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On One-Parameter Proofs of Almost Sure Convergence of Multiparameter Processes

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The martingale convergence theorem of R. Cairoli [4] was proved assuming that the filtration satisfies a conditional independence assumption usualy called (F4). It belongs to the folklore of the subject (see also [17]) that (F4) can be restated as a condition on commutation of conditional expectation operators. This formulation will allow us to derive Cairoli's theorem from a simple general argument about operators on Orlicz spaces. The advantage is that one obtains a unified proof of multi-parameter versions of several other results: the theorems of Rota, Dunford-Schwartz, Akcoglu, and Stein.

H. Föllmer was able to apply Proposition 2.1 below to random fields.

1. A General Argument

Let I be a countable set, partially ordered by $\leqq$ and filtering to the right. Let $(T_s, s\in I)$ be a net of positive linear operators from an Orlicz space L_Φ to a larger Orlicz space $L(1)$, of a probability space. We assume that Φ satisfies the Δ_2 condition (is "moderate"). (For a discussion of Orlicz spaces, see e.g., the appendix to [19]. The Δ_2 condition is that at infinity: cf. [14, p. 120].) Consider the following assumption.

(i) For each random variable X in L_Φ, $\lim_I T_s X$ exists a.s. and is in $L(1)$. Denote this limit $T_\infty X$.

We observe that this implies

(ii) If $(S_n, n\in\mathbb{N}=\{1,2,\ldots\})$ is a sequence of random variables in L_Φ such that

$$\lim_{\mathbb{N}} \downarrow S_n = 0 \quad \text{a.s.},$$

then

$$\lim T_\infty S_n = 0 \quad \text{a.s.}$$

Indeed, (ii) holds for *every* positive operator T from L_Φ to $L(1)$. To see this, observe that such an operators is necessarily continuous. Now $S_n \downarrow 0$ and the

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Δ_2 condition imply that $\|S_n\|_\Phi \downarrow 0$, hence $\|TS_n\|_{L(1)} \downarrow 0$. Now use the Fatou property of $\| \ \|_{L(1)}$.

Results related to the following proposition are known: Blackwell-Dubins [2], Maker [15], and Hunt [12, p. 47]. The Blackwell-Dubins theorem would be sufficient to obtain the two-parameter case of Cairoli's theorem.

1.1. **Proposition.** *Suppose that a net of operators* $(T_s, s \in I)$ *satisfies the condition* (i). *If* $(X_n, n \in \mathbb{N})$ *is a sequence of random variables in* L_Φ *such that* $\sup_n |X_n| \in L_\Phi$ *and* $\lim_{\mathbb{N}} X_n = X_\infty$ a.s., *then*

$$\lim_{\substack{s \in I \\ n \in \mathbb{N}}} T_s X_n = T_\infty X_\infty \quad \text{a.s.}$$

Proof. For each fixed $n \in \mathbb{N}$, letting

$$S_n = \sup_{p \geqq n} |X_p - X_\infty|,$$

we have

$$\begin{aligned} \limsup_{s,p} |T_s X_p - T_\infty X_\infty| &\leqq \limsup_{s,p} |T_s(X_p - X_\infty)| + \limsup_s |T_s X_\infty - T_\infty X_\infty| \\ &\leqq \limsup_s T_s S_n = T_\infty S_n. \end{aligned}$$

We let $n \to \infty$ and apply (ii). $\square$

Now for each fixed $m \in \mathbb{N}$ define:

$$I^m = I_1 \times I_2 \times \ldots \times I_m,$$

with $I_k = \mathbb{N}$ for $k = 1, \ldots, m$. The partial order $\leqq$ on I^m is given by:

$$s = (s_1, \ldots, s_m) \leqq t = (t_1, \ldots, t_m)$$

if $s_k \leqq t_k$ for $k = 1, \ldots, m$. Let $L(1) \supset L(2) \supset \ldots \supset L(m)$ be Orlicz spaces with Δ_2 and let $T(k,n)$, $k = 1, \ldots, m$; $n \in \mathbb{N}$ be positive and linear (hence bounded) operators from $L(k)$ to $L(1)$.

Consider the following property of the system:

For every $k = 1, \ldots, m$, if $X \in L^+(k)$, then

(iii) (a) $\lim_n T(k,n) X = T(k,\infty) X$ exists a.s.

