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Strassen's Law for Local Time

Carl Mueller

Dept. of Mathematics, The University of Texas at Austin, Austin, TX 78712, USA

1. Introduction

Let $B(t)$ be Brownian motion. The law of the iterated logarithm asserts that

$$\limsup_{t\uparrow\infty} \frac{|B(t)|}{(2t\log\log t)^{1/2}} = 1.$$

This theorem and many others follow from Strassen's functional law of the iterated logarithm [6]. For $0 \leqq s \leqq 1$, set

$$\bar{B}_t(s) = \frac{B(s\,t)}{(2t\log\log t)^{1/2}}.$$

Strassen's law states that the set $\{\bar{B}_t\}$ is relatively compact in the uniform topology, and as $t\uparrow\infty$, the set of limit points is the set K of absolutely continuous functions $g(x)$ satisfying $g(0)=0$ and $\int_0^1 (g'(x))^2\,dx \leqq 1$. If Φ is a functional continuous in the uniform topology, this implies that

$$\limsup_{t\uparrow\infty} \Phi(\bar{B}_t) = \sup_{g\in K} \Phi(g).$$

Chung's law of the iterated logarithm [1] states that

$$\liminf_{t\uparrow\infty} \left(\frac{\log\log t}{t}\right)^{1/2} \sup_{0\leqq s\leqq t} |B(s)| = \frac{\pi}{\sqrt{8}}.$$

In an unpublished paper, Wichura [7] proves a functional form of Chung's theorem analogous to Strassen's law. For $0 \leqq s \leqq 1$, let

$$W_t(s) = \left(\frac{\log\log t}{t}\right)^{1/2} \sup_{x\leqq st} |B(x)|.$$

Let G be the set of nondecreasing nonnegative functions g on $[0,1]$ satisfying

$$\int_0^1 \frac{1}{g(x)^2}\,dx \leqq \frac{8}{\pi^2}.$$

Then, with probability 1, the set of limit points of $\{W_t\}$ in the weak topology, as $t \nearrow \infty$, is G.

For the remainder of the paper, we will chiefly consider

$$B_t(s) = \left(\frac{\log\log t}{t}\right)^{1/2} B(s\,t).$$

Donsker and Varadhan [3], using their powerful asymptotic methods, give another functional form of Chung's law. Let L_t be the occupation measure

$$L_t(A) = \int_0^1 \chi_A(B_t(s))\,ds$$

and let l_t be its density, so that

$$L_t(A) = \int_A l_t(y)\,dy.$$

In the topology of uniform convergence on bounded intervals, $\{l_t\}$ is relatively compact, and as $t \uparrow \infty$ the set of limit points is the set C of subprobability densities $g(y)$ such that

$$I(g) \equiv \tfrac{1}{8} \int_{-\infty}^{\infty} \frac{(g'(y))^2}{g(y)}\,dy \leqq 1.$$

Then, for suitable functionals Φ,

$$\liminf_{t \uparrow \infty} \Phi(l_t) = \inf_{\substack{g \in C \\ \int g = 1}} \Phi(g).$$

The purpose of this paper is to prove a functional form of Chung's theorem which contains both Wichura's, and Donsker and Varadhan's results. Furthermore, we deal with functionals which do not necessarily depend on the local time density l_t.

For $0 \leqq s \leqq 1$, let $\hat{B}_t(s)$ be the spacetime process

$$\hat{B}_t(s) = \left(s, \left(\frac{\log\log t}{t}\right)^{1/2} B(s\,t)\right).$$

If A is a subset of $[0,1] \times \mathbb{R}$, let $\hat{L}_t(A)$ be the occupation measure

$$\hat{L}_t(A) = \int_0^1 \chi_A(\hat{B}_t(s))\,ds.$$

Although L_t can be recovered from $\hat{L}_t$, the reverse is not true. Since $\hat{L}_t$ does not have a density, we must use a different topology than uniform convergence on bounded intervals.

Suppose that $0 \leqq a < b \leqq 1$ and $D \subseteq \mathbb{R}$, and let

$$\hat{L}_t^{a,b}(D) = \hat{L}_t((a,b) \times D) = \int_a^b \chi_D(B_t(s))\, ds.$$

By the existence of local time for Brownian motion, we can write

$$\hat{L}_t^{a,b}(D) = \int_D l_t^{a,b}(y)\, dy$$

where $l_t^{a,b}$ is a nonnegative function.

