Z. Wahrscheinlichkeitstheorie verw. Geb. 20, 58–74 (1971)

Ladder Phenomena in Stochastic Processes with Stationary Independent Increments

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1. Introduction

In this paper we introduce a concept of ladder phenomena for continuous time processes with stationary independent increments (s.i.i.) as follows. Let $\{X(t);\, t \geqq 0\}$ be a real valued stochastic process with s.i.i. on a probability space $(\Omega, \mathscr{F}, P)$. We assume that $X(t)$ is measurable, separable and centered, and that its sample functions are right continuous, with $X(0)=0$ a.s. The characteristic function (c.f.) of $X(t)$ is given by

$$E\{e^{i\omega X(t)}\} = e^{t\phi(\omega)} \qquad (\omega \text{ real}), \tag{1}$$

where $\phi(\omega)$ is the exponent function in the Lêvy-Khintchin representation for a c.f. of an infinity divisible distribution. Throughout this paper we shall use Feller's version ([5], p. 533) of $\phi(\omega)$, which is

$$\phi(\omega) = i\omega b + \int_{-\infty}^{\infty} (e^{i\omega x} - 1 - i\omega \sin x)\, x^{-2} M\{dx\}. \tag{2}$$

Here b is a real constant and M is a measure on $(-\infty, \infty)$ which is finite on finite intervals, and has the property that

$$M^+(x) = \int_{x-}^{\infty} y^{-2} M\{dy\} \quad \text{and} \quad M^-\{x\} = \int_{-\infty}^{-x+} y^{-2} M\{dy\}$$

converge for all $x>0$.

For the process $X(t)$, we shall call it the *basic process*, we define a family of sets $\mathscr{L} = \{L(t);\, t>0\}$ where

$$L(t) = \{\omega \in \Omega \colon X(t,\omega) \geqq X(s,\omega)\ (0 \leqq s \leqq t)\}, \tag{3}$$

and term it the (ascending) *ladder phenomenon* of $X(t)$. It will be found that $\mathscr{L}$ is regenerative in the sense of Kingman [6], and we shall devote the main body of this paper to study its properties in the light of Kingman's theory of regenerative phenomena.

To explain the motivation for this study let us describe briefly the concept of a ladder process. Consider a discrete time random walk $\{S_n;\, n \geqq 0\}$, where $S_0 = 0$ and S_n $(n \geqq 1)$ is the n-th partial sum of a sequence of independent identically distributed random variables. Let N_1 be the first entry time, of this random walk, into the set $(0, \infty]$, and Z_1 the value assumed at that time. Similarly, let N_2 be the first time the random walk enters the set $(Z_1, \infty]$, and Z_2 its state at that time. In this manner, we define the pairs $(N_3, Z_3), (N_4, Z_4), \ldots$. The process $\{(N_k, Z_k); k \geqq 1\}$ is called the ladder process of the random walk $\{S_n\}$. The concept was introduced by Blackwell [2], and later extended by Feller ([4], [5]) who applied it to the fluctuation theory of random walks.

Consider now the basic process $X(t)$ as a natural continuous extension of $\{S_n\}$. Suppose that we attempt to define a ladder process for $X(t)$ in an analogous way to the discrete construction. Letting $T_1 = \inf\{t: X(t) > 0\}$, be the first entry time into $(0, \infty]$, we run into the immediate difficulty that T_1 may be zero with probability one. We may overcome this difficulty by introducing instead, the ladder phenomenon $\mathscr{L}$, as a possibly meaningful construction of a continuous time ladder process even in cases when $T_1 = 0$ a.s. Accordingly we can define *ladder epochs* (weak ascending), for the basic process $X(t)$ as the time points at which $\mathscr{L}$ occurs (we say that $\mathscr{L}$ occurs at t whenever $\omega \in L(t)$). We find that these ladder epochs, and the ladder epochs $\{N_k; k \geqq 1\}$ in the discrete case, share two basic properties. Both are points of local maxima with respect to the past of their basic processes, and both have the regnerative property in the sense of Kingman. The last statement means that $\{N_k\}$ is a discrete time regenerative phenomenon, or in the common terminology—a recurrent event; and $\mathscr{L}$, as we shall show, is a continuous time regenerative phenomenon. The difference between the two cases is connected with the set of time points which are ladder epochs. While in the discrete case the number of these points in finite intervals is finite, this number may be uncountable in the continuous case. We shall find, however, that for a large class of processes with s.i.i. this will cause no serious difficulty.

We note that our treatment here is concerned with ladder epochs only, and leaves the problem of constructing a meaningful continuous analogues to the process $\{(N_k, Z_k)\}$, for future investigations. Progress in this direction, has been made in certain special classes of processes by Worthington [16], Prabhu [9] and Prabhu and Rubinovitch [10].

An outline of the paper now follows. In Section 2 we survey the basic elements of the theory on regenerative phenomena. In Section 3 we establish that ladder phenomena are regenerative, obtain some properties of the function $l(t) = \Pr\{L(t)\}$, and derive its Laplace transform. In Section 4 we characterize the local behavior of ladder phenomena in terms of the basic process, and in Section 5 we give their limiting properties. In Section 6 we restrict our attention to a certain class of basic processes and investigate the connection between the ascending ladder phenomenon $\mathscr{L}$ and a process of descending ladder epochs to be defined later.

