Sinai [3] (who used the Markov partition and Bernstein's classical method) is hoped to be obtained by basing on the periodic continuation of the Markov approximation and on the probability theory of random walks with internal degrees of freedom. In this approach the index of the elementary lattice cube, where the Lorentz particle gets actually reflected ($\in Z^\nu$) is the random walk component, while the index of the element of the Markov partition in which the particle is contained in the moment of reflection is the internal state. Even in the framework of this sketchy description we should remark that this process itself is not a Markov chain but by properties of the Markov partition it can be sufficiently well approximated by Markov chains.

To realize this program one should first work out a probability theory for random walks with internal degrees of freedom and this is the aim of the present paper. Its present part has been devoted to local limit theorems. They are planned to apply to obtain a local version of the Bunimovich-Sinai central limit theorem for the Lorentz process [3], that would imply, in particular, Pólya type recurrence behaviour, i.e. recurrence of the Lorentz process for $\nu=2$ and non-recurrence for $\nu \geqq 3$ provided the Markov partition will also be constructed for this latter case, too. In the second part of the paper we calculate certain ruin probabilities and expectations (case $\nu=1$) from which Fourier's law of heat conduction, also in a Lorentz process setting, is hoped to be deduced.

4. When applying our results to the Lorentz process one has a sequence of Markov processes with an increasing number of states. The analogue of Lemma 5.1 will be true with a uniform remainder term and, what is more, the uniform Doeblin condition verified in [9] for the Markov approximation of

the Lorentz process also ensures that the gap in the spectrum of Q is uniform. Consequently, (5.1) will hold uniformly in the Markov approximation. The main difficulty is to achieve uniformity in the consequences of the Wielandt-lemma in this approximation. This problem of arithmetics also arises when one wants to extend our results to the case $d=\infty$. The first progress in this direction is achieved in the author's forthcoming paper "How to prove the CLT for the Lorentz process by using perturbation theory?" (to appear in Proceedings of the 3rd PSMS, Reidel Publ.).

5. It is worth stressing that an important novelty of Theorem 5.2 lies in the constructive description given by Theorems 4.6 and 4.7 of the group $\mathscr{S}$. As a comparison see the conditions of classical local limit theorems for Markov chains by Kolmogorov [10] and Statulevicius [11]. Of course, our conditions are based on the additional structure of the Markov chain we treat.

Added in Proof. As a matter of fact Gyires (cf. Gyires, B.: On a generalization of the local version of the CLT. (In Hungarian). MTA III. Oszt. Közl. **10**, 469–479 (1960) and Gyires, B.: Eine Verallgemeinerung des zentralen Grenzwertsatzes. Acta Math. Acad. Sci. Hungar. **13**, 69–80 (1962)) in his studies on Toeplitz type hypermatrices, proved a local central limit theorem closely related to the Theorem of [9]. His papers refer to a remark of Rényi, who also found a probabilistic interpretation of Gyires's result, namely just in terms of random walks with internal states.

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