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An Asymptotic Expansion for One-Sided Brownian Exit Densities

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Summary. Let $p(t)$ be the density of the first-exit time of a Brownian motion over the one-sided moving boundary given by $x=f(t)$. We derive the following formal expansion for p:

$$p(t) \sim \varphi(t^{-1/2} f(t)) \left[t^{-3/2} \lambda(t) - \sum_{n=1}^{\infty} c_n t^{n/2} m_n(t^{-1/2} \lambda(t)) \right].$$

Here $\lambda(t)=f(t)-tf'(t)$, φ is the standard normal density, m_n is the Hermite function of order $(-n)$, and the coefficients c_n are functions of the derivatives of f at t. We give bounds for the error incurred by approximating p by the first n terms of the series, and examples in which the series provides an asymptotic expansion for p.

1. Introduction

Let $f(t)$ be an infinitely differentiable function defined on an open interval $(0, T)$. Let τ be the first-exit time of a standard Brownian motion $W(t)$ over the moving boundary given by $x=f(t)$:

$$\tau = \inf\{t \mid t>0, W(t) \geqq f(t)\}.$$

We exclude the uninteresting case $\tau=0$ a.s. It is known that τ has a continuous density $p(t)$. In [4] it was shown that the "normalized density" $r(t)$, defined by

$$r(t)=p(t)/\varphi(t^{-1/2} f(t))$$

satisfies the integral equation

$$r(t)=t^{-3/2} \lambda(t)-(2\pi)^{-1/2} \int_0^t \sigma^{-2} y e^{-\Delta} r(s)\, ds. \tag{1.1}$$

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Here $\lambda(t)=f(t)-tf'(t)$ is the intercept on the vertical axis of the tangent to the curve at t; t is regarded as fixed, and σ, y, and Δ are functions of s given by

$$\sigma=(t-s)^{1/2},$$
$$y=(t-s)^{-1/2}(f(t)-f(s)-(t-s)\,f'(t))$$

and

$$\Delta=\left(\frac{f(s)}{s}-\frac{f(t)}{t}\right)^2 \bigg/ \left(\frac{1}{s}-\frac{1}{t}\right).$$

(This equation has been found independently in a more general context by Durbin (1981).) In [4] we discussed the use of $r_1(t)=t^{-3/2}\,\lambda(t)$ as an approximation for $r(t)$. We now sharpen this approximation by the addition of higher-order terms. (Other second- and third-order terms, asymptotically equivalent to those given below in certain situations, have been found by Jennen (1981).)

At first sight it seems strange to try to approximate $p(t)$, which depends on the whole curve $f(s)$, $s\in(0,t)$, by quantities which reflect only the local behavior of f near t. The key to the possibility of such an approximation lies in the quantity Δ. Roughly speaking, Δ measures the degree to which the point $(t,f(t))$ is isolated from earlier points $(s,f(s))$ on the curve. If Δ is large, except near t, then the integrand in (1.1) is concentrated near t, and a local approximation becomes feasible. These questions are discussed in more detail in [4].

To explain how the higher-order terms are obtained let us define the linear operator L acting on a function g by

$$Lg=(2\pi)^{-1/2}\int_0^t g(s)\,e^{-\Delta}\,r(s)\,ds.$$

(If g is given explicitly as a function of several variables, for instance $g(s)=(t-s)^{-1/2}$, we make the convention that s is always the variable of integration.) In this notation, the integral Eq. (1.1) becomes

$$r(t)=t^{-3/2}\,\lambda(t)-L(\sigma^{-2}\,y). \tag{1.2}$$

The idea is now to try to evaluate the unknown quantity $L(\sigma^{-2}\,y)$ by expanding $\sigma^{-2}\,y$ in terms of functions whose L-transforms are known. Such functions are easily found. Let m_n be the Hermite function of order $(-n)$:

$$m_0(x)=(1-\Phi(x))/\varphi(x),$$
$$m_1(x)=-x\,m_0(x)+1,$$
$$m_2(x)=(x^2+1)\,m_0(x)-x$$

and so forth. (Here Φ and φ are the standard normal distribution function and density. Section 5 contains the definition of the Hermite functions and a summary of their relevant properties.) Then an easy argument (Proposition 2.5) using the strong Markov property shows that

$$t^{n/2}\,m_n(t^{-1/2}\,\lambda(t))=L(\sigma^n\,m_n(y)) \qquad n=0,1,2,\ldots. \tag{1.3}$$

The idea behind this equation is not new. Special cases have been used by Daniels (1969), Durbin (1971), and Park and Schuurmann (1976).) Now $\sigma^{-2}\,y$

can be expanded, at least in the sense of an asymptotic expansion at $\sigma=0$, as a series in the functions $\sigma^n m_n(y)$:

$$\sigma^{-2} y = \sum_{k=1}^{\infty} c_k \sigma^k m_k(y), \tag{1.4}$$

where the c_k's are functions of the derivatives of f at t (Proposition (2.7)). The first few coefficients are

$$c_1 = -f^{(2)}(t)/2,\ c_2 = 0,\ c_3 = f^{(3)}(t)/12,\ c_4 = (f^{(2)}(t))^2/12.$$