2. Some Results on Regenerative Phenomena

To help make the paper self contained we outline in this section some relevant results on regenerative phenomena. The concept was introduced by Bartlett [1] but the systematic theory was developed by Kingman ([6], [7]) as a continuous time analog of Feller's recurrent events. All the results in this section are due to Kingman.

Definition. A *regenerative phenomenon* on a probability space $(\Omega, \mathscr{F}, P)$ is a family $\mathscr{E} = \{E(t); t > 0\}$ of $\mathscr{F}$-measurable subsets of Ω with the property that whenever $t_0=0, 0<t_1<\cdots<t_n$ $(n\geqq 1)$ then

$$\Pr\left\{\bigcap_{j=1}^{n} E(t_j)\right\}=\prod_{j=1}^{n} p(t_j-t_{j-1}). \tag{4}$$

Here $p(t)=\Pr\{E(t)\}$ $(t>0)$ and is called the *p-function* of $\mathscr{E}$.

A regenerative phenomenon is said to be *standard* whenever $p(t)\to 1$ as $t\to 0$, and the class of all p-functions with this property is denoted by $\mathscr{P}$. If $p(t)\in\mathscr{P}$ then $p(0)=1$ is defined by continuity at the origin.

We shall now state the results of interest to us as three theorems. The first lists some properties of standard p-functions. The second describes sample functions behavior of standard regenerative phenomena, and the third summarizes their ergodic properties.

Theorem K1. *Let $p(t)$ be any function in $\mathscr{P}$. Then we have the following:*

(a) *$p(t)>0$ for $t\geqq 0$.*

(b) *$p(t)$ is absolutely continuous in $t\geqq 0$.*

(c) *The limit*

$$q=\lim_{t\to 0} t^{-1}[1-p(t)] \tag{5}$$

exists and $0\leqq q\leqq\infty$.

(d) *There exists a unique positive measure μ on $(0,\infty]$ with $\mu\{(0,\infty]\}=q$ and*

$$\int_{(0,\infty]} (1-e^{-x})\,\mu\{dx\}<\infty, \tag{6}$$

such that, for all $\theta>0$

$$\int_0^\infty e^{-\theta t} p(t)\,dt=\Big\{\theta+\int_{(0,\infty]}(1-e^{-\theta x})\,\mu\{dx\}\Big\}^{-1}. \tag{7}$$

(e) *The function $p(t)$ satisfies the Volterra equation*

$$1-p(t)=\int_0^t p(t-s)\,\mu\{(s,\infty]\}\,ds. \tag{8}$$

The measure μ associated with a regenerative phenomenon $\mathscr{E}$ is called its *canonical measure.* When $\mu\{(0,\infty]\}<\infty$ $\mathscr{E}$ is said to be *stable* and when $\mu\{(0,\infty]\}=\infty$, $\mathscr{E}$ is called *instantaneous.*

Theorem K2. *Let $\mathscr{E}$ be a standard regenerative phenomenon and let*

$$S=S(\omega)=\{t>0\colon \omega\in E(t)\}. \tag{9}$$

(a) *For each $t>0$*

$$\begin{aligned}\Pr\{\lim_{\varepsilon\to 0}(1/2\varepsilon)|S\cap(t-\varepsilon,t+\varepsilon)|=1\,|\,E(t)\}=1\\ \Pr\{\lim_{\varepsilon\to 0}(1/\varepsilon)|S\cap(0,\varepsilon)|=1\}=1,\end{aligned} \tag{10}$$

where $|I|$ denotes the Lebesgue measure of I.

(b) *The set S is a.s. metrically dense in itself.*

(c) *For each $t>0$*

$$\Pr\{s\in S\ (t<s<t+T)\,|\,E(t)\}=e^{-qT}\qquad (T>0). \tag{11}$$

Consider now the random variables $\{T_n: n \geqq 0\}$, where $T_0 = 0$ and

$$\begin{aligned} T_{2n-1} &= \sup\{t: s \in S\ (T_{2n-2} < s < t)\} \\ T_{2n} &= \inf\{t > T_{2n-1}: t \in S\}. \end{aligned} \tag{12}$$

Also let $X_n = T_n - T_{n-1}$ $(n \geqq 1)$. It follows from the last theorem that when $\mathscr{E}$ is stable $(q < \infty)$ the random variables $\{X_n; n \geqq 1\}$ are mutually independent and

$$\Pr\{X_{2n-1} \leqq x\} = 1 - e^{-qx}, \qquad \Pr X_{2n} \leqq x\} = q^{-1}\mu\{(0, x]\}. \tag{13}$$

Conversely, one may define a sequence of independent random variables $\{X_n; n \geqq 1\}$ which satisfy (13) with some finite measure μ and $q = \mu\{(0, \infty]\}$. Construct the sequence $\{T_n; n \geqq 1\}$ as $T_n = \sum_1^n X_j$; $T_0 = 0$, and the random set $S = \bigcup_{n=0}^{\infty} (T_{2n}, T_{2n+1})$. It may then be shown that $\mathscr{E} = \{E(t); t > 0\}$ with $E(t) = \{\omega: t \in S(\omega)\}$ is equivalent to a stable regenerative phenomenon, in the sense that there exists a stable regenerative phenomenon $\mathscr{E}_1 = \{E_1(t); t > 0\}$, such that

$$\Pr\{[E_1(t) \cap E^c(t)] \cup [E_1^c(t) \cap E(t)]\} = 0$$

for all $t > 0$. We see that sample functions of stable phenomena behave in a rather simple way and are, in fact, the same as alternating renewal processes of a special kind. Sample function behavior of instantaneous phenomena is more complex since here the random variables T_n all vanish a.s., and the random set S has a void interior a.s. This is discussed in greater detail in [6] where a general model for instantaneous phenomena is also given.

