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Almost Sure Convergence of Branching Processes

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A general theorem concerning the almost sure convergence of some nonhomogeneous Markov chains, whose conditional distributions satisfy a certain convergence condition, is given. This result applied to branching processes with infinite mean yields almost sure convergence for a large class of processes converging in distribution, as well as a characterization of the limiting distribution function.

1. Introduction

Let $(\Omega, \mathscr{F}, P)$ be a probability space and $\{X_n(\omega): n \geqq 0\}$ a nonhomogeneous Markov chain assuming a countable state space E, defined on this space. Write A^c for the complementary set of A with respect to Ω. We shall further attach to a sequence of events $\{A_n: n \geqq 0\}$ the events $\liminf A_n$ and $\limsup A_n$, defined in the standard way. In the case when $A = \liminf A = \limsup A_n$ we shall agree to write $\lim_{n\to\infty} A_n = A$. $[a]$ will stand for the integral part of a. We shall say that U is a *slowly varying* function if U is measurable, positive, defined on $[A, \infty)$ with $A > 0$ and for each $\lambda > 0$

$$\lim_{x\to\infty} \frac{U(\lambda x)}{U(x)} = 1. \tag{1}$$

Let $\{Z_n: n \geqq 0\}$ denote a supercritical Galton-Watson process with $Z_0 = 1$, $1 < m = E(Z_1) \leqq \infty$. In the case when $m < \infty$, $\{Z_n/m^n\}$ was proved to converge almost surely by Doob, who noticed that this sequence is a martingale (see [4] p. 13). However, when $E(Z_1 \log Z_1) = \infty$, $\{Z_n/m^n\}$ was shown to tend to 0 a.s. by Kesten and Stigum [5]. Seneta [7] has shown that there exists some norming constants $\{C_n\}$ such that $\{Z_n/C_n\}$ converges in distribution to a nondegenerate limit law. Subsequently, Heyde [3] has found a bounded positive martingale derived from the sequence $\{Z_n/C_n\}$ and obtained the almost sure convergence, by using the martingale convergence theorem.

There were recently some results asserting the convergence in distribution of $\left\{\frac{\log(Z_n+1)}{C_n}\right\}$ in the case $m=\infty$, due to Darling [2] and Seneta [8]. In view of the results mentioned above in the case $m<\infty$, it is natural to raise the question whether in such cases the almost sure convergence also obtains. The situation is of course different from the finite mean case and a martingale argument does not seem likely to be workable. The aim of this paper is to give a general theorem regarding the almost sure convergence of nonhomogeneous Markov chains which can be applied to the case $m=\infty$. We shall show that the conditions of our Theorem are satisfied by the sequences $\{U(Z_n)/C_n\}$ where U is a slowly varying function, provided that the convergence in distribution holds. The proof of the main Theorem 1 is ultimately based on martingales, but the conditions of the theorem involves only the conditional distributions of the given sequence of random variables.

2. Results

Theorem 1. *Let $\{X_n\colon n\geqq 0\}$ be a nonnegative nonhomogeneous Markov chain converging in distribution to a distribution function F and suppose that there exist the limits*

$$\lim_{n\to\infty} P\{X_n\in I\,|\,X_m=i\}=a_{m,i} \tag{2}$$

for any state i with $P\{X_m=i\}>0$, where $I=[0,x)$ or (x,∞), x a continuity point of F with $x>0$ and $0<F(x)<1$. Assume, further that the sequence $\{a_{m,i}\colon m\geqq 0\}$ converges to 1 uniformly with respect to $i\in I_\varepsilon$, where $I_\varepsilon=[0,x-\varepsilon)$ in the case when $I=[0,x)$ or $I_\varepsilon=(x+\varepsilon,\infty)$ in the case when $I=(x,\infty)$, for any $\varepsilon>0$. Then X_n converges almost surely.