Substituting (1.4) into (1.2) and using (1.3) produces the formal expansion

$$r(t) \sim t^{-3/2} \lambda(t) - \sum_{k=1}^{\infty} c_k t^{k/2} m_k(t^{-1/2} \lambda(t)). \tag{1.5}$$

The derivation sketched above is carried out in detail in Sect. 2. Let r_n be the approximation to r obtained by taking the first n terms of the series:

$$r_n(t) = t^{-1/2} \lambda(t) - \sum_{k=1}^{n-1} c_k t^{k/2} m_k(t^{-1/2} \lambda(t)). \tag{1.6}$$

Section 3 is devoted to the problem of estimating the error $(r - r_n)$. The bounds obtained are unfortunately rather complicated, but asymptotically, for example as the boundary recedes to infinity, the situation is simpler. Section 4 gives some examples.

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2. Derivation of the Expansion

Since, as just explained

$$r(t) = t^{-3/2} \lambda(t) - L(\sigma^{-2} y)$$

it is natural to investigate functions m for which Lm can be evaluated. The following lemma provides a source of such m's. We use N to denote a standard normal random variable and $\mathscr{E}$ to denote expectations with respect to the distribution of N. Recall the convention that s denotes the integration variable in the definition of L.

(2.1) **Lemma.** *Let b be a non-negative measurable function such that $b(x)=0$ if $x<0$. Let μ be defined by*

$$\mu(\alpha^2, x) = \mathscr{E} b(\alpha N - x)/\varphi(x/\alpha).$$

Then, for each fixed t,

$$\mu(t, f(t)) = L\mu(t-s, f(t)-f(s)).$$

More generally, for any θ,

$$\mu(t, f(t) - t\theta) = L\mu(t-s, f(t)-f(s)-(t-s)\theta). \tag{2.2}$$

Proof. The proof is simple, though regrettably somewhat overburdened with notations. Define $b_\theta(x)=e^{\theta x}\, b(x)$ and

$$\beta_\theta(\alpha^2, x)=\mathscr{E}\, b_\theta(\alpha N-x).$$

We claim that

$$\beta_\theta(t, f(t))=\int_0^t \beta_\theta(t-s, f(t)-f(s))\, p(s)\, ds. \tag{2.3}$$

To see this, let $E_{(s,x)}$ denote expectation for the Brownian motion starting at (s,x). Since $b_\theta(x)=0$ if $x<0$, we have, by the strong Markov property,

$$\begin{aligned} E_{(0,0)}\, b_\theta(W(t)-f(t)) &= E_{(0,0)}\, E_{(\tau, f(\tau))}\, b_\theta(W(t)-f(t)) \\ &= \int_0^t E_{(s, f(s))}\, b_\theta(W(t)-f(t))\, p(s)\, ds. \end{aligned}$$

Since, for the Brownian motion starting at (s,x), $W(t)$ has the same distribution as $(t-s)^{1/2} N+x$, this can be written as

$$\mathscr{E}\, b_\theta(t^{1/2} N-f(t))=\int_0^t \mathscr{E}\, b_\theta((t-s)^{1/2} N-(f(t)-f(s)))\, p(s)\, ds.$$

This proves (2.3).

A routine computation shows that

$$\begin{aligned} \beta_\theta(\alpha^2, x) &= \exp(-x\theta+\alpha^2\theta^2/2)\, \beta_0(\alpha^2, x-\alpha^2\theta) \\ &= \varphi(x/\alpha)\, \mu(\alpha^2, x-\alpha^2\theta). \end{aligned}$$

Substituting this expression into (2.3) and using the identity

$$(2\pi)^{-1/2}\, e^{-\Delta}=\varphi(s^{-1/2} f(s))\, \varphi((t-s)^{-1/2}(f(t)-f(s)))/\varphi(t^{-1/2} f(t))$$

yields (2.2). QED

We apply the lemma with $b(x)=x^n\, 1_{\{x>0\}}$ and $\theta=f'(t)$. We define

$$m_n(x)=\mathscr{E}((N-x)^+)^n/\varphi(x) \qquad n=0,1,\ldots. \tag{2.4}$$

Writing, as before,

$$\begin{aligned} \sigma &= (t-s)^{1/2}, \\ y &= (t-s)^{-1/2}(f(t)-f(s)-(t-s)\, f'(t)) \end{aligned}$$

we get the following basic result.

(2.5) **Proposition.** $t^{n/2}\, m_n(t^{-1/2}\, \lambda(t))=L(\sigma^n\, m_n(y)) \qquad n=0,1,2,\ldots.$

As already mentioned, the m_n's are the Hermite functions of negative integral order (see Sect. 5).

Because f is infinitely differentiable, $\sigma^{-2} y$ has a formal Taylor series expansion in powers of $(t-s)$ and thus also in powers of σ. Since the m_n's are infinitely differentiable, with $m_n(0) \neq 0$, it follows that $\sigma^{-2} y$ can also be formally expanded in terms of $\sigma^n\, m_n(y)$. We define c_k, $k=1,2,\ldots$, to be the coefficients of this expansion:

(2.6) $$\sigma^{-2} y = \sum_{k=1}^{\infty} c_k \sigma^k m_k(y),$$

where the equality means that the two sides are identical when expanded as formal power series in σ. Determining the c_k's is a matter of bookkeeping: we simply expand both sides and equate coefficients.