Let $\mathscr{E}$ be a standard regenerative phenomenon. Let $p(t)$ be its p-function and μ its canonical measure. There are three different possibilities for the ergodic behavior of $\mathscr{E}$:

(I) $\mu\{\infty\} > 0$,

(II) $\mu\{\infty\} = 0$ and $\int_{(0,\infty)} x\,\mu\{dx\} = \infty$,

(III) $\mu\{\infty\} = 0$ and $\int_{(0,\infty)} x\,\mu\{dx\} < \infty$.

In case (I) $\mathscr{E}$ is called *transient*, in (II) *null* and in (III) *positive*.

Theorem K 3. *We have*

(a) *As* $t \to \infty$, $p(t) \to p(\infty)$ *where*

$$p(\infty) = \Big[1 + \int_{(0,\infty]} x\,\mu\{dx\}\Big]^{-1} \tag{14}$$

if the integral converges, and $p(\infty) = 0$ *if the integral diverges.*

(b) (i) $\mathscr{E}$ *is transient if* $p(\infty) = 0$ *and* $\int_0^\infty p(t)\,dt < \infty$; (ii) $\mathscr{E}$ *is null if* $p(\infty) = 0$ *and* $\int_0^\infty p(t)\,dt = \infty$; (iii) $\mathscr{E}$ *is positive if* $p(\infty) > 0$.

(c) *The set S is a.s. bounded if $\mathscr{E}$ is transient and unbounded if $\mathscr{E}$ is positive or null. Its Lebesgue measure I has an exponential distribution with mean*

$$E(I)=[\mu\{\infty\}]^{-1}=\int_0^\infty p(t)\,dt \tag{15}$$

when $\mathscr{E}$ is transient, while I is a.s. unbounded when $\mathscr{E}$ is positive or null.

This concludes our outline of regenerative phenomena.

3. Preliminary Properties of Ladder Phenomena

Let $\{X(t);\, t\geqq 0\}$ be the basic process defined in Section 1, and $\mathscr{L}=\{L(t);\, t>0\}$ the corresponding ladder phenomenon according to (3). If $\mathscr{L}$ occurs at time t we shall say that t is a *ladder epoch* of $X(t)$. As we assumed that the basic process is separable, it follows that $\{L(t);\, t>0\}$ are measurable sets, and we can define the function

$$l(t)=\Pr\{L(t)\} \qquad (t>0), \tag{16}$$

called the *ladder function* of $X(t)$. In this section we establish some properties of this ladder function, prove that $\mathscr{L}$ is a regenerative phenomenon and obtain the Laplace transform of $l(t)$.

We begin with a Lemma in which a fundamental property of discrete random walks (see for example Feller [5], p. 378) is extended to continuous time.

Lemma. *For every $t>0$ we have*

$$l(t)=\Pr\{X(s)\geqq 0 \;\; (0\leqq s\leqq t)\}. \tag{17}$$

Proof. Fix t and let $\{X^*(s);\, 0\leqq s\leqq t\}$ be defined by

$$X^*(s)=X(t)-X(t-s).$$

Let $0=s_0<s_1<\cdots<s_{n+1}=t$ and consider

$$\begin{aligned}
&\Pr\{\inf_{1\leqq j\leqq n} X^*(s_j)\geqq 0\}\\
&\quad=\Pr\{X(s_{n+1})-X(s_{n+1}-s_j)\geqq 0 \;\; (1\leqq j\leqq n)\}\\
&\quad=\Pr\left\{\sum_{i=0}^{j-1}[X(s_{n+1}-s_i)-X(s_{n+1}-s_{i+1})]\geqq 0 \;\; (1\leqq j\leqq n)\right\}\\
&\quad=\Pr\left\{\sum_{i=0}^{j-1}X(s_{i+1}-s_i)\geqq 0 \;\; (1\leqq j\leqq n)\right\}\\
&\quad=\Pr\{\inf_{1\leqq j\leqq n} X(s_j)\geqq 0\}.
\end{aligned}$$

It follows by separability that

$$\Pr\{\inf_{0<s<t} X^*(s)\geqq 0\}=\Pr\{\inf_{0<s<t} X(s)\geqq 0\}.$$

This implies (17) since $X(0)=0$ by assumption, and it is known (Doob [3], p. 408) that sample functions of the basic process have left hand limits and for each t

$$X(t-)=X(t)=X(t+) \quad \text{a.s.}$$

Proposition 1. *$l(t)$ is monotone nonincreasing in $t \geqq 0$.*

Proof. This result is a direct consequence of the Lemma.