Proof. Let us take x a continuity point of F and denote $P^*\{I\}=F(x)$ or $1-F(x)$ according as $I=[0,x)$ or $I=(x,\infty)$. We get

$$\lim_{m\to\infty}\lim_{n\to\infty} P\{X_m\in I)\cap(X_n\in I)\}=$$

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i\in I} P\{(X_n\in I\,|\,X_m=i\}\,P\{X_m=i\}\geqq$$

$$\lim_{\varepsilon\to 0}\lim_{m\to\infty}\sum_{i\in I_\varepsilon} a_{m,i}\,P\{X_m=i\}=P^*\{I\}.$$

The converse inequality is obviously true and therefore we have

$$\lim_{m\to\infty}\lim_{n\to\infty} P\{A_n^c(x)\cap A_m(x)\}=0$$

where we have denoted $A_n(x)=\{X_n\in I\}$.

Write now $\alpha_m=\lim_{n\to\infty} P\{A_n^c(x)\cap A_m(x)\}$ and choose a sequence of positive numbers $\{\varepsilon_n\}$ such that $\sum_{n=1}^{\infty}\varepsilon_n<\infty$. Define further successively the sequence of positive integers $\{n_k\}$ in the following way: choose n_1 such that $\alpha_{n_1} < \varepsilon_1$, n_2 such that

$$P\{A^c_{n_2}(x) \cap A_{n_1}(x)\} < \varepsilon_1 \quad \text{and} \quad \alpha_{n_2} < \varepsilon_2, \text{ etc.}$$

It is easy to see that we can define in this way a sequence $\{n_k\}$ such that

$$P\{A^c_{n_k}(x) \cap A_{n_{k-1}}(x)\} < \varepsilon_k, \quad k = 1, 2, \ldots.$$

Now, we are in a position to apply Lemma 4.1 of Barndorff-Nielsen [1] p. 997 and deduce that $\lim_{k\to\infty} A_{n_k}(x) = A(x)$ (say) exists a.s.

Further, we can write

$$a_{m,i} = \lim_{k\to\infty} P\{A_{n_k}(x) \mid X_m = i\} = P\{A(x) \mid X_m = i\}. \tag{3}$$

Take now a number δ, $0.5 < \delta < 1$. By (3) and the assumptions of the Theorem we get that for any $\varepsilon > 0$

$$B_n(x) = \{i\colon P\{A(x) \mid X_n = i\} > \delta\} \supset I_\varepsilon \tag{4}$$

for n sufficiently large.

By the Markov property we get $P\{A(x) \mid X_n = i\} = P\{A(x) \mid \mathscr{F}_0^n\}$ on the set $\{X_n = i\}$ where $\mathscr{F}_0^n$ is the σ-field generated by $X_0, X_1, \ldots, X_n$. According to the well-known martingales convergence theorem (see e.g. [6] p. 409) we get that

$$\lim_{n\to\infty} P\{A(x) \mid \mathscr{F}_0^n\}(\omega) = 1_{A(x)}$$

where $1_{A(x)} = 1$ or 0 according as $\omega \in A(x)$ or $\omega \in A^c(x)$. (Here we used that $A(x)$ belongs to $\mathscr{F}_0^\infty$.) Therefore $\lim_{n\to\infty} \{X_n \in B_n(x)\} = A(x)$ a.s. The above considerations hold for $I = [0, x)$ or (x, ∞). We fix now $I = [0, x)$ and denote $I' = (x, \infty)$. Further the dash will indicate that the sets correspond to I'. By (4) we have $B_n(x) \supset [0, x - \varepsilon)$ and $B'_n(x) \supset (x + \varepsilon, \infty)$. But $A(x)$ and $A'(x)$ are disjoint a.s. Indeed, we have constructed above the sets $\{A_{n_k}(x)\}$ converging a.s. to $A(x)$ and we can identify $A'(x)$ as the a.s. limit of some sets $\{A'_{n'_k}(x)\}$, taking $\{n'_k\}$ as a subsequence of $\{n_k\}$ by following the same procedure. But

$$A'_{n'_k}(x) \cap A_{n'_k}(x) = \phi,$$

because I and I' are disjoint and therefore $A(x)$ and $A'(x)$ are disjoint a.s.

Now, we can show that $B_n(x)$ and $B'_n(x)$ are also disjoint, because if we suppose the contrary i.e. that there exists $i \in B_n(x) \cap B'_n(x)$, then we get

$$P\{A(x) \cup A'(x) \mid X_n = i\} = P\{A(x) \mid X_n = i\} + P\{A'(x) \mid X_n = i\} > 2\delta > 1$$

which is absurd. It follows that $B_n(x)$ can be taken either as $[0, x)$ or $[0, x]$.