(b) $\sup_n T(k,n) X \in L(k-1)$ for $k \geqq 2$.

(c) $T(1,\infty)$ maps $L(1)$ to $L(1)$.

Set for $X \in L(m)$, $U_s X = T(1,s_1) T(2,s_2) \ldots T(m,s_m) X$.

1.2. **Theorem.** *Suppose that the system satisfies* (iii). *Then for each* $X \in L(m)$, *almost surely*

$$\lim_{I^m} U_s X = T(1,\infty) \ldots T(m,\infty) X.$$

Proof. By induction on m. For $m = 2$, $X \in L(2)$ and therefore $\sup_n T(2,n) X \in L(1)$. Apply Proposition 1.1 with $L_\Phi = L(1)$. Now we suppose that the assertion holds for I^m, and we prove it for I^{m+1}. For $t \in I^{m+1}$, $X \in L(m+1)$, one has

$$\begin{aligned} U_t X &= T(1, t_1) \ldots T(m, t_m)[T(m+1, t_{m+1}) X] \\ &= U_s[T(m+1, t_{m+1}) X], \end{aligned}$$

with $s=(t_1, \ldots, t_m)$. By the hypothesis

$$\sup_n T(m+1, n) X \in L(m)$$

and $T(m+1, n) X$ converges a.s.

Now apply Proposition 1.1 with

$$L_\Phi = L(m). \quad \square$$

Given an Orlicz function Φ with $\Phi' = \varphi$ and the conjugate function Ψ, set

$$\xi(u) = u\,\varphi(u) - \Phi(u) = \Psi[\varphi(u)].$$

To apply the previous theorem, the following will be useful.

1.3. **Proposition.** *Let X and Y be positive random variables with $X \in L_\Phi$, $\|X\|_\Phi = 1$, and suppose that for every constant $\lambda > 0$,*

$$P(Y > \lambda) \leqq \frac{1}{\lambda} E(X\, 1_{\{Y > \lambda\}}). \tag{1}$$

Then for any constant $\rho > 1$, one has

$$E\left[\xi\left(\frac{Y}{\rho}\right)\right] \leqq \frac{1}{\rho - 1}. \tag{2}$$

Proof. The case when $Y \in L_\infty$ is proved (but not stated) in Neveu [19, pp. 218–219]. To obtain the general case, observe that for each positive constant C, (1) implies the same inequality with Y replaced by $Y \wedge C$. Now let in (2) $C \uparrow \infty$. $\square$

If $\Phi(u) = u(\log^+ u)^m$, we denote the corresponding Orlicz space by $L \log^m L$. In that case, $\xi(u) = m\,u(\log^+ u)^{m-1}$. Hence Proposition 1.3 has the following:

1.4. **Corollary.** *If X and Y satisfy* (1) *and $X \in L \log^m L$, then $Y \in L \log^{m-1} L$.*

There is also a short direct proof of the Corollary by a standard argument giving a less precise estimate than (2):

$$\begin{aligned} 2\lambda P(Y > 2\lambda) &\leqq \int_{\{Y > 2\lambda\}} X\, dP \\ \leqq \int_{\{Y > \lambda\}} X\, dP + &\int_{\{X < \lambda, Y > 2\lambda\}} X\, dP \leqq \int_{\{X > \lambda\}} X\, dP + \lambda P(Y > 2\lambda), \end{aligned}$$

from which one obtains:

$$P(Y > 2\lambda) \leqq \frac{1}{\lambda} \int_{\{X > \lambda\}} X\, dP.$$

Hence applying Fubini:

$$E\left[\xi\left(\frac{Y}{2}\right)\right]=E\left[\int_0^{Y/2} d\xi(\lambda)\right]=\int_0^{\infty} P(Y>2\lambda)\, d\xi(\lambda)$$

$$\leqq \int_0^{\infty} d\xi(\lambda)\frac{1}{\lambda}\int_{\{X>\lambda\}} X\, dP=\int X\int_0^X \frac{d\xi(\lambda)}{\lambda} dP=E[X\,\varphi(X)].$$

Now let $\Phi(u)=u(\log^+ u)^m$. □

Remark. The particular property of Orlicz spaces as Banach lattices used in this section, namely that $X_i \downarrow 0$ implies $\|X_i\| \to 0$, is called "order-continuous norm." 1.1 and 1.2 depend only on this property, which for Orlicz spaces is a consequence of Δ_2. The results apply to general Banach lattices, provided that the one-dimensional limit operators $T(k,\infty)$ preserve monotone sequential convergence to zero, i.e., if (ii) holds.