Proposition 2. *Either $l(t) \equiv 0$ or $l(t) > 0$ for all $t > 0$. In the latter case*

$$\Pr\{L(t)\, L(t+s)\} = l(t)\, l(s) \qquad (t>0;\; s>0). \tag{18}$$

Proof. Let t, s and x be any positive real numbers and suppose that $l(t)>0$. Using well known properties of processes with s.i.i. we may write

$$\begin{aligned}
&\Pr\{L(t+s) | L(t);\; X(t)=x\} \\
&\quad = \Pr\{\inf_{0 \leqq \tau \leqq t+s} [X(t+s) - X(\tau)] \geqq 0 | \inf_{0 \leqq \tau \leqq t} [X(t) - X(\tau)] \geqq 0;\; X(t)=x\} \\
&\quad = \Pr\{X(t+s) - X(\tau) \geqq 0 \;(t \leqq \tau \leqq t+s) | X(t)=x\} \\
&\quad = \Pr\{X(s) - X(\tau) \geqq 0 \;(0 \leqq \tau \leqq s) | X(0)=0\} = l(s).
\end{aligned}$$

Hence we find that

$$\begin{aligned}
\Pr\{L(t)\, L(t+s)\} &= \int_0^\infty \Pr\{L(t+s) | L(t);\; X(t)=x\}\, d_x \Pr\{L(t);\; X(t) \leqq x\} \\
&= l(t)\, l(s).
\end{aligned}$$

In particular, this gives $l(2t) \geqq [l(t)]^2 > 0$ and by an induction argument $l(kt) > 0$ for every integer $k \geqq 1$. This together with the monotone property of $l(t)$ (Proposition 1) proves that $l(t) > 0$ for all $t > 0$, and the proposition follows.

Theorem 1. *$\mathscr{L}$ is a regenerative phenomenon. Its p-function $l(t)$ is uniformly continuous in $t \geqq 0$.*

Proof. According to the definition of a regenerative phenomenon we have to show that whenever $t_0 = 0$, $0 < t_1 < \cdots < t_n$ $(n \geqq 1)$ then

$$\Pr\left\{\bigcap_{j=1}^n L(t_j)\right\} = \prod_{j=1}^n l(t_j - t_{j-1}). \tag{19}$$

This is trivially true when $l(t) \equiv 0$, so suppose that $l(t) > 0$ for all $t > 0$ (Proposition 2). The argument leading to (18) could now be extended to show that for all sequences of real numbers $t_0 = 0$, $0 < t_1 < \cdots < t_n$ we have

$$\Pr\left\{\bigcap_{j=1}^n L(t_j)\right\} = l(t_n - t_{n-1}) \Pr\left\{\bigcap_{j=1}^{n-1} L(t_j)\right\}.$$

From this (19) follows by induction. Also, since $l(t)$ is nonincreasing (Proposition 1), it is necessarily a multiple of a standard p-function (Kingman [8]) and hence uniformly continuous by Theorem K 1 (b).

We now proceed to derive the Laplace transform of $l(t)$. The argument will involve a limiting technique and the use of known results on ladder processes for discrete random walks.

Proposition 3. *For $t \geqq 0$ let*

$$l_n(t) = \Pr\{X(k/2^n) \geqq 0 \;(0 \leqq k \leqq [2^n t])\} \qquad (n \geqq 1), \tag{20}$$

where $[a]$ *denotes the integral part of* a. *Then, as* $n\to\infty$, $l_n(t)\to l(t)$ *uniformly in any finite interval.*

Proof. Fix $t>0$ and let

$$A_n(t)=\{x\colon x=k/2^n\ (0\leqq k\leqq[2^n t])\}.$$

Suppose that $x\in A_n(t)$; then there exists an integer k with $k\leqq[2^n t]$ such that $x=k/2^n$. If we set $k'=2k$ then $x=k'/2^{n+1}$ with $k'\leqq[2^{n+1}t]$, since $2k\leqq 2[2^n t]\leqq[2^{n+1}t]$. This shows that $x\in A_{n+1}(t)$ and $A_n(t)\subset A_{n+1}(t)$. It follows that

$$l_n(t)=\Pr\{X(s)\geqq 0,\ s\in A_n(t)\}$$

is a monotone nonincreasing sequence. Since $l_n(t)$ is also bounded, the limit

$$\lim_{n\to\infty} l_n(t)=l_\infty(t) \qquad \text{(say)}$$

exists, and $0\leqq l_\infty(t)\leqq 1$. Let

$$A=\{x\colon x=k/2^n \text{ with some integers } k\geqq 0,\ n\geqq 1\};$$

also let $B=\{\omega\colon X(s,\omega)\geqq 0,\ s\in A\cap(0,t)\}$ and $C=\{\omega\colon X(s,\omega)\geqq 0,\ s\in(0,t)\}$. By separability of the basic process there exists a set $A\in\mathscr{F}$ with $\Pr\{A\}=0$, such that $B\subset[A\cup C]$. Since $C\subset B$ it follows that $\Pr\{B\}=\Pr\{C\}$. This gives

$$\begin{aligned} l_\infty(t)&=\Pr\{B\}=\Pr\{C\}\\ &=\Pr\{X(s)\geqq 0\ \ (0\leqq s\leqq t)\}=l(t). \end{aligned}$$

Here again we used the assumption that $X(0)=0$ and the fact that for each t, $X(t-)=X(t)=X(t+)$ a.s., we have thus shown pointwise convergence. Uniform convergence in finite intervals follows since $\{l_n(t)\}$ is a sequence of monotone functions which converge to a continuous limit (Rudin [13], p. 156).