(2.7) **Proposition.** *Let* $a_{2n}=(-1)^{n+1} f^{(n)}(t)/n!$, $a_{2n+1}=0$, $n \geqq 0$. *Then the* c_n, $n \geqq 1$, *satisfy the recurrence relation*

(2.8) $$c_n = ((n-1)!!)^{-1} \left[(-1)^n a_{n+3} - \sum_{k=1}^{n-1} c_k \sum_{l=0}^{[(n-k)/3]} ((k+l-1)!!/l!) \sum{}' a_{m_1} \dots a_{m_l}\right],$$

where $\sum'$ *denotes summation over all* $m_1, \dots, m_l \geqq 4$ *with* $m_1 + \dots + m_l = n-k+l$.

Proof. The Taylor series for f yields, in our notation,

$$y = \sum_{m=4}^{\infty} a_m \sigma^{m-1}$$

so that

$$\sigma^{-2} y = \sum_{n=1}^{\infty} a_{n+3} \sigma^n.$$

Writing $b_n = m_0^{(n)}(0) = (-1)^n m_n(0)$ (by (5.1)), we have similarly, using (5.2),

$$m_k(y) = \sum_{l=0}^{\infty} (b_{k+l}/l!)\, x^l.$$

Substituting into (2.6) and rearranging, we get

(2.9) $$\begin{aligned}\sum_{n=1}^{\infty} a_{n+3} \sigma^n &= \sum_{k=1}^{\infty} c_k \sigma^k \sum_{l=0}^{\infty} (b_{k+l}/l!) \left(\sum_{m=4}^{\infty} a_m \sigma^{m-1}\right)^l \\ &= \sum_{n=1}^{\infty} \sigma^n \sum_{k=1}^{\infty} c_k \sum_{l=0}^{[(n-k)/3]} (b_{k+l}/l!) \sum{}' a_{m_1} \dots a_{m_l}.\end{aligned}$$

The final sum $\sum'$ is 0 unless $n-k+l$ is even, so that $n \equiv k+l \pmod 2$. In this case we find from (5.3), that

$$\frac{b_{k+l}}{b_n} = \frac{(k+l-1)!!}{(n-1)!!}.$$

Equating coefficients of σ^n in (2.9) and using this fact, we obtain (2.8). QED

Applying the proposition we find the following values for $c_1, \dots, c_{10}$.

(2.10) $$c_1 = -\frac{f^{(2)}(t)}{2}, \quad c_2 = 0, \quad c_3 = \frac{f^{(3)}(t)}{12},$$

$$c_4 = \frac{(f^{(2)}(t))^2}{12}, \quad c_5 = -\frac{f^{(4)}(t)}{192}, \quad c_6 = -\frac{f^{(2)}(t)\, f^{(3)}(t)}{72},$$

$$c_7 = -\tfrac{5}{1152} (f^{(2)}(t))^3 + \tfrac{1}{5760} f^{(5)}(t),$$

$$c_8 = \tfrac{23}{40{,}320} f^{(2)}(t)\, f^{(4)}(t) + \tfrac{2}{2520} (f^{(3)}(t))^2,$$

$$c_9 = \frac{5}{6912}(f^{(2)}(t))^2 f^{(3)}(t) - \frac{1}{276{,}480} f^{(6)}(t),$$
$$c_{10} = \frac{17}{1{,}209{,}600} f^{(2)}(t) f^{(5)}(t) - \frac{1}{40{,}320} f^{(3)}(t) f^{(4)}(t) + \frac{79}{725{,}760}(f^{(2)}(t))^4.$$

The following property of the c_n's follows immediately from the recursion relation (2.8).

(2.11) **Proposition.** *Each c_n is a linear combination of products of the form $f^{(k_1)}(t)\dots f^{(k_l)}(t)$, where $2 \leqq k_i \leqq (n+3)/2$ for $i=1,\dots,l$, and*

$$\sum_{i=1}^{l}(2k_i - 1) = n+2. \tag{2.12}$$

From (2.12) it follows that $l \leqq (n+2)/3$, since

$$n+2 = \sum_{i=1}^{l}(2k_i - 1) \geqq 3l.$$

Since $(n+2)/3 \leqq (n+1)$ for $n \geqq 0$, we get the following simple bound for c_n, which will be needed later.