Notice now that

$$\begin{aligned} F(x+\varepsilon) - F(x-\varepsilon) &= P\{A(x+\varepsilon) - A(x-\varepsilon)\} \\ &= P\{\lim_{n\to\infty} [B_n(x+\varepsilon) - B_n(x-\varepsilon)]\}. \end{aligned}$$

Because $x \in B_n(x+\varepsilon) - B_n(x-\varepsilon)$ for any $\varepsilon > 0$ we get $P\{X_n = x \text{ i.o.}\} = 0$ and therefore

$$\lim_{n\to\infty} \{X_n \in [0, x)\} = A(x) \quad \text{a.s.} \tag{5}$$

We shall choose further two monotone sequences of constants $\{a_n\}$ and $\{b_n\}$ consisting of continuity points of F and such that $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} b_n = +\infty$. By (5) we get

$$\lim_{n\to\infty} \{X_n \in (a_m, b_m)\} = C^{(m)} \quad \text{(say) a.s.}$$

whereas $C^{(m)} \subset C^{(m+1)}$ for $m = 1, 2, \ldots$ and

$$P\left\{\bigcup_{m=1}^{\infty} C^{(m)}\right\} = 1 - F(0+). \tag{6}$$

It is easy to see that $P\{\lim_{n\to\infty} X_n(\omega) = 0\} = F(0+)$. Let us take the sequence $x_0, \ldots, x_{k_m}$ of continuity points of F such that

$$a_m = x_0 < x_1 < \cdots < x_{k_m} = b_m \quad \text{and} \quad F(x_j) - F(x_{j-1}) < \varepsilon$$

for a given $\varepsilon > 0$, $i = 1, \ldots, k_m$. By (5) we have

$$\lim_{n\to\infty} \{x_j < X_n < x_{j+1}\} = C_j^{(m)} \quad \text{(say) a.s.}$$

and obviously $\bigcup_{j=1}^{k_m} C_j^{(m)} \subset C^{(m)}$. But

$$P\left\{\bigcup_{j=1}^{k_m} C_j^{(m)}\right\} = \sum_{j=1}^{k_m} (F(x_j) - F(x_{j-1})) = F(b_m) - F(a_m).$$

Therefore $\bigcup_{j=1}^{k_m} C_j^{(m)}$ may differ from $C^{(m)}$ only by a set of null probability. Further

$$\limsup_{n\to\infty} X_n(\omega) - \liminf_{n\to\infty} X_n(\omega) < \varepsilon$$

for almost all $\omega \in \bigcup_{j=1}^{k_m} C_j^{(m)}$ and therefore for almost all $\omega \in C^{(m)}$. But ε and m were arbitrarily chosen and taking into account (6) we conclude the proof.

Theorem 2. *Suppose that $\{Z_n: n \geq 0\}$ is a Galton Watson process with $m = \infty$, U a strictly increasing, continuous, slowly varying function with $\lim_{x\to\infty} U(x) = \infty$ and $\{C_n\}$ an increasing sequence of constants with $\lim_{n\to\infty} C_n = \infty$. Suppose further that $\{U(Z_n)/C_n: n \geq 0\}$ converges in distribution to a distribution function F which is continuous and strictly increasing on $(0, \infty)$. Then $\{U(Z_n)/C_n: n \leq 0\}$ converges almost surely, $\lim_{n\to\infty} C_n/C_{n-1} = \alpha > 1$ exists and is finite and F satisfies the conditions*

$$\lim_{n\to\infty} F_{(\alpha^n x)}^{[U^{-1}(C_n(x-\varepsilon))]} = 1$$

and

$$\lim_{n\to\infty} F_{(\alpha^n x)}^{[U^{-1}(C_n(x+\varepsilon))]}=0.$$

for any $x>0$ *and* $\varepsilon>0$.