Notice that $\hat{L}_t$ determines $l_t^{a,b}$. We will say that $\hat{L}_{t_i}$ converges to L in the topology $\mathcal{T}$ if for all $0 \leqq a < b \leqq 1$, $l_{t_i}^{a,b}$ converges to $l^{a,b}$ uniformly on bounded intervals.

By an abuse of notation, we will call L a subprobability if for all $0 \leqq t_1 < t_2 \leqq 1$

$$\int_{t_1 < t < t_2} \int_{x \in \mathbb{R}} L(dx\, dt) \leqq t_2 - t_1.$$

Next, denote by $\hat{I}(L)$ the supremum over all partitions

$$0 = a_1 < a_2 < \ldots < a_{n+1} = 1, \qquad n = 1, 2, \ldots,$$

of

$$\sum_{k=1}^{n} I(l^{a_k, a_{k+1}}).$$

Theorem 1.1. *In the topology $\mathcal{T}$, the set $\{\hat{L}_t\}$ is relatively compact. As $t \uparrow \infty$, the set of limit points is the set $\hat{C}$ of subprobabilities L such that*

$$\hat{I}(L) \leqq 1.$$

2. The Exponential Estimate

Let $\mathcal{U}$ be the set of functions $u(x,y)$ on $(0,1) \times \mathbb{R}$ having two bounded continuous derivatives in y and one in x, and for each of which there are two numbers α and β such that for all y, $0 < \alpha \leqq u(x,y) \leqq \beta < \infty$.

Below we give another definition of $\hat{I}$, equivalent to the first. Since the proof of equivalence is similar to an argument of Donsker and Varadhan ([2] p. 27), we will omit it.

For L a subprobability, let

$$\hat{I}(L) = -\inf_{u \in \mathcal{U}} \int_0^1 \int_{-\infty}^{\infty} \frac{u_{yy}}{2u}\, dL(x,y).$$

If C is a set of probabilities on $(0,1)\times\mathbb{R}$, and if $\tilde{L}_t$ is the occupation density for $(s, B(st))$, let

$$Q_{z,t}(C)=P_z\{\tilde{L}_t\in C\}$$

where P_z denotes the Brownian motion measure with $B(0)=z$.

Note that with respect to $f(x,y)$, the infinitesimal generator (see [4]) of $(s, B(st))$ is

$$\frac{\partial}{\partial x}+\frac{t}{2}\frac{\partial^2}{\partial y^2}.$$

For $u\in\mathscr{U}$, let

$$\psi(0,y,t)=E_y\left\{u(s,B(st))\ \exp\left[-\int_0^1\left(\frac{t\,u_{yy}+2u_x}{2u}\right)\cdot(s,B(st))\,ds\right]\right\}.$$

By the Feynman-Kac formula,

$$\frac{\partial\psi}{\partial t}=\frac{1}{2}\frac{\partial^2\psi}{\partial y^2}+\frac{\partial\psi}{\partial x}-\frac{t\,u_{yy}+2u_x}{2u}\psi.$$

Since $\psi(0,y,t)=u(0,y)$ is a solution of this equation, we have

$$E_y\left\{u(s,B(st))\exp\left[-\int_0^1\left(\frac{t\,u_{yy}+2u_x}{2u}\right)(s,B(st))\,ds\right]\right\}=u(0,y).$$

But since $u(x,y)\geqq\alpha>0$,

$$E_y\left\{\exp\left[-\int_0^1\left(\frac{t\,u_{yy}+2u_x}{2u}\right)(s,B(st))\,ds\right]\right\}\leqq\frac{u(0,y)}{\alpha}.$$

Let γ be the bound of u_x. (Recall that γ depends on u.) Therefore,

$$E_y\left\{\exp\left[-\int_0^1\left(\frac{t\,u_{yy}+2u_x}{2u}\right)(s,B(st))\,ds\right]\right\}$$
$$\geqq e^{-\frac{\gamma}{\alpha}}E_y\left\{\exp\left[-t\int_0^1\int_{-\infty}^{\infty}\left(\frac{u_{yy}}{2u}\right)(x,y)\,d\tilde{L}_t(x,y)\right]\right\}.$$