(2.13) **Corollary.** *For each $n \geqq 1$ there is a constant γ_n such that, if $A \geqq 1$ and*

$$|f^{(k)}(t)| \leqq A \quad \text{for } 2 \leqq k \leqq (n+1)$$

then

$$|c_n| \leqq \gamma_n A^{[(n+2)/3]}.$$

Another consequence of Proposition (2.11) is the invariance of the asymptotic expansion under Brownian rescaling. (Actually this invariance can be seen directly, and the relation (2.12) follows, but the argument is hard to formalize.) Let us introduce new coordinates $\tilde{x} = \alpha x$ and $\tilde{t} = \alpha^2 t$, where $\alpha > 0$. The original Brownian motion is transformed into a new Brownian motion, the original boundary f is replaced by a new boundary given by $\tilde{f}(\tilde{t}) = \alpha f(t)$, and so on. Since $\tilde{\tau} = \alpha^2 \tau$, it is clear that $\tilde{p}(\tilde{t}) = \alpha^{-2} p(t)$. Now let $r_n(t)$ be the n-th approximation to $r(t)$, as in (1.6), and let

$$p_n(t) = r_n(t)\,\varphi(t^{-1/2} f(t)).$$

By invariance under Brownian rescaling we mean that $p_n(t)$ satisfies the same transformation rule as p.

(2.14) **Proposition.** $\qquad \tilde{p}_n(\tilde{t}) = \alpha^{-2} p_n(t).$

Proof. First consider the factor $\varphi(t^{-1/2} f(t))$. Since $\tilde{t}^{-1/2}\tilde{f}(\tilde{t}) = t^{-1/2} f(t)$, we need only show that $\tilde{r}_n(\tilde{t}) = \alpha^{-2} r_n(t)$. Since $\tilde{f}'(\tilde{t}) = \alpha^{-1} f'(t)$, we see that $\tilde{\lambda}(\tilde{t}) = \alpha\lambda(t)$, whence $\tilde{t}^{-1/2}\tilde{\lambda}(\tilde{t}) = t^{-1/2}\lambda(t)$ and $\tilde{t}^{-3/2}\tilde{\lambda}(\tilde{t}) = \alpha^{-2}(t^{-3/2}\lambda(t))$. Thus we must prove that $\tilde{c}_n \tilde{t}^{n/2} = \alpha^{-2}(c_n t^{n/2})$ for all n, that is $\tilde{c}_n = \alpha^{-(n+2)} c_n$. But c_n is a linear combination of terms of the form $f^{(k_1)}(t)\dots f^{(k_l)}(t)$, and since $\tilde{f}^{(k)}(\tilde{t}) = \alpha^{-(2k-1)} f^{(k)}(t)$, it follows from (2.12) that $\tilde{f}^{(k_1)}(\tilde{t})\dots\tilde{f}^{(k_l)}(\tilde{t}) = \alpha^{-(n+2)} f^{(k_1)}(t)\dots f^{(k_l)}(t)$. Thus $\tilde{c}_n = \alpha^{-(n+2)} c_n$. QED

3. Error Bounds

We wish to estimate the error $(r(t)-r_n(t))$ of the n-th order approximation r_n. We have

$$r(t)-r_n(t)=-(2\pi)^{-1/2}\int_0^t \rho_n(s)\,e^{-\Delta}\,r(s)\,ds \tag{3.1}$$

where

$$\rho_n(s)=\sigma^{-2}y-\sum_{k=1}^{n-1} c_k\,\sigma^k\,m_k(y).$$

In spite of strenuous efforts at simplification, the error bounds obtained below remain complicated. As the lesser of two evils we have chosen to introduce a considerable amount of abbreviating notation, in order to make the final formulas easier to read.

To begin with, we divide the interval of integration $(0,t)$ into two parts. We assume that the point of division u satisfies $0<u<t$, $u>(t-1)$ and $u>t/2$. For the applications which we have in mind, the integrand in (3.1) is much smaller on $(0,u)$ than on (u,t). We define

$$\varepsilon=\sup_{s\in(0,u)}\left(s^{-3/2}\,e^{-\Delta}\sup_{v\in(0,s)}\lambda(v)\right),$$

$$\eta_n=\sup_{s\in(0,u)}\rho_n(s).$$

Usually, quite crude bounds for ε and η_n are enough to dispose of the contribution of the interval $(0,u)$.

The situation on (u,t) is more delicate. The relevant descriptive parameters are:

$$\Lambda=\sup_{s\in(0,t)}|\lambda(s)|,$$

$$\Lambda_0=\inf_{s\in(u,t)}\lambda(s),$$

$$y_0=\inf_{s\in(u,t)}y(s),$$

$$F_{n+1}=\max_{2\leqq k\leqq n+1}\ \sup_{s\in(u,t)}|f^{(k)}(s)|,$$

$$G_n=\max\{F_{n+1},|c_k|\,(F_{n+1}\vee 1)^{[(n-k+2)/3]}\,k=1,\dots,(n-1)\}.$$

(3.2) **Theorem.** *There exist universal constants K_n, $n\geqq 1$, such that, if $\Lambda_0>0$, the following inequalities hold.*

$$|r(t)-r_1(t)|\leqq K_1F_2\,t^{3/2}\,\Lambda\Lambda_0^{-3}+t\varepsilon\eta_1, \tag{3.3}$$

$$|r(t)-r_2(t)|\leqq K_2(F_3\,t^{7/2}\,\Lambda\Lambda_0^{-5}+F_2^2\,t^{9/2}\,\Lambda\Lambda_0^{-6}\,m_2(y_0))+t\varepsilon\eta_2, \tag{3.4}$$

$$|r(t)-r_n(t)|\leqq K_nG_n\,t^{n+1/2}\,\Lambda\,\Lambda_0^{-n-2}\,m_n(y_0)+t\varepsilon\eta_n \quad \text{for } n\geqq 4. \tag{3.5}$$

(The case $n=3$ is omitted because $r_3=r_2$.)