Theorem 2. *For* $\theta>0$, *let*

$$r(\theta)=\int_0^\infty e^{-\theta t}\, l(t)\, dt. \tag{21}$$

Then

$$\theta r(\theta)=\exp\left[-\int_0^\infty t^{-1} e^{-\theta t}\Pr\{X(t)<0\}\, dt\right] \tag{22}$$

if the integral in the exponent converges, and $r(\theta)=0$ *if the integral diverges.*

Proof. It follows from Proposition 3 that

$$r(\theta)=\lim_{n\to\infty}\int_0^\infty e^{-\theta t}\, l_n(t)\, dt.$$

Also, if $k\geqq 1$ is an integer then by (20), $l_n(t)=l_n(k/2^n)$ whenever $k/2^n\leqq t<(k+1)/2^n$. Therefore

$$\begin{aligned} r(\theta)&=\lim_{n\to\infty}\sum_{k=0}^{\infty} l_n(k/2^n)\int_{k/2^n}^{(k+1)/2^n} e^{-\theta t}\, dt\\ &=\lim_{n\to\infty}\theta^{-1}(1-e^{-\theta/2^n})\sum_{k=1}^{\infty} l_n(k/2^n)\, e^{-\theta k/2^n}. \end{aligned} \tag{23}$$

For each fixed $n \geqq 1$ let $\{X_k^{(n)}; k \geqq 1\}$ be a sequence of independent random variables each with the same distribution function (d.f.) as $X(1/2^n)$. Let $S_0^{(n)}=0$; $S_k^{(n)}=X_1^{(n)}+\cdots+X_k^{(n)}$ $(k \geqq 1)$, and set $u_0^{(n)}=1$ and

$$u_k^{(n)}=\Pr\{S_j^{(n)} \geqq 0 \ (1 \leqq j \leqq k)\} \qquad (k \geqq 1).$$

It is known (Feller [5], Chapter XII) that $u_k^{(n)}$ equals the probability that k is a ladder epoch for the random walk $\{S_k^{(n)}\}$, and that

$$\sum_{k=0}^{\infty} u_k^{(n)} z^k = \exp\left[\sum_{k=1}^{\infty} k^{-1} z^k \Pr\{S_k^{(n)} \geqq 0\}\right] \qquad (|z|<1).$$

But by (20) $u_k^{(n)}=l_n(k/2^n)$, and also $S_k^{(n)}$ has the same distribution as $X(k/2^n)$. It follows that

$$\sum_{k=1}^{\infty} l_n(k/2^n)\, e^{-\theta k/2^n} = \exp\left[\sum_{k=1}^{\infty} k^{-1} e^{-\theta k/2^n} \Pr\{X(k/2^n) \geqq 0\}\right] - 1.$$

Substituting this into (23) we obtain

$$\begin{aligned}\theta r(\theta) &= \lim_{n\to\infty} (1-e^{-\theta/2^n}) \exp\left[\sum_{k=1}^{\infty} k^{-1} e^{-\theta k/2^n} \Pr\{X(k/2^n) \geqq 0\}\right] \\ &= \lim_{n\to\infty} \exp\left[-\sum_{k=1}^{\infty} k^{-1} e^{-\theta k/2^n} + \sum_{k=1}^{\infty} k^{-1} e^{-\theta k/2^n} \Pr\{X(k/2^n) \geqq 0\}\right].\end{aligned}$$

For each finite n the first sum in the exponent dominates the second sum, and in fact, converges. Thus, both sums converge and we have

$$\theta r(\theta) = \exp\left[-\lim_{n\to\infty} \sum_{k=1}^{\infty} (k/2^n)^{-1} e^{-\theta k/2^n} \Pr\{X(k/2^n) < 0\}\, 2^{-n}\right].$$

When $l(t)$ does not vanish, the left-hand side is positive for every $\theta > 0$, and therefore the right-hand side must converge. The required result now follows and the proof is complete.

We now define the descending ladder phenomenon $\mathscr{L}^* = \{L^*(t); t>0\}$ by

$$L^*(t) = \{\omega \in \Omega : X(t,\omega) \leqq X(s,\omega) \ (0 \leqq s \leqq t)\}, \tag{24}$$

and the descending ladder function $l^*(t)=\Pr\{L^*(t)\}$ $(t>0)$. Results similar to those obtained above for $\mathscr{L}$, obviously hold also for $\mathscr{L}^*$.

4. Characterizing a Ladder Phenomena – Local Time Behavior

In order to apply the general theory on regenerative phenomena to study the behavior of a ladder phenomenon $\mathscr{L}$, it is necessary, first to determine whether $\mathscr{L}$ is standard, and if so whether it is stable or instantaneous. These are the problems to be discussed in this section. The main results are Theorem 3 and Theorem 4 which give, respectively, necessary and sufficient conditions for $\mathscr{L}$ to be standard and stable.

Let $X(t)$ be the basic process as defined in Section 1. Let $\mathscr{L}$ be its ascending ladder phenomenon, $l(t)$ the corresponding ladder function and $r(\theta)$ its Laplace transform. Whenever $l(t)$ vanishes identically (Proposition 2) we shall say that $\mathscr{L}$ is *degenerate*.

Theorem 3. *$\mathscr{L}$ is standard if, and only if,*

$$\int_0^1 t^{-1} \Pr\{X(t)<0\}\, dt<\infty. \tag{25}$$

If $\mathscr{L}$ is not standard then it is degenerate.