Therefore, for any measurable set C,

$$Q_{y,t}(C)\leqq\frac{u(0,y)}{\alpha}e^{\frac{\gamma}{\alpha}}\exp\left[t\sup_{F\in C}\int_0^1\int_{-\infty}^{\infty}\left(\frac{u_{yy}}{2u}\right)(x,y)\,dF(x,y)\right].$$

This formula holds for all $u\in\mathscr{U}$, so

$$\limsup_{t\uparrow\infty}\frac{1}{t}\log Q_{y,t}(C)\leqq\inf_{u\in\mathscr{U}}\sup_{F\in C}\int_{[0,1]\times\mathbb{R}}\left(\frac{u_{yy}}{u}\right)(x,y)\,dF(x,y),$$

and also, if $\{C_i\}_{i=1}^n$ is a finite cover of C, then

$$\limsup_{t\uparrow\infty}\frac{1}{t}\log Q_{y,t}(C)\leqq\max_{1\leqq i\leqq n}\inf_{u\in\mathscr{U}}\sup_{F\in C_i}\int_{[0,1]\times\mathbb{R}}\left(\frac{u_{yy}}{u}\right)(x,y)\,dF(x,y).$$

Suppose that for any set C closed in the vague topology and for any finite cover $\{C_i\}_{i=1}^n$ of C, we could show

$$\max_{1\leqq i\leqq n}\inf_{u\in\mathscr{U}}\sup_{F\in C_i}\int_{-\infty}^{\infty}\left(\frac{u''}{u}\right)(x,y)\,dF(x,y)$$
$$\leqq\sup_{F\in C}\inf_{u\in\mathscr{U}}\int_{[0,1]\times\mathbb{R}}\left(\frac{u''}{u}\right)(x,y)\,dF(x,y).$$

Then we would have shown:

Theorem 2.1. *Suppose that C is closed in the vague topology. Then*

$$\limsup_{t\uparrow\infty}\frac{1}{t}\log Q_{y,t}\leqq-\inf_{F\in C}\hat{I}(F).$$

Proof. Let

$$M=\sup_{F\in C}\inf_{u\in\mathscr{U}}\int_{[0,1]\times\mathbb{R}}\left(\frac{u''}{u}\right)(x,y)\,dF(x,y)$$

and fix $\varepsilon>0$. Note that the set of subprobabilities is compact in the vague topology, so C must be compact. For each $F\in C$, we can clearly choose $u_F\in\mathscr{U}$ such that

$$\lim_{|(x,y)|\uparrow\infty}\left(\frac{u_F''}{u_F}\right)(x,y)=0$$

and

$$\int_{[0,1]\times\mathbb{R}}\left(\frac{u_F''}{u_F}\right)(x,y)\,dF(x,y)\leqq M+\varepsilon.$$

Now, if

$$C_F=\left\{G\text{ a subprobability: }\int_{[0,1]\times\mathbb{R}}\left(\frac{u_F''}{u_F}\right)(x,y)\,dG(x,y)<M+\varepsilon\right\}$$

then C_F is open in the vague topology. Since C is compact, we can choose a finite set $\{F_i\}_{i=1}^n$ such that $\{C_{F_i}\}_{i=1}^n$ is a cover of C. Set $N_i=C_{F_i}$.

Thus,

$$\sup_{G\in N_i}\int_{[0,1]\times\mathbb{R}}\left(\frac{u_{F_i}''}{u_{F_i}}\right)(x,y)\,dG(x,y)\leqq M+\varepsilon,$$

$$\inf_{u\in\mathscr{U}}\sup_{G\in N_i}\int_{[0,1]\times\mathbb{R}}\left(\frac{u''}{u}\right)(x,y)\,dG(x,y)\leqq M+\varepsilon,$$

$$\max_{1\leqq i\leqq n}\inf_{u\in U}\sup_{G\in N_i}\int_{[0,t]\times\mathbb{R}}\left(\frac{u''}{u}\right)(x,y)\,dG(x,y)\leqq M+\varepsilon.$$

Since ε was arbitrary, this proves Theorem 2.1.

Although we could also obtain a lower estimate, it is easier to directly use Donsker and Varadhan's theorem.

3. Completion of the Proof of Theorem 1

Lemma 3.1. *The set of limit points of* $\{\hat{L}_t\}$ *as* $t\uparrow\infty$ *contains* $\hat{C}$.