Given a specific curve, it is easy enough to evaluate, or at least to estimate, all the quantities appearing on the right-hand side of these inequalities, except for G_n. The difficulty with G_n is that we have no explicit formula for the coefficients c_k. We can avoid this difficulty, at the cost of some loss of accuracy, by using the a priori bounds for c_k given in Corollary (2.13). We find that

$$G_n \leqq (\max_{1 \leqq k \leqq n-1} \gamma_k)(F_{n+1} \vee 1)^{[(n+4)/3]}$$

where the γ_k's are absolute constants. This yields the following simplified form of (3.5).

(3.6) **Corollary.** *There exist absolute constants M_n, $n \geqq 4$, such that, if $\Lambda_0 > 0$, the following inequality holds for $n \geqq 4$.*

(3.7) $$|r(t) - r_n(t)| \leqq M_n (F_{n+1} \vee 1)^{[(n+4)/3]} t^{n+1/2} \Lambda \Lambda_0^{-n-2} m_n(y_0) + t\varepsilon\eta_n.$$

The remainder of this section is devoted to the proof of Theorem (3.2). The reader may prefer to skip to the next section, which contains examples.

As already explained, the first step in the proof is to divide the interval of integration in (3.1) into two parts. Using the elementary inequality

$$r(s) \leqq s^{-3/2} \sup_{v \in (0,s)} \lambda(s)$$

(see [4]), we have, trivially from the definitions of ε and η_n,

$$\left|\int_0^u \rho_n(s)\, e^{-\Delta} r(s)\, ds\right| \leqq t\varepsilon\eta_n.$$

We must now bound the integral over (u, t). In [4] it was shown that if $|h(s)| \leqq H\sigma^n$ for $s \in (u, t)$, then

(3.8) $$\int_u^t h(s)\, e^{-\Delta} r(s)\, ds = O(H t^{n+1/2} \Lambda \Lambda_0^{-n-2}).$$

(Here, and in the rest of this section, we make the convention that the constants implied in the O-notation depend only on n.) To prove the theorem it therefore suffices to show that, for $s \in (u, t)$

(3.9) $$\rho_1(s) = O(F_2 \sigma),$$

(3.10) $$\rho_2(\sigma) = O(F_3 \sigma^3 + F_2^2 m_2(y_0) \sigma^4),$$

and, for $n \geqq 4$,

(3.11) $$\rho_n(\sigma) = O(G_n m_n(y_0) \sigma^n).$$

We treat the simpler cases (3.9) and (3.10) first. Expanding y in a Taylor series around $\sigma = 0$ (i.e. $s = t$) as in (2.7) we have, for any $l \geqq 2$

(3.12) $$y = \sum_{k=1}^{(l-1)} (-1)^{k+1} f^{(k)}(t)\, \sigma^{2k-1}/k! + (-1)^l f^{(l)}(s^*)\, \sigma^{2l-1}/l!$$

for some $s^* \in (s, t)$. Taking $l = 2$ we get (3.9). Taking $l = 3$ we find

(3.13) $$y = -f^{(2)}(t)\, \sigma^3/2 + O(F_3 \sigma^5).$$

On the other hand

$$m_1(y)=m_1(0)+ym_1'(y^*)=1-ym_2(y^*)$$
$$=1+O(F_2\,m_2(y_0)\,\sigma^3)$$

so that

$$\begin{aligned}\rho_2(\sigma)&=\sigma^{-2}y-c_1\,\sigma\,m_1(y)\\&=[-f^{(2)}(t)\,\sigma/2+O(F_3\,\sigma^3)]-(-f^{(2)}(t)/2)\,\sigma(1+O(F_2\,m_2(y_0)\,\sigma^3))\\&=O(F_3\,\sigma^3+F_2^2\,m_2(y_0)\,\sigma^4)\end{aligned}$$

which proves (3.10).

In proving (3.11) we must of course make use of the fact that the coefficients c_k have been so defined that the coefficients of σ^k in the Taylor expansion of ρ_n vanish for $k<n$. To exploit this fact we introduce the following notation. Given a function $g(\sigma)$ with n derivatives, we write

$$R_n\,g(\sigma)=\sigma^{-n}\left[g(\sigma)-\sum_{k=0}^{n-1}g^k(0)\,\sigma^k/k!\right].$$

Clearly the operator R_n is linear, and $R_n(\sigma^k g)=R_{n-k}\,g$ for $k\leqq n$. Thus the quantity $\sigma^{-n}\rho_n$, which we wish to estimate, can be written as

$$\sigma^{-n}\rho_n=R_n(\sigma^{-2}y)-\sum_{k=1}^{n-1}c_k\,R_{n-k}(m_k(y)).$$

Our task is to show that each of the summands on the right is $O(G_n m_n(y_0)\,\sigma^n)$. By (3.8) this will complete the proof.