Proof. Using (22) we may write

$$\begin{aligned} l(0) &= \lim_{t\to 0} l(t) = \lim_{\theta\to\infty} \theta r(\theta) \\ &= \lim_{\theta\to\infty} \exp\left[-\int_0^\infty t^{-1} e^{-\theta t} \Pr\{X(t)<0\}\, dt\right]. \end{aligned}$$

It follows that

$$l(0)=\exp\left[-\lim_{\theta\to\infty} \int_0^1 t^{-1} e^{-\theta t} \Pr\{X(t)<0\}\, dt\right], \tag{26}$$

since

$$\lim_{\theta\to\infty} \int_1^\infty t^{-1} e^{-\theta t} \Pr\{X(t)<0\}\, dt=0.$$

There are two possibilities, either the integral in (25) diverges, or it is convergent. Suppose it diverges, then

$$\int_0^1 t^{-1} e^{-\theta t} \Pr\{X(t)<0\}\, dt \geqq e^{-\theta} \int_0^1 t^{-1} \Pr\{X(t)<0\}\, dt=\infty$$

and by (26), $l(0)=0$ which implies that $l(t)=0$ for all $t\geqq 0$ (Proposition 1). If on the other hand, the integral in (25) converges, then by the dominated convergence theorem

$$\lim_{\theta\to\infty} \int_0^1 t^{-1} e^{-\theta t} \Pr\{X(t)<0\}\, dt=0.$$

Using again (26), we conclude that $l(0)=1$. The required results now follow.

Let us now consider the ladder phenomena of two special processes, namely the compound Poisson and the Brownian motion. It is well known that every process with s.i.i. may be represented as a convolution, or as a limit of a sequence of convolutions, of two such processes. In this sense the compound Poisson and the Brownian motion processes are extremes which stand at opposite ends of the spectrum of all processes with s.i.i., and further insight may be gathered by looking at their ladder phenomena.

Consider first the ladder phenomena $\mathscr{L}$ of a compound Poisson process $X(t)$. Let

$$g(t)=\Pr\{X(s)=0\ (0\leqq s\leqq t)\} \qquad (t>0), \tag{27}$$

then $g(t)=e^{-\lambda t}$ with some $\lambda>0$. Using (17) we see that

$$l(t)\geqq g(t) \tag{28}$$

and $g(t)\to 1$ as $t\to 0$. Therefore $\mathscr{L}$ is standard. Moreover because of (28) $l'(0)$ is finite and $\mathscr{L}$ is also stable. On the other hand, when $X(t)$ is a Brownian motion process, then $\Pr\{X(t)<0\}=\frac{1}{2}$ (all t), the integral in (25) diverges and by Theorem 3,

$\mathscr{L}$ is degenerate. Another way to prove this may rest on the fact that for Brownian motion processes

$$\liminf_{t\to\infty} \frac{X(t)}{t} = -\infty \qquad \text{a.s.} \tag{29}$$

(Doob [3], p. 394). Therefore, with probability one $X(t)$ assumes negative values in any neighborhood of the origin. This implies that $l(t) = \Pr\{X(s) \geqq 0\ (0 \leqq s \leqq t)\} = 0$ for all t, so that $\mathscr{L}$ is degenerate. It is easy to see that the trouble with Brownian motion processes lies in the unpleasant way their sample functions oscillate. In fact, since (29) is true for any process with s.i.i. which is of unbounded variation (Rogozin [12]), the same argument applies and shows that $\mathscr{L}$ is degenerate wherever $X(t)$ is such a process. The converse of this, however, is not true as the next proposition states.

Proposition 4. *If $X(t)$ is a process of unbounded variation then $\mathscr{L}$ is degenerate. If $X(t)$ is of bounded variation, then $\mathscr{L}$ is standard whenever $b>0$ (b is the drift of the process) and degenerate when $b<0$. No conclusion can be reached when $b=0$.*

Proof. The first part of the statement has already been proved, so consider the case when $X(t)$ is a process of bounded variation with drift b ($-\infty < b < \infty$). Here we have

$$\lim_{t\to 0} \frac{X(t)}{t} = b \qquad \text{(a.s.)} \tag{30}$$

(Shtatland [14]). Suppose that $b>0$ and let $t_0 = t_0(\omega) = \sup\{t\colon X(s,\omega) > 0\ (0 < s \leqq t)\}$. Because of (30) there exists, for almost all ω, a positive number $t(\omega)$ such that $X(s,\omega)/s > b/2 > 0$ for all $0 < s \leqq t(\omega)$. This implies that $t_0 > 0$ a.s. and we may write

$$\lim_{t\to 0} \Pr\{t_0 > t\} = 1.$$

On the other hand, using (17) we see that $l(t) \geqq \Pr\{t_0 > t\}$ and it follows that $\mathscr{L}$ is standard. When $b<0$ we find in a similar way, that $t_1 = t_1(\omega) = \sup\{t\colon X(s,\omega) < 0\ (0 < s \leqq t)\}$ is positive a.s. Therefore

$$l(t) \leqq \Pr\{X(t) \geqq 0\} \leqq 1 - \Pr\{t_1 > t\} \to 0 \qquad \text{as } t \to 0$$

which shows that $\mathscr{L}$ is degenerate. To prove the last statement we note that the compound Poisson process discussed before is an example of a process of bounded variation for which $\mathscr{L}$ is standard and $b=0$. At the end of this section we shall construct a class of processes of bounded variation and $b=0$ for which the ascending ladder phenomena are degenerate. The completion of the proof is deferred until then.