We will say that a subprobability density $g(x,y)$ is contained in $\mathscr{G}$ if $\hat{I}(g)<1$, g has compact support, and if for some partition $0=a_1<\ldots<a_{n+1}=1$, $g(x,y)$ is a constant function of x when $a_k\leqq x<a_{k+1}$. $\mathscr{G}$ is clearly dense in $\hat{C}$. We will show that the set of limit points contains $\mathscr{G}$.

Considering the definition of $\mathscr{T}$, it suffices to show for any refinement $0=b_1<\ldots<b_{m+1}=1$ of the previous partition, the set of limit points of $\{(l_t^{b_1,b_2},\ldots,l_t^{b_m,b_{m+1}})\}$ contains $(g(b_1,\cdot),\ldots,g(b_m,\cdot))$.

Our strategy will be to choose a sequence $\{t_n\}$ such that t_{n+1}/t_n sufficiently large, and then use the Borel-Cantelli lemma. Now if the $l_t^{b_i,b_{i+1}}$ were independent, we could use estimates of Donsker and Varadhan to approximate the distribution of $(l_t^{b_1,b_2},\ldots,l_t^{b_m,b_{m+1}})$. For each $i=1,\ldots,n$ choose a point q_i in the support of $g(b_i,\cdot)$. Let τ_i be the first time greater than b_i that B_t hits q_i. Also, set $b_1=\varepsilon/2$.

Now since Brownian motion has independent increments, it follows that if $\tau_i<b_{i+1}$, then the $l_t^{\tau_i,b_{i+1}}$, $i=1,\ldots,n$ are independent. Our aim is to show that τ_i-b_i becomes so small as $t\uparrow\infty$ that $l_t^{\tau_i,b_{i+1}}$ is a good approximation of $l_t^{b_i,b_{i+1}}$.

Assume that all of the functions $g(b_i,\cdot)$ are supported in the interval $(\alpha,\alpha+\Delta)$. Clearly, if τ_Δ is the first hitting time of Δ by B_t, then τ_Δ is stochastically larger than each τ_i-b_i in the sense that for all $\varepsilon>0$,

$$P\{\tau_\Delta>\varepsilon\}\geqq P\{\tau_i-b_i>\varepsilon\}.$$

Note also that τ_Δ is $1/t$ times the first hitting time by $B(t)$ of

$$\Delta\left(\frac{t}{\log\log t}\right)^{1/2}.$$

Therefore, as $t\uparrow\infty$, by standard theory,

$$P\{\tau_\Delta>\varepsilon\}=o(t).$$

Therefore, for any $\varepsilon>0$,

$$P\{\sup_i(\tau_i-b_i)>\varepsilon\}=o(t).$$

Let $0<a<b\leqq 1$, and let E be a set of subprobabilities on $\mathbb{R}$, open in the vague topology, satisfying

$$L(\mathbb{R})\leqq b-a \quad \text{if } L\in E.$$

Then, by the asymptotics of Donsker and Varadhan,

$$\liminf_{t\uparrow\infty}\frac{1}{\log\log t}\log P_x(\hat{L}_t^{a,n}\in E)>-\inf_{L\in E} I(L). \tag{**}$$

The $\log\log t$ term appears because we are using $B_t(s)$ instead of $B(s)$.

Now fix $\varepsilon>0$, and let $t_n=\left(\frac{2}{\varepsilon}\right)^n$. Let V_i be a vague neighborhood of $g(b_i,\cdot)$, and let V_i^ε be an ε-enlargement of V_i. That is, replace each equation

$$M<\int_{-\infty}^{\infty} h(x)\, g(b_i,x)\, dx<N$$

occurring in the definition of V_i by the equation

$$M-\varepsilon G<\int_{-\infty}^{\infty} h(x)\, g(b_i,x)\, dx<N+\varepsilon G$$

where $G=\sup_x |h(x)|$. Let A_n be the event

$$A_n=\left\{\frac{\varepsilon}{2}<\tau_1<\varepsilon, \tau_i-b_i<\varepsilon \text{ for } i=2,\dots,n,\right.$$
$$\left.\text{and } l_{t_n}^{\tau_i,b_{i+1}}\in V_i \text{ for } i=1,\dots,n\right\}.$$

Note that these conditions insure that for points in A_n,

$$l^{b_i,b_{i+1}}\in V_i^\varepsilon \quad \text{for } i=1,\dots,n.$$

Our aim is to show via the Borel-Cantelli lemma that A_n occurs infinitely often, almost surely. The A_n are independent, and for n large enough (**) shows that

$$P(A_n)\geqq C(\varepsilon)\, n^{-\sum_{i=1}^{n} I(g(b_i,\cdot))}.$$

But since $\hat{I}(g)<1$, the sum $\sum_{n=1}^{\infty} P(A_n)$ diverges, and so by the Borel-Cantelli lemma, A_n occurs infinitely often. Since ε was arbitrary, this shows that in the vague topology, the set of limit points of $\{\hat{L}_t\}$ contains $\hat{C}$.