Proof. First, we shall show that for any $x>0$ and $a>0$, we get

$$\lim_{n\to\infty} P\left\{\frac{Z_n}{U^{-1}(C_n x)}\leqq a\right\}=F(x). \tag{7}$$

Indeed, we can write

$$P\left\{\frac{Z_n}{U^{-1}(C_n x)}\leqq a\right\}=P\left\{\frac{U(Z_n)}{C_n}\leqq\frac{U(a\,U^{-1}(C_n x))}{C_n}\right\}.$$

Making appeal to (1), we obtain

$$\lim_{n\to\infty}\frac{U(a\,U^{-1}(C_n x))}{C_n}=\lim_{n\to\infty}\frac{U(a\,U^{-1}(C_n x))}{U(U^{-1}(C_n x))}\,\frac{C_n x}{C_n}=x.$$

Now, because F is continuous we get (7).

Write further $X_n=U(Z_n)/C_n$ for $n=1,2,\ldots$. Then we get

$$P\{X_n\in[0,x)\,|\,Z_m=i\}=P\left\{\frac{Z_1^{(n-m)}+\cdots+Z_i^{(n-m)}}{U^{-1}(C_n x)}\in[0,1)\right\} \tag{8}$$

where $Z_1^{(n-m)}, Z_2^{(n-m)}\ldots$ are independent, identically distributed random variables with the same distribution function as Z_{n-m}. Next, we shall show that we can have neither $\liminf_{n\to\infty} C_n/C_{n-1}=1$ nor $\limsup_{n\to\infty} C_n/C_{n-1}=\infty$. Indeed, denote $\beta_n=C_n/C_{n-1}$. Then $C_n/C_{n-m}=\beta_{n-m+1}\ldots\beta_n$. Because $U^{-1}(C_n x)=U^{-1}(C_{n-1}x(C_n/C_{n-1}))$ we get by (7) and (8) that for any sequence $\{n_k\}$ such that $\lim_{k\to\infty}\beta_{n_k}=1$

$$\lim_{k\to\infty} P\{X_{n_k}\in[0,x)\,|\,Z_1=i\}=F^i(x)\qquad i=1,2,\ldots \tag{9}$$

Set now $P\{Z_1=i\}=p_i$, $i=0,1,\ldots$. By (9) we get

$$F(x)=\sum_{i=0}^{\infty}F^i(x)\,p_i. \tag{10}$$

But if we denote by $f(s)$ the generating function of Z_1, then we get that $F(x)$ must be a solution of the equation $f(s)=s$ and in the supercritical case there is only one solution in the interval $(0,1)$ (see e.g. [4]) which contradicts the nondegeneracy of F.

Therefore $\liminf_{n\to\infty}\beta_n=1$ is impossible. We can easily see that neither $\limsup_{n\to\infty}\beta_n=\infty$ is possible, because in such a case we would get for $\lim_{k\to\infty}\beta_{n_k}=\infty$

$$\lim_{k\to\infty} P\{X_{n_k}\in[0,x)\,|\,Z_1=i\}=1$$

for any $i\geqq 0$ which also is in contradiction with the nondegeneracy of F.

Let us take now a subsequence $\{n_k\}$ of the set of positive integers (depending on m), such that $\lim_{k\to\infty} C_{n_k}/C_{n_k-m}=\alpha_m$. As we have seen above $1<\alpha_m<\infty$. By (7), (8) and the continuity of F we get

$$\lim_{k\to\infty} P\{X_{n_k}\in[0,x)\,|\,Z_m=i\}=\lim_{k\to\infty} P^i\left\{\frac{Z_1^{(n_k-m)}}{U^{-1}[C_{n_k-m}(\alpha_m x)]}<1\right\} \qquad (11)$$

$$=F^i(\alpha_m x).$$

We can easily see that

$$F_{(\alpha_m x)}^{[U^{-1}(G_m(x-\varepsilon))]}\leqq F_{(\alpha_m x)}^i \quad \text{for } i<U^{-1}(C_m(x-\varepsilon))$$

$$F_{(\alpha_m x)}^{[U^{-1}(C_m(x+\varepsilon))]}\geqq F_{(\alpha_m x)}^i \quad \text{for } i>U^{-1}(C_m(x+\varepsilon)). \qquad (12)$$

Because $P\{X_n\in[0,x)\,|\,X_m=0\}=1$ for any m and n, due to the fact that $\{Z_n=0\}\subset\{Z_{n+1}=0\}$, the conditions of the Theorem 1 will be verified if

$$\lim_{m\to\infty} F_{(\alpha_m x)}^{[U^{-1}(C_m(x-\varepsilon))]}=1 \quad \text{and} \quad \lim_{m\to\infty} F_{(\alpha_m x)}^{[U^{-1}(C_m(x+\varepsilon))]}=0$$

for any $\varepsilon>0$, $x>0$ and if the limit in (11) will be proved to be the same for all subsequences $\{n_k\}$.