Recall now the measure M and the functions $M^+(x)$ and $M^-(x)$ arising in connection with the c.f. (2) of $X(t)$. It is known that when $M\{0\}=0$, $X(t)$ does not have a diffusion component, and that, except for a possible drift, changes of state occur only through jumps. The number of jumps in $(0,t)$ which are of magnitude $\leqq a < 0$ follow a Poisson distribution with mean $tM^-(a)$. This is also true for $a=0$ whenever $M^-(0) < \infty$ (Doob [3], p. 423). We can now state

Theorem 4. *A standard ladder phenomenon $\mathscr{L}$ with a canonical measure μ is stable if, and only if, $M^-(0) < \infty$. When this is the case then $\mu\{(0,\infty]\} = M^-(0)$.*

Proof. Assume that $\mathscr{L}$ is standard. In order for $\mathscr{L}$ to be also stable it is necessary and sufficient that for every $t>0$

$$\Pr\{s\in S\,(0<s<t)\}=e^{-qt} \tag{31}$$

where

$$S=S(\omega)=\{t\colon \omega\in L(t)\}$$

and $q=\mu\{(0,\infty]\}$ [see Theorem K2 (c)]. On the other hand, since $\mathscr{L}$ is standard, we must have $M\{0\}=0$ and $b\geqq 0$ (Proposition 4). We conclude, that except for a possible positive drift, $X(t)$ changes states only through jumps. Therefore the left-hand side of (31) equals the probability that $X(t)$ has no negative jumps in $(0,t)$, which equals zero if $M^-(0)=\infty$ and equals $e^{-tM^-(0)}$ if $M^-(0)<\infty$. This proves the required result.

We can also express the results of Theorem 4 in a different way. It is known that the exponent function $\phi(\omega)$ of a basic process which is of bounded variation is given by

$$\phi(\omega)=i\omega b+\int_0^\infty (e^{i\omega x}-1)\,x^{-1}M_1\{dx\}-\int_0^\infty (e^{i\omega x}-1)\,x^{-1}M_2\{dx\}, \tag{32}$$

where M_1 and M_2 are measures on $(0,\infty)$ such that $(1+x)^{-1}$ is integrable with respect to M_1 and M_2 (Skorokhod [15], p. 94)[1]. It is not hard to verify that the condition $M^-(0)<\infty$ is equivalent to

$$\lambda=\int_0^\infty x^{-1}M_2\{dx\}<\infty. \tag{33}$$

This means that the second integral may be written as

$$\lambda\int_0^\infty (e^{i\omega x}-1)\,K\{dx\} \tag{34}$$

where $K\{dx\}=(\lambda x)^{-1}M_2\{dx\}$. We recognize (34) as the exponent function of a compound Poisson process whose state space is $[0,\infty)$, and the first integral in (32) as the exponent function of a basic process which has positive jumps only (Feller [5], p. 539). We conclude that in order for $\mathscr{L}$ to be standard and stable it is necessary and sufficient that $b\geqq 0$ and that the c.f. of $X(t)$ be representable as the product of a c.f. of a compound Poisson process on $(-\infty,0]$, and a process with s.i.i. on $[0,\infty)$. This includes the situation in which the compound Poisson component vanishes. Here sample functions of $X(t)$ are a.s. nondecreasing, $l(t)=1$ identically, and $\mathscr{L}$ is trivially standard and stable.

We have thus given a complete characterization of the local time behavior of an ascending ladder phenomena $\mathscr{L}$. The same techniques, when applied to the process $-X(t)$, would yield an analogous characterization for a descending ladder phenomena $\mathscr{L}^*$. However, the natural problem to pose now is about the mutual behavior of $\mathscr{L}$ and $\mathscr{L}^*$. Could they both be standard and both stable? The next proposition gives the answer which is quite surprising.

[1] The formula for $\phi(\omega)$ in [15] differs from (32) due to the different measure used. See the remark on p. 536 of Feller [5].

Proposition 5. *The following statements are equivalent.*

(i) *Both $\mathscr{L}$ and $\mathscr{L}^*$ are standard.*

(ii) *Both $\mathscr{L}$ and $\mathscr{L}^*$ are standard and stable.*

(iii) *$X(t)$ is a compound Poisson process.*

Proof. In the discussion proceeding Theorem 3 we have in fact shown that (iii) implies (ii). Since (ii) implies (i) it is sufficient to prove that (i) implies (iii). Assume that (i) holds and let $g(t)$ be the function defined in (27). Suppose that $g(t) \equiv 0$; this implies that for all $t>0$

$$\Pr\{X(s) \neq 0 \text{ for some } 0<s \leqq t\}=1. \tag{35}$$

If we let $T^+ = \inf\{s>0\colon X(s)>0\}$ and $T^- = \inf\{s>0\colon X(s)<0\}$, then (35) implies that either $T^+ \leqq t$ or $T^- \leqq t$ with probability one. On the other hand

$$\Pr\{T^+ \leqq t\} = 1 - \Pr\{X(s) \leqq 0 \ (0 \leqq s \leqq t)\} = 1 - l^*(t)$$

and similarly

$$\Pr\{T^- \leqq t\} = 1 - l(t).$$

Hence for all $t>0$ we have

$$\begin{aligned} 1 &= \Pr\{[T^+ \leqq t] \cup [T^- \leqq t]\} \\ &\leqq 1 - l(t) + 1 - l^*(t) \to 0 \quad \text{as} \quad t \to 0 \end{aligned}$$

by assumption. This contradiction shows that $g(t)$ cannot vanish identically. Let $t>0$ be such that $g(t)>0$; by a straightforward argument one may show that $g(t)$ has the property that for every $s>0$ $g(t+s)=g(t)\,g(s)$. We conclude that $g(t)=e^{-\alpha t}$ with some $\alpha>0$ and a process with s.i.i. for which this holds must be a compound Poisson process. This completes the proof.