Lemma 3.2. *In the vague topology, $\{\hat{L}_t\}$ is compact and its limit points as $t\uparrow\infty$ are contained in $\hat{C}$.*

Proof. For $\varepsilon>0$, let $\hat{C}^\varepsilon$ be the set of subprobabilities L satisfying

$$\hat{I}(L)\leqq 1+\varepsilon.$$

Let N_ε be a neighborhood of $\hat{C}^\varepsilon$, chosen such that

$$\bigcap_{\varepsilon>0} N_\varepsilon=\hat{C}.$$

This is possible because

$$\bigcap_{\varepsilon>0} \hat{C}^\varepsilon=\hat{C}.$$

It suffices to show that with probability 1, there exists a time T_ε such that $t>T_\varepsilon$ implies $\hat{L}_t\notin N_\varepsilon$.

Now fix $\delta>0$ and let $t_n=(1+\delta)^n$. We will show that almost surely, $\hat{L}_{t_n}\notin N_\varepsilon$ only finitely often. Note that

$$\inf_{L\notin N_\varepsilon} \hat{I}(L)\geqq 1+\varepsilon.$$

By Theorem 2.1, for n sufficiently large,

$$P\{\hat{L}_{t_n}\notin N_\varepsilon\}\leqq n^{-(1+\varepsilon/2)}.$$

Since the sum of these probabilities converges, the Borel-Cantelli lemma assers that $\hat{L}_{t_n}\notin N_\varepsilon$ only finitely often.

By the definition of $\hat{L}_t$, we see that we can find neighborhoods $N(\varepsilon,\delta)$ of N_ε such that if $\hat{L}_{t_n}\in N_\varepsilon$ then for all $t_{n-1}<t<t_n$, $\hat{L}_t\in N(\varepsilon,\delta)$, and also that

$$\bigcap_{\varepsilon,\delta>1} N(\varepsilon,\delta)=\hat{C}.$$

But by the previous result, we see that $\hat{L}_t\notin N(\varepsilon,\delta)$ only finitely often. Since ε and δ were arbitrary, this proves the theorem.

4. The Topology $\mathcal{T}$

To finish the proof of Theorem 1, we must show

Lemma 4.1. *$\{\hat{L}_t\}$ has the same limit points in both the vague topology and the topology $\mathcal{T}$.*

Proof. Since the vague topology is weaker than $\mathcal{T}$, it suffices to show that if L is a limit point in the vague topology, then it is also a limit point in $\mathcal{T}$.

Suppose that $\hat{L}_{t_n}\to L$ vaguely. It suffices to show that in $\mathcal{T}$, $\hat{L}_{t_n}$ has a convergent subsequence. We need a theorem of Donsker and Varadhan ([3], Theorem 3.8).

Theorem 4.2. (Donsker and Varadhan). *For each $a>0$, there exists $\delta>0$ such that*

$$P\{\limsup_{t\uparrow\infty}\sup_{|y_1-y_2|\leqq\delta}|l_t(y_1)-l_t(y_2)|\geqq a\}=0.$$

Let $0=a_1<\ldots<a_{n+1}=1$ be a partition of $[0,1]$. Then, as a corollary of Theorem 4.2, we have that for each $a>0$, there exists a $\delta>0$ such that

$$P\{\limsup_{t\uparrow\infty}\sup_{1\leqq i\leqq n}\sup_{|y_1-y_2|\leqq\delta}|l_t^{a_i,a_{i+1}}(y_1)-l_t^{a_i,a_{i+1}}(y_2)|\geqq a\}=0. \tag{*}$$

Since $\mathcal{T}$ depends on compact subsets of $[0,1]\times\mathbb{R}$, we need only consider $\hat{L}_{t_n}|_K$, that is $\hat{L}_{t_n}$ restricted to a compact subset K. In this setting condition (*) and Ascoli's theorem imply that $\hat{L}_{t_n}|_K$ has a convergent subsequence. Since this is true for all K, the proof of Lemma 4.2 is complete.