First we consider the term $R_n(\sigma^{-2}y)$. By the Taylor series (3.12) with $l=[(n+4)/2]$ we get, remembering that $0\leqq\sigma\leqq 1$,

$$R_n(\sigma^{-2}y)=O(F_{n+1})=O(G_n).$$

Further, $y_0\leqq 0$, so, by (5.5), $m_n(y_0)\geqq m_1(y_0)\geqq m_1(0)=1$; thus we have

$$R_n(\sigma^{-2}y)=O(G_n\,m_n(y_0)).$$

Now we must treat the terms $c_k\,R_{n-k}(m_k(y))$. This is more complicated, because we must first expand m_k in powers of y and then expand in powers of σ. We have

$$m_k(y)=\sum_{i=0}^{l-1}m_k^{(i)}(0)\,y^i/i!+m_k^{(l)}(y^*)\,y^l/l!.$$

Since the expansion of y in powers of σ begins with σ^3 we take l as small as possible such that $(n-k)\leqq 3l$, i.e. $l=[(n-k+2)/3]$. Since $k\leqq(n-1)$ it follows that $k+l\leqq n$, so that

$$|m_k^{(l)}(y^*)|=m_{k+l}(y^*)\leqq m_{k+l}(y_0)\leqq m_n(y_0).$$

Further, by (3.13), $y=O(F_{n+1})$, so we get

$$R_{n-k}(m_k(y))=O\left(\sum_{i=1}^{l-1}R_{n-k}(y^i)+F_{n+1}^l\,m_n(y_0)\right). \tag{3.14}$$

To estimate $R_{n-k}(y^i)$ note that, regarding y now as function of σ,

$$R_{n-k}(y^i)=(y^i)^{(n-k)}(\sigma^*)/(n-k)! \tag{3.15}$$

for some $0\leqq\sigma^*\leqq\sigma$. To evaluate the derivatives of y^i with respect to σ, let us write $y=g(\sigma)/\sigma$, where

$$g(\sigma)=f(t)-f(s)-(t-s)\,f'(t)=f(t)-f(t-\sigma^2)-\sigma^2 f'(t).$$

Using the chain rule it is easy to show that

$$\sup_{s\in(u,t)} g^{(m)}(\sigma)=O(F_{n+1}) \quad \text{for } 1\leqq m\leqq(n+1).$$

For any function g with $(m+1)$ derivatives and $g(0)=0$ one has the elementary inequality

$$|(g(\sigma)/\sigma)^{(m)}|\leqq \sup_{\zeta\in(0,\,\sigma)} |g^{(m+1)}(\zeta)|.$$

Therefore

$$\sup_{s\in(u,t)} y^{(m)}(\sigma)=O(F_{n+1}) \quad 0\leqq m\leqq n.$$

By Leibniz' rule for the derivative of a product

$$(y^i)^{(m)}=\sum \binom{n}{j_1\dots j_i} y^{(j_1)}\dots y^{(j_i)}$$

the sum being extended over all $j_1\geqq 0\dots j_i\geqq 0$ with $j_1+\dots+j_i=m$. It follows that

$$\sup_{s\in(u,t)} (y^i)^{(m)}=O(F^i_{n+1}) \quad \text{for } 0\leqq m\leqq n.$$

Using this with (3.15) in (3.14) we see that

$$R_{n-k}(m_k(y))=O((F_{n+1}\vee 1)^l\, m_n(y_0)) \quad \text{for } k=1,\dots,(n-1),$$

where $l=[(n-k+2)/3]$. Therefore, by definition of G_n

$$c_k R_{n-k}(m_k(y))=O(G_n\, m_n(y_0)).$$

This completes the proof. QED

4. Applications and Examples

We now give some examples in which the formal expansion (1.5) yields an asymptotic expansion for $r(t)$. Typical situations are a fixed curve with $t\to 0$ or $t\to\infty$, or a fixed t with a family of curves receding to infinity. We are forced to use the term "asymptotic expansion" in a somewhat looser sense than is usual. To explain the difficulty, let us denote the general term of the expansion by T_n:

$$T_0=t^{-3/2}\,\lambda(t),$$
$$T_n=c_n\, t^{n/2}\, m_n(t^{-1/2}\,\lambda(t)) \quad n\geqq 1,$$

so that

$$r_n = T_0 + T_1 + \dots + T_{n-1}.$$

For an asymptotic expansion in the classical sense, one would need

(4.1) $$T_n = o(T_{n-1}) \quad n \geqq 1$$

and

(4.2) $$r = r_n + o(T_n) \quad n \geqq 1.$$

Unfortunately, it is sometimes the case that (4.1) is false for some values of n, and Theorem (3.2) yields, instead of (4.2), in general only an error

(4.3) $$r = r_m + O(B_m) \quad m \geqq 1$$

with B_n of larger order of magnitude than T_n. These problems are difficult to handle in a general context; the principal obstacle is the lack of information on the precise order of magnitude of the coefficients, the estimates of Sect. 2 yielding only upper bounds. Nevertheless, if we are dealing with a fixed curve and a fixed n, so that the coefficients are known explicitly, these drawbacks are less serious than might be supposed. In fact, one can often derive (4.2) from the apparently weaker (4.3). This happens for those n for which we can find an $m = n+k$ such that (4.3) holds, and such that B_{n+k}, $T_{n+1}, \dots, T_{n+k-1}$ are all $O(T_n)$. For then we have