Suppose we can choose a sequence $\{m_l\}$ such that

$$\lim_{l\to\infty} F_{(\alpha_{m_l} x)}^{[U^{-1}(C_{m_l} x_0)]}=\delta>0$$

for a certain x_0. We shall show that

$$\lim_{l\to\infty} F_{(\alpha_{m_l} x)}^{[U^{-1}(C_{m_l} x')]}=1$$

for any $x'<x_0$, whereas if

$$\lim_{l\to\infty} F_{(\alpha_{m_l} x)}^{[U^{-1}(C_{m_l} x_0)]}=0$$

then

$$\lim_{l\to\infty} F_{(\alpha_{m_l} x)}^{[U^{-1}(C_{m_l} x')]}=0$$

for any $x'>x_0$. Because we can write the exponent $[U^{-1}(C_{m_l}x')]$ as

$$\left[U^{-1}(C_{m_l}x_0)\,\frac{U^{-1}(C_{m_l}x')}{U^{-1}(C_{m_l}x_0)}\right]$$

it will be enough to prove that

$$\lim_{n\to\infty}\frac{U^{-1}(C_n a)}{U^{-1}(C_n b)}=0 \quad \text{for } a<b. \qquad (13)$$

Suppose the contrary. Write $U^{-1}(C_n a)=\gamma_n U^{-1}(C_n b)$ or equivalently $C_n a=U(\gamma_n U^{-1}(C_n b))$. Then

$$\frac{a}{b}=\frac{U[\gamma_n U^{-1}(C_n b)]}{C_n b}. \qquad (14)$$

Because $0<\gamma_n \leqq 1$, there must exist a sequence of positive integers say $\{n_k\}$ such that $\gamma_{n_k} \to \gamma$ as $k\to\infty$, with $\gamma>0$. A slight manipulation of condition (1) yields

$$\lim_{k\to\infty} \frac{U[\gamma_{n_k} U^{-1}(C_{n_k} b)]}{C_{n_k} b} = 1$$

and (14) would be invalidated unless $\gamma=0$. Therefore there must be a certain x_0 such that

$$\lim_{l\to\infty} F_{(\alpha_{m_l} x)}^{[U^{-1}(C_{m_l} x')]} = \begin{cases} 1 & \text{if } x'<x_0 \\ 0 & \text{if } x'>x_0 \end{cases}. \tag{15}$$

If we apply now the total probability formula and make use of the convergence in distribution we get

$$\begin{aligned} F(x) &= \lim_{l\to\infty} \lim_{k\to\infty} \sum_{i\in[0,\infty)} \mathrm{P}\{X_{n_k}\in[0,x) \mid X_{m_l}=i\}\, P\{X_{m_l}=i\} \\ &= \lim_{l\to\infty} \lim_{k\to\infty} \Big[\sum_{i\in[0,x_0-\varepsilon)} P\{X_{n_k}\in[0,x) \mid X_{m_l}=i\}\, P\{X_{m_l}=i\} \\ &\quad + \sum_{i\in[x_0-\varepsilon,x_0+\varepsilon)} P\{X_{n_k}\in[0,x) \mid X_{m_l}=i\}\, P\{X_{m_l}=i\} \\ &\quad + \sum_{i\in[x_0+\varepsilon,\infty)} P\{X_{n_k}\in[0,x) \mid X_{m_l}=i\}\, P\{X_{m_l}=i\}\Big]. \end{aligned}$$

We can easily see by (12) and (15) that the last two sums of above are going to 0 as $k\to\infty$, $l\to\infty$ and $\varepsilon\to 0$, and therefore we get

$$F(x) = \lim_{\varepsilon\to 0} F(x_0-\varepsilon) = F(x_0).$$

Hence $x_0 = x$.