For certain applications, we need the following theorem. Suppose that g,h are continuous functions on $[0,1]$ such that $g(0)<0<h(0)$ and $h(x)-g(x)>0$ for all $x\in[0,1]$. Also assume that $L\in\hat{C}$ and that for all $0\leqq t_1<t_2\leqq 1$,

$$\int_{t_1<t<t_2}\int_{x\in\mathbb{R}} L(dx\,dt)=t_2-t_1.$$

Let

$$R = \{(t, x) \in [0, 1] \times \mathbb{R} : g(t) < x < h(t)\}$$

and

$$R^\varepsilon = \{(t, x) \in [0, 1] \times \mathbb{R} : g(t) - \varepsilon < x < h(t) + \varepsilon\}.$$

With these definitions, we have

Theorem 4.3. *With $L \in \hat{C}$ as above, if L is supported in R, then for almost all ω there is a sequence t_n such that*

(i) *$\hat{L}_{t_n}$ is supported in R^ε,*
(ii) *$\hat{L}_{t_n} \to L$ in the topology $\mathscr{T}$.*

Proof. It suffices to consider the following case. Let $0 = a_1 < \ldots < a_{n+1} = 1$ be a partition, and let $g < h$ be discontinuous functions on $[0, 1]$ such that $g(0) < 0 < h(0)$, and g, h are constant on each interval $[a_i, a_{i+1})$. Also assume that the intervals $(g(a_i), h(a_i))$ and $(g(a_{i+1}), h(a_{i+1}))$ are not disjoint for any i.

Let $L \in \hat{C}$ satisfy

$$\int_{x \in \mathbb{R}} L(dx\, dt) = dt \qquad \text{for all } t \in [0, 1],$$

and let L have supported contained in

$$R = \{(t, x) \in [0, 1] \times \mathbb{R} : g(t) < x < h(t)\},$$

and suppose that $-\hat{I}(L) < \frac{1}{\theta}$, for some $\theta > 1$. Let $t_n = 2^{n^\theta}$, and fix $\delta > 0$. Now let $\tau_1 = t_{n-1}/t_n$, and for $i = 1, \ldots, n$ let τ_i be the first time $s > q_i$ such that $B_{t_n}(s)$ hits

$$\frac{h(a_{i+1}) - g(a_{i+1})}{2}, \quad g(a_i) - 2\varepsilon, \qquad \text{or } h(a_i) + 2\varepsilon.$$

If none of these points have been hit before time a_{i+1}, set $\tau_i = a_{i+1}$. Let $\mathscr{N}$ be a weak neighborhood of L supported in S, such that

$$\sup_{N \in \mathscr{N}} -\hat{I}(N) < \frac{1}{\theta}.$$

Let A_n be the event that

(i) For $0 < s < \tau_1$, $g(0) - \varepsilon < B_t(s) < h(0) + \varepsilon$.
(ii) For $\tau_i < s < a_{i+1}$, $g(a_i) - \varepsilon < B_t(s) < h(a_i) + \varepsilon$, $i = 1, \ldots, n$.
(iii) $B_t(\tau_i) = \dfrac{h(a_{i+1}) - g(a_{i+1})}{2}$, $i = 2, \ldots, n$.
(iv) $\hat{L}_t \in \mathscr{N}$.

Let E_n be the event that condition (i) is satisfied. Using the same argument as in the proof of Theorem 1.1, we see that

$$P\{\sup_i (\tau_i - a_i) > \delta\} = o(t).$$

Also, by the definition of τ_i,

$$P\left\{B_t(\tau_i)=\frac{h(a_{i+1})-g(a_{i+1})}{2},\ i=2,\dots,n\right\}>C>0.$$

We now need a theorem of Donsker and Varadhan ([3], Lemma 2.12).

Lemma 4.4. *Suppose that β is a probability measure supported in an interval (a,b), and that $\mathcal{N}$ is a weak neighborhood of β. If τ is the first exit time s of $B_t(s)$ from $(a-\varepsilon, b+\varepsilon)$ then*

$$\liminf_{t\uparrow\infty} \log \inf_{x\in(a,b)} P_x\{L_t\in\mathcal{N}, \tau>t\} \geqq -\inf_{N\in\mathcal{N}} I(N).$$

Using these two facts and the strong Markov property, we see that for n large enough, there exists $C>0$ such that

$$P\{A_n|E_n\}\geqq C n^{\theta \sup_{N\in\mathcal{N}} -I(N)}.$$

This diverges, and we wish to show that A_n occurs infinitely often. The following conditional Borel-Cantelli lemma is used in Donsker and Varadhan [3].