(4.4) $$\begin{aligned} r &= r_{n+k} + O(B_{n+k}) \\ &= r_n + T_n + T_{n+1} + \dots + T_{n+k-1} + O(B_{n+k}) \\ &= r_n + O(T_n). \end{aligned}$$

As our first example, however, we take a simple situation in which none of the above difficulties occur, and for which we can give a direct proof of the asymptotic expansion independent of all the repellent machinery of the previous section. We consider a fixed curve $x = g(s)$ translated upwards a fixed distance v: thus $f(s) = f_v(s) = g(s) + v$. In the interests of uncluttered notation we refrain from encumbering every f, λ, and r with a subscript v. We also rescind the convention of the previous section that the constants in the O-notation depend only on n. We denote the relative error of the approximation by δ_n:

$$\delta_n = (r(t) - r_n(t))/r(t).$$

(4.5) **Proposition.** *Let $t > 0$ be fixed. Let $f(s) = f_v(s) = g(s) + v$, where g is a C^∞ function on $[0, t]$. Then as $v \to \infty$, so that $\lambda(t) \to \infty$, the relative error δ_n of the approximation r_n satisfies*

$$\delta_n = O(\lambda^{-n-2}(t)) \quad n \geqq 1.$$

(Since $c_2 = 0$, so that $r_2 = r_3$, we have $\delta_2 = O(\lambda^{-5}(t))$.)

Proof. Since y and the coefficients c_k do not depend on v, the same is true of the constants

$$a_n = \sup_{s \in (0,t)} |\rho_n(s)| / (\sigma^n m_n(y)),$$

which are finite by definition of the c_k's. Therefore, using (2.5) and (5.4), we have, as $v \to \infty$,

$$\begin{aligned} |r(t)-r_n(t)| &= |L\rho_n| \\ &\leqq a_n\,|L\sigma^n m_n(y)| \\ &= a_n\, t^{n/2}\, m_n(t^{-1/2}\,\lambda(t)) \\ &\leqq a_n n!\, t^{n+1/2}\,\lambda^{-n-1}(t) \\ &= O(\lambda^{-n-1}(t)). \quad \text{QED} \end{aligned}$$

For a concrete numerical example illustrating Proposition (4.5) we consider the family of curves

(4.6) $$f(s)=f_v(s)=(v-\tfrac{7}{3})-\tfrac{5}{4}s^2+\tfrac{1}{6}s^3$$

and focus our attention on $r(t)$ for $t=2$. The parameter v is then equal to $\lambda(2)$. Table (4.7) shows the relative error δ_n at $t=2$ as a function of v for $n=1,2,4,5$.

(4.7) **Table.** The relative error $(r_n(2)-r(2))/r(2)$ for the curves (4.6) for various values of $v=\lambda(2)$.

n	v				
	2.5	3.0	4.0	6.0	8.0
1	1.27	0.19	0.043	0.0102	0.0041
2	1.10	0.13	0.019	0.0021	0.0005
4	1.02	0.11	0.011	0.0008	0.0001
5	0.98	0.10	0.009	0.0004	–

(I am indebted to Dr. Jennen for computing the values of $r(2)$ used in constructing this table.)

In order to have $f(0)>0$, we must take $v>\frac{7}{3}$. When v is only slightly greater than this, say $v=2.5$, all the approximations are poor, but the quality improves rapidly with increasing v, and the differing orders of approximation become apparent.

In Proposition (4.5) we have an asymptotic expansion in the classical sense; for, since $m_n(x) \sim n!\, x^{-n-1}$ as $x \to \infty$ (see (5.6)), we have

$$T_n = c_n\, t^{n/2}\, m_n(t^{-1/2}\,\lambda(t)) \sim c_n n!\, t^{n+1/2}\,\lambda^{-n-1}(t),$$

so that, with the trivial exception of when $c_{n-1}=0$, we have $T_n=o(T_{n-1})$ for all $n \geqq 1$.

By contrast, consider what happens when the boundary recedes "multiplicatively", instead of "additively": $f(s)=f_v(s)=v\,g(s)$. Assuming that $g(t)-tg'(t)>0$, we see that $\lambda(t)$ is proportional to v. The same is true of all the non-zero derivatives of f. Consulting the list of c_n-values (2.10) we see that each c_n is either 0 or proportional to a power $v^{a(n)}$, as in the following table

n	1	2	3	4	5	6	7	8	9	10
$a(n)$	1	–	1	2	1	2	1 or 3	2	1 or 3	2 or 4

For example, c_7 is proportional to v^3 if $g^{(2)}(t)\neq 0$, and to v if $g^{(2)}(t)=0$ but $g^{(5)}(t)\neq 0$. Thus T_n is proportional to $v^{b(n)}$, with $b(n)=a(n)-(n+1)$. Taking the larger values of $a(n)$ in the above table (the "general case"), we get the following values for $b(n)$.

n	1	2	3	4	5	6	7	8	9	10
$b(n)$	-1	$-$	-3	-3	-5	-5	-5	-7	-7	-7

Clearly it is only sensible to use such r_n's for which the last (non-zero) term included is of greater order of magnitude than the first (non-zero) term omitted. In the case under discussion this means that of the r_n with $n\leqq 10$, only the approximations r_1, r_2, r_5, and r_8 are usable. The next proposition gives a specific instance of this situation.