Now the limit obtained in (15) with $x=x_0$ does not depend on the particular choice of $\{m_l\}$ and therefore

$$\lim_{m\to\infty} F_{(\alpha_m x)}^{[U^{-1}(C_m x')]} = \begin{cases} 1 & \text{if } x'<x \\ 0 & \text{if } x'>x \end{cases}. \tag{16}$$

In the end, if $\{\alpha'_m\}$ were the limits corresponding to other sequences $\{n_k\}$ say $\{n'_k\}$ then by (16) $\lim_{m\to\infty} (\alpha'_m - \alpha_m) = 0$.

Suppose that there exist two distinct numbers β_1 and β_2 such that

$$\lim_{k\to\infty} \beta_{n_k} = \beta_1, \quad \lim_{k\to\infty} \beta_{n'_k} = \beta_2$$

for some subsequences $\{n_k\}$ and $\{n'_k\}$. By the well known diagonal procedure we can obtain subsequences of $\{n_k\}$ and $\{n'_k\}$ (which by brevity will be also denoted by $\{n_k\}$ and $\{n'_k\}$) such that

$$\lim_{k\to\infty} C_{n_k}/C_{n_k-m} = \alpha_m, \qquad \lim_{k\to\infty} C_{n'_k}/C_{n'_k-m} = \alpha'_m$$

exist for each $m=1, 2, \ldots$. Further, we have

$$\lim_{k\to\infty} C_{n_k-1}/C_{n_k-1-m}=\alpha_{m+1}/\beta_1 \quad \text{and} \quad \lim_{k\to\infty} C_{n'_k-1}/C_{n'_k-1-m}=\alpha'_{m+1}/\beta_2.$$

But as we have seen above, we must have

$$\lim_{m\to\infty} (\alpha_m-\alpha_{m'})=0,$$

as well as

$$\lim_{m\to\infty} (\alpha_m/\beta_1-\alpha'_m/\beta_2)=0$$

which is impossible in the case when $\beta_1 \neq \beta_2$, because $\{\alpha_m\}$ and $\{\alpha'_m\}$ are bounded away from 0. Therefore $\lim_{n\to\infty} \beta_n=\alpha$ exists and the proof is complete.

Remark 1. The almost sure convergence known for the simple branching processes with finite mean, could also be obtained by applying Theorem 1. Evidently, the martingale argument mentioned in the introduction for such a case is simpler, but it seems of interest to have an almost sure convergence criterion which covers both the finite and the infinite mean case.

Suppose that $1<m<\infty$. Then by Seneta's result [7], $\{Z_n/C_n\}$ converges in distribution to a nondegenerate limit W. But, because $\lim_{m\to\infty} C_n/C_{n-1}=m$, we get

$$\begin{aligned}\lim_{n\to\infty} P\{Z_n/C_n\in[0,x)|Z_k/C_k=j\} &= \lim_{n\to\infty} P\left\{\frac{Z_1^{(n-k)}+\cdots+Z_{C_kj}^{(n-k)}}{C_n}\in[0,x)\right\}\\ &=P\left\{\frac{W_1+\cdots+W_{C_kj}}{m^k}\in[0,x)\right\}.\end{aligned}$$

Using now a result of Seneta [9] asserting that $\frac{W_1+\cdots+W_{[C_n]}}{m^n}\to 1$ in probability as $n\to\infty$, we get

$$\lim_{k\to\infty} P\left\{\frac{W_1+\cdots+W_{[C_k(x-\varepsilon)]}}{m^k}\in[0,x)\right\}=1$$

which can be easily shown to entail the condition of Theorem 1.

Remark 2. We notice that the proof sketched above for the finite mean case yields also that $P\{W\in(a,b)\}>0$ for any $a<b$, due to the condition of Theorem 1 regarding $\{a_{m,i}\}$. In the branching processes literature, this result follows from complicated arguments proving the positivity of the density function of W.

Remark 3. Without imposing the condition that $\{X_n\}$ be a Markov chain in Theorem 1, we can get, by using a similar reasoning, the convergence in probability.

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