2. Applications

We will use the notation of Theorem 1.2 letting at first $L(1)=L_1, \ldots, L(m) = L\log^{m-1} L$. Let for each fixed k, $\mathfrak{F}_n^k$, $n\in\mathbb{N}$ be either increasing or decreasing sub-σ-fields of $\mathfrak{F}$, let $\mathfrak{F}_\infty^k=\bigvee_n \mathfrak{F}_n^k[\bigwedge_n \mathfrak{F}_n^k]$, $E_n^k=E[\cdot\,|\mathfrak{F}_n^k]$ for $n\in\mathbb{N}\cup\{\infty\}$, $U_s = E_{s_1}^1 E_{s_2}^2 \ldots E_{s_m}^m$ for $s=(s_1,\ldots,s_m)\in I^m$.

The weak maximal inequality (1) in Proposition 1.3 holds for martingales, that is if $Y=\sup_n E[X\,|\mathfrak{F}_n^k]$, hence one obtains:

2.1. **Proposition.** *If*

$$X\in L\log^{m-1} L,$$

then

$$E_{s_1}^1 E_{s_2}^2 \ldots E_{s_m}^m X$$

converges to

$$E_\infty^1 \ldots E_\infty^m X$$

when the indices $s_i \to \infty$ *independently.*

Let $(\mathfrak{F}_s, s\in I^m)$ be an increasing (decreasing) net of sub-σ-fields, of $\mathfrak{F}$. A martingale (reversed martingale) indexed by I^m is defined by the property: if $s\leqq t$ $[s\geqq t]$ then $E[X_t|\mathfrak{F}_s]=X_s$. Let $\mathfrak{F}_\infty=\bigvee \mathfrak{F}_s[\bigcap \mathfrak{F}_s]$.

Now for $s=(s_1,\ldots,s_m)\in I^m$, $\mathfrak{F}_s^k$ is defined as the σ-field obtained by lumping together the σ-fields on all the axes except for the k-th one. That is,

$$\mathfrak{F}_s^k=\bigvee \mathfrak{F}_{(s_1,s_2,\ldots,s_k,\ldots,s_m)},$$

where $\bigvee$ is taken over all $s_1\in I_1, \ldots, s_{k-1}\in I_{k-1}, s_{k+1}\in I_{k+1}, \ldots, s_m\in I_m$. Let for $k\leqq m$, $s\in I^m$

$$E_s^k=E[\cdot\,|\mathfrak{F}_s^k].$$

The *commutation assumption* is the assumption that the operators $E_s^k=E_{s_k}^k$ on L_1 commute; then

$$E[\cdot\,|\mathfrak{F}_s]=E_{s_1}^1 E_{s_2}^2 \ldots E_{s_m}^m.$$

We now state Cairoli's convergence theorem. A proof via Banach valued martingale sequences was given by Chaterji [5]. For arguments reducing two-parameter theorems to amarts with respect to totally ordered filtrations, also in the continuous parameter case, see Millet-Sucheston [18].

2.2. **Cairoli's Theorem.** *Let* $(X_s,\ s\in I^m)$ *be a martingale or a reversed martingale such that* $\sup_s E[|X_s|(\log^+|X_s|)^{m-1}]<\infty$.

Suppose that $(\mathfrak{F}_s)$ satisfies the commutation assumption. Then $\lim_{I^m} X_s$ exists a.s.

Proof. Since (X_s) is bounded in $L\log^{m-1}L$ for $m\geqq 2$, it is uniformly integrable and admits a representations $X_s=E[X|\mathfrak{F}_s]$ for an $X\in L\log^{m-1}L$. Then the previous proposition is applicable. (The limit of X_s is necessarily $E[X|\mathfrak{F}_\infty]$.) □

A related result is Rota's theorem. Recall that it says that if T_i are positive operators on L_1 such that $T_i 1=1$ and $T_i^* 1=1$ (bistochastic operators), and $U_n=T_1\ \dots\ T_n T_n^*\ \dots\ T_1^*$, then $U_n X$ converges a.s. for each $X\in L\log L$.