Lemma 4.5. *Suppose that $\mathfrak{F}_n$ is an increasing sequence of σ-fields with $A_n\in\mathfrak{F}_n$, and that for almost all w,*

$$\sum_{n=1}^{\infty} P\{A_{n+1}|\mathfrak{F}_n\}=\infty.$$

Then A_n occurs infinitely often, almost surely.

To apply the lemma, it suffices to show that E_n^c occurs only finitely often, where E_n^c is the complement of E_n. But for some $C>0$,

$$\begin{aligned} P\{E_n^c\} &\leqq P\{\sup_{0<s<t_{n-1}/t_n} |B_{t_n}(s)|>C\} \\ &\leqq P\left\{\sup_{0<s<1} |B(s)|>C\left(\frac{t_n}{t_{n-1}}\right)^{1/2} (\log\log t_n)^{1/2}\right\} \\ &\leqq P\{\sup_{0<s<1} |B(s)|>C 2^{\frac{(n-1)^{\theta-1}}{2}}\} \\ &\leqq C' \exp(-C 2^{\frac{(n-1)^{\theta-1}}{2}}). \end{aligned}$$

The sum over n converges, so this completes the chain of reasoning involved in the proof of Theorem 4.3.

5. Applications

(1) Chung's Theorem and Wichura's Law

These theorems were stated in the introduction. We will prove Wichura's law, which implies Chung's theorem. We need a lemma.

Lemma 5.1. *Let φ be a probability density supported in the interval $[a,b]$. Then*

$$I(\varphi) \geqq \frac{\pi}{(b-a)^2 \sqrt{8}}$$

and equality is attained for some density $\psi_{a,b}$.

Donsker and Varadhan [3] show the result for $b-a=1$. Lemma 5.1 is established by considering the density

$$g(y) = (b-a)\,\varphi((b-a)\,y)$$

which is supported on an interval of length 1. Indeed,

$$I(\varphi) = \frac{1}{(b-a)^2} I(g).$$

This proves the lemma, Now we turn to Wichura's theorem, which was stated in the introduction.

Theorem 5.2. (Wichura) *Let G and $W_t(x)$ be as in the introduction. With probability 1, the set of limit points of $\{W_t\}$ in the weak topology, as $t \nearrow \infty$, is G.*

Proof. Suppose $g(x)$ is a nonnegative nondecreasing function on $[0,1]$. Applying lemma 5.1 to the density

$$\hat{\psi}(t,x) = \psi_{-g(t),\,g(t)}(x)$$

where $(t,x) \in [0,1] \times \mathbb{R}$, we see that

$$I(\psi(t,x)) = \frac{\pi^2}{8} \int_0^1 \frac{1}{g(t)^2}\, dt.$$

Now suppose that $g \in G$, and that g is continuous; such functions are dense in G under the weak topology. We will show that g is a limit point of $\{W_t\}$. Fix $\varepsilon > 0$, and let L be the measure with density $\hat{\psi}(t,x) = \psi_{-g(t),\,g(t)}(x)$, which has support on $R = \{(t,x) \in [0,1] \times \mathbb{R} \colon |x| \leqq g(t)\}$. By Theorem 4.3, since $\mu \in \hat{C}$, we may choose a sequence $\hat{L}_{t_n}$ tending to L in the topology $\mathscr{T}$, such that $\hat{L}_{t_n}$ is supported on

$$R^\varepsilon = \{(t,x) \in [0,1] \times \mathbb{R} \colon |x| \leqq g(t) + \varepsilon\}.$$

Thus, $\sup\limits_{0 \leqq s \leqq x} |B_{t_n}(s)| \leqq g(x) + \varepsilon$.

To show that g is a weak limit point, it suffices to show that for all $N > 0$, and all $k = 1, \ldots, N$, that if n is large enough,

$$\sup_{\frac{k-1}{N} \leqq s \leqq \frac{k}{N}} |B_{t_n}(s)| \geqq g\left(\frac{k}{N}\right) - \varepsilon$$

so that $\hat{L}_{t_n}$ is supported on $R^\varepsilon = \{(t,x) \in [0,1] \times \mathbb{R} \colon |x| < g(t) + \varepsilon\}$. Thus, $\sup\limits_{0 \leqq s \leqq x} |B_{t_n}(s)| \leqq g(x) + \varepsilon$.