(4.8) **Proposition.** *Let $t>0$ be fixed. Let $f(s)=f_v(s)=vs^{\beta}$, where $0<\beta<\frac{1}{2}$. Then as $v\to\infty$, so that $\lambda(t)\to\infty$, we have*

(4.9) $$\delta_1=0(\lambda^{-2}(t)),$$

(4.10) $$\delta_2=0(\lambda^{-4}(t)),$$

(4.11) $$\delta_5=0(\lambda^{-6}(t)),$$

(4.12) $$\delta_8=0(\lambda^{-8}(t))$$

and

(4.13) $$\delta_n=0(\lambda^{-e(n)}(t)) \qquad \text{for } n\geqq 9$$

where $e(n)=n+2-[(n+4)/3]$.

Proof. We apply Theorem (3.2) and Corollary (3.6). Let u be arbitrary satisfying the conditions of the theorem. An easy exercise shows that the term $t\varepsilon\eta_n$ bounding the contribution of the interval $(0,u)$ to the error is $O(\lambda^{-M}(t))$ for every $M\geqq 0$. Since the boundary curve is concave, y is non-negative, so $m_n(y_0)=m_n(0)$. The quantities F_n, Λ, and Λ_0 are all of order $\lambda(t)$. Thus (4.9) follows directly from (3.3), (4.10) from (3.4), and (4.13) from (3.7). Finally, (4.11) and (4.12) follow from (4.13) by the argument (4.4). QED

By Proposition (2.14), the relative error of r_n is invariant under Brownian rescaling. A standard rescaling argument the yields the following reformulation of Proposition (4.8), in which we now keep the curve fixed and let $t\to 0$.

(4.14) **Corollary.** *Let $f(s)=s^{\beta}$, where $0<\beta<\frac{1}{2}$. Let $\gamma=(1-2\beta)/2$. Then as $t\to 0$ we have*

$$\delta_1=O(t^{2\gamma}) \quad \delta_2=O(t^{4\gamma}) \quad \delta_5=O(t^{6\gamma}) \quad \delta_8=O(t^{8\gamma})$$

and

$$\delta_n=O(t^{\gamma e(n)}) \quad \text{for } n\geqq 9.$$

We close with an example in which the asymptotic expansion holds as $t\to\infty$. The proof is a straight-forward application of Theorem (3.2).

(4.15) **Proposition.** *Let* $f(s)=a_0+a_1 s+\ldots+a_m s^m$, *where* $m\geqq 2$, $a_0>0$, *and* $a_m<0$. *Then as* $t\to\infty$ *the relative error satisfies*

$$\delta_1=0(t^{-2m-1/2}),$$
$$\delta_2=0(t^{-3m+1/2})$$

and

$$\delta_n=0(t^{-h(n)}) \quad \textit{for } n\geqq 4$$

where $h(n)=n^2+n-(n-2)\,[(n+4)/3]-\frac{1}{2}$.

As was the case in Proposition (4.8), these bounds can be considerably sharpened for particular polynomials.

5. Appendix: Properties of the Functions $m_n(x)$

Here we assemble the properties of the Hermite functions m_n which are made use of in the paper (see Lebedev (1965), Chap. 10). First of all, from the Definition (2.4)

$$m_n(x)=\mathscr{E}((N-x)^+)^n/\varphi(x)$$

one can show that

(5.1) $$m'_n=-m_{n+1}.$$

Thus an alternative definition of m_n is

(5.2) $$m_n(x)=(-1)^n((1-\Phi(x))/\varphi(x))^{(n)}.$$

The m_n's are positive and decreasing. They satisfy a three-term recurrence relation

$$m_{n+1}(x)=-x m_n(x)+n m_{n-1}(x) \qquad n\geqq 1.$$

It follows that m_n has the form

$$m_n(x)=P_n(x)\, m_0(x)+Q_n(x),$$

where P_n and Q_n are polynomials of degree n and $(n-1)$, respectively. Another consequence is that

$$m_{n+1}(0)=n m_{n-1}(0)$$

so that

(5.3) $$m_n(0)=\begin{matrix}(\pi/2)^{1/2}(n-1)!! & n \text{ even}\\ (n-1)!! & n \text{ odd.}\end{matrix}$$

Finally, we have the elementary inequalities

(5.4) $$m_n(x)\leqq n!\, x^{n+1} \qquad x>0,$$

(5.5) $$m_{n+1}(x)>m_n(x) \qquad n\geqq 1 \quad x\leqq 0$$

and the asymptotic relation

(5.6) $$m_n(x)\sim n!\, x^{-n-1} \qquad x\to\infty.$$

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