The proof of this result (see [20, 6], or Doob [7]) shows that U_n can be represented as $E^{\mathfrak{B}}E^{\mathfrak{C}_n}$, where $\mathfrak{C}_n\downarrow$ and $\mathfrak{B}$ is a fixed σ-field. Since $E^{\mathfrak{B}}$ is a contraction on each L_Φ, the previous discussion shows that if $X\in L\log^m L$ then $\sup_n U_n X\in L\log^{m-1}L$. Hence Theorem 1.2 is applicable with $L(m)=L\log^m L$, and we obtain

2.3. **Multiparameter Rota's Theorem.** *Let for* $i=1, 2, \dots, m$ *and for* $n\in\mathbb{N}$, T_n^i *be a bistochastic operator and set*

$$U_n^i=T_1^i\dots T_n^i(T_n^i)^*\dots(T_1^i)^*.$$

If $X\in L\log^m L$, *then*

$$\lim U_{s_1}^1\dots U_{s_m}^m X$$

exists a.s. as the indices $s_i\to\infty$ *independently.*

We will now discuss ergodic averages. Let $T_1,\dots,T_m$ be bi-substochastic operators: positive linear contractions on L_1 such that $T_i 1\leqq 1$ for $i=1,\dots,m$. The following theorem, due to Fava [10], is a multiparameter version of the Dunford-Schwartz theorem, or, equivalently, the operator version of Dunford [8] and Zygmund [23].

2.4. **Theorem.** *If* $X\in L\log^{m-1}L$, *then*

$$\lim \frac{1}{s_1\dots s_m}\sum_{k_1=0}^{s_1-1}\dots\sum_{k_m=0}^{s_m-1} T_1^{k_1}\dots T_m^{k_m}X \tag{3}$$

exists a.s. as the indices $s_i\to\infty$ *independently.*

Proof. The result again follows from Theorem 1.2. The weak inequality (1) in Proposition 1.3 holds in the one-parameter case with $X\in L_1^+$,

$$Y=\sup_n \frac{1}{n}\sum_{k=0}^{n-1} T^i X$$

if T is bi-substochastic: this is due to Hopf-Dunford-Schwartz (see, e.g., Garsia [11], p. 24). Let $T(k,n)=(1/n)\sum_{j=0}^{n-1} T_k^j$.

Note that 2.4 remains true if T_i are *not* necessarily *positive* contractions of L_1 and L_∞. The passage from non-positive operators to positive "modulus" operators is by standard arguments which we will not give here.

It should be noted that there is another mutliparameter version of the Dunford-Schwartz theorem in which convergence is "restricted"; it is over "squares" rather than "rectangles", but X is allowed to be only integrable (cf. Dunford-Schwartz [9]; also Brunel [3]). This deep result most likely cannot be simply reduced to the one-parameter case.

Finally, multiparameter versions of Akcoglu's theorem [1], and of several theorems of Stein [22, pp. 86–87] are also available by the method of this note. The proofs are, if anything, easier, because all the Orlicz spaces $L(i)$ in Theorem 1.2 are the same: for Akcoglu's theorem, they are L_p for a fixed p, $1<p<\infty$; for Stein's theorems they are L_2. We omit the details, but state here:

2.5. **Multiparameter Akcoglu's Theorem.** *Let p be fixed with $1<p<\infty$ and let $T_1, \dots, T_m$ be Akcoglu's operators: positive linear contraction on L_p. Then for each $X \in L_p$, the averages* (3) *converge a.s. as $s_i \to \infty$ independently.*

This result is known. A proof was obtained by S.M. McGrath [16, Theorem 3].

3. Infinite Measure Spaces

We finally comment on extensions of present results to infinite measure spaces. This is completely routine when the operators are contractions on only one space, as in the theorem of Akcoglu, since an equivalent change of measure transforms an operator acting on L_p of a σ-finite measure space to an isomorphic one acting on L_p of a probability space. The case of bi-stochastic or bi-substochastic operators is more difficult, and the right setting are not the spaces $L \log^m L$, but spaces R_m introduced by Fava [10], not Orlicz spaces, but intersections of Orlicz spaces. The methods of the present note extend, and in an article in preparation we obtain not only the theorem of Fava, i.e., the multiparameter infinite measure version of the Dunford-Schwartz theorem, but also analogous versions of other ergodic and martingale theorems.

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