To show that g is a weak limit point, it suffices to show that for all $N>0$, and all $k=1,\ldots,N$, that if n is large enough,

$$\sup_{\frac{k-1}{N}\leqq s\leqq\frac{k}{N}} |B_{t_n}(s)| \geqq g\left(\frac{k-1}{N}\right)-\varepsilon.$$

But this follows because $L_{t_n}^{\frac{k-1}{N},\frac{k}{N}}$ tends to $L^{\frac{k-1}{N},\frac{k}{N}}$ in the topology of uniform convergence on compact intervals, and because the support of $L^{\frac{k-1}{N},\frac{k}{N}}$ contains the interval $\left(-g\left(\frac{k-1}{N}\right), g\left(\frac{k-1}{N}\right)\right)$.

Now we will show that if $g\notin G$, then g is not a weak limit point. This is obvious if g fails to be nonnegative or nondecreasing, so we may assume that

$$\int_0^1 \frac{1}{g(t)^2}\,dt > \frac{8}{\pi^2}.$$

Suppose that W_{t_n} tends weakly to g. By Theorem 1.1, we may choose a subsequence t_n' such that $\hat{L}_{t_n}$ tends to a limit L in the topology $\mathscr{T}$, and such that $\hat{I}(L)\leqq 1$. Since W_{t_n} tends to g, L must have mass 1. Now by Lemma 5.1 and the definition of $\hat{I}$, this implies that L is not supported on

$$R^{\varepsilon}=\{(t,x)\in[0,1]\times\mathbb{R}: |x|\leqq g(t)+\varepsilon\}$$

for some $\varepsilon>0$. Thus, for n large enough, $\hat{L}_{t_n}$ is not supported on R^{ε}, but this contradicts the fact that W_{t_n} converges weakly to g.

(2) Functionals

Let $V(x,y)$ be continuous on $[0,1]\times\mathbb{R}$ and assume $V(x,y)\uparrow\infty$ as $|(x,y)|\uparrow\infty$. If L is a subprobability, let

$$\Phi(L)=\int_{[0,1]\times\mathbb{R}} V(x,y)\,dL(x,y).$$

We wish to show that if $\mathscr{L}$ is the set of probabilities in $\hat{C}$, then

Theorem 5.5.

$$\liminf_{t\uparrow\infty}\Phi(\hat{L}_t)=\inf_{L\in\mathscr{L}}\Phi(L).$$

Proof. Let $\mathscr{L}^c$ be the set of probabilities in $\mathscr{L}$ with compact support. Then, by Theorem 4.3,

$$\liminf_{t\uparrow\infty}\Phi(\hat{L}_t)\leqq\inf_{L\in\mathscr{L}^c}\Phi(L)$$

and since $\mathscr{L}^c$ is dense in $\mathscr{L}$,

$$\liminf_{t\uparrow\infty}\Phi(\hat{L}_t)\leqq\inf_{L\in\mathscr{L}}\Phi(L).$$

To prove the reverse inequality, let $\hat{L}_{t_n}$ be a sequence such that $\Phi(\hat{L}_{t_n})$ tends to its infimum. Choose a convergent subsequence J_n converging to L. If $L \in \mathscr{L}$, then the reverse inequality is immediate, so assume that L has mass $1-\delta$. However, for all $N>0$,

$$\lim_{t\uparrow\infty} \Phi(J_n) \geqq \delta \inf_{|(x,y)|>N} V(x,y)$$
$$\uparrow \infty \quad \text{as } N \uparrow \infty.$$

This proves Theorem 5.5.

Donsker and Varadhan [3] showed that

$$\liminf_{t\uparrow\infty} \frac{\log\log t}{t^2} \int_0^t |B(s)|^2\, ds = \tfrac{1}{2} \text{ a.s.}$$

Theorem 5.5 implies that

$$\liminf_{t\uparrow\infty} \frac{\log\log t}{t^3} \int_0^t s\,|B(s)|^2\, ds = \tfrac{2}{9}.$$

Thus, we can handle functionals which do not depend only on local time.

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