We can combine the results of Propositions 4 and 5 in the following statement. *There are three classes of processes with s.i.i.:* (i) *processes in which both $\mathscr{L}$ and $\mathscr{L}^*$ are standard,* (ii) *processes in which exactly one of $\mathscr{L}$ and $\mathscr{L}^*$ is standard and the other is degenerate, and* (iii) *processes in which both are degenerate.* Processes of unbounded variation belong to the third class, while processes of bounded variation and nonzero drift belong to the second. The only members of the first class are compound Poisson processes which have no drift.

We are now in a position to complete the proof of Proposition 4, by constructing a basic process of bounded variation and $b=0$ whose ascending ladder phenomena are degenerate. Let

$$\phi_1(\omega) = \int_0^\infty (e^{i\omega x} - 1)\, x^{-1} M_1\{dx\}$$

where M_1 is the same as in (32) with the additional property that $\int_0^\infty x^{-1} M_1\{dx\} = \infty$. Let $\{X_i(t)\colon t \geqq 0\}$ $(i=1,2)$ be two independent processes each with the c.f. $e^{t\phi_1(\omega)}$ and set $X(t)=X_1(t)-X_2(t)$ $(t>0)$. The c.f. of $X(t)$ is a special case of (32) so that $X(t)$ is of bounded variation; and it also has no drift. Consider now the ladder phenomena $\mathscr{L}$ and $\mathscr{L}^*$ of this process. By reasons of symmetry they are both either standard or degenerate. The first possibility is ruled out since it implies (Proposition 5) that $X(t)$ is compound Poisson, and the additional requirement on M_1 assures us that this is not the case. We are therefore left with the second possibility and conclude that $\mathscr{L}$ is degenerate.

5. Characterizing a Ladder Phenomenon – Ergodic Properties

In this section we give an expression for the limiting value of a standard ladder function, and find necessary and sufficient conditions for a standard ladder phenomenon to be positive, null, or transient. We can restrict attention to basic processes of bounded variation, since only for such processes could a ladder phenomenon be standard. Let $X(t)$ be such a process and let us say that the ascending ladder phenomenon $\mathscr{L}$ of $X(t)$ is standard. Let

$$A=\int_1^\infty t^{-1}\Pr\{X(t)<0\}\,dt \quad \text{and} \quad B=\int_1^\infty t^{-1}\Pr\{X(t)>0\}\,dt.$$

Proposition 6. *We have*

$$l(\infty)=\lim_{t\to\infty} l(t)=\exp\left[-\int_0^\infty t^{-1}\Pr\{X(t)<0\}\,dt\right]>0 \tag{36}$$

if $A<\infty$, *and* $l(\infty)=0$ *if* $A=\infty$.

Proof. The limit exists by Theorem K3 since $\mathscr{L}$ is standard. Hence we may write

$$l(\infty)=\lim_{\theta\to 0+}\theta\, r(\theta)=\lim_{\theta\to 0+}\exp\left[-\int_0^\infty t^{-1}e^{-\theta t}\Pr\{X(t)<0\}\,dt\right].$$

However, by Theorem 3

$$\int_0^1 t^{-1}\Pr\{X(t)<0\}\,dt<\infty$$

and the value of the limit will be determined by the value of A. The proposition now follows.

It is interesting to note that $l(\infty)=1$ if, and only if, $l(t)=1$ identically. To see this we note that in order to have $l(\infty)=1$ it is necessary and sufficient that

$$\int_0^\infty t^{-1}e^{-\theta t}\Pr\{X(t)<0\}\,dt=0,$$

or, equivalently, that $\Pr\{X(t)<0\}=0$ for all t. This brings us to the trivial case cited in Section 4 when almost all sample functions of $X(t)$ are nondecreasing and $l(t)=1$ for all t.

Theorem 5. *We have*

(i) $\mathscr{L}$ *is transient if* $A=\infty$ *and* $B<\infty$;

(ii) $\mathscr{L}$ *is null if* $A=\infty$ *and* $B=\infty$;

(iii) $\mathscr{L}$ *is positive if* $A<\infty$.

Proof. Proposition 6 together with Theorem K3 prove (iii). When $A=\infty$, it will suffice to show that $\mathscr{L}$ is transient if, and only if, $B<\infty$. Substituting into (22) for θ, according to the identity

$$\theta=\exp\left[-\int_0^\infty (e^{-\theta t}-e^{-t})\,t^{-1}\,dt\right] \qquad (\theta>0)$$

we obtain

$$r(\theta)=\exp\left[-\int_1^\infty t^{-1}e^{-t}\,dt+\int_0^1 (e^{-\theta t}-e^{-t})\,t^{-1}\,dt\right.$$
$$-\int_0^1 t^{-1}e^{-\theta t}\Pr\{X(t)<0\}\,dt+\int_1^\infty t^{-1}e^{-\theta t}\Pr\{X(t)=0\}\,dt$$
$$\left.+\int_1^\infty t^{-1}e^{-\theta t}\Pr\{X(t)>0\}\,dt\right].$$

As $\theta\to 0$, the second and fourth integrals in the last expression always converge, while the third integral also converges since $\mathscr{L}$ is standard. It follows that $\lim_{\theta\to 0} r(\theta)<\infty$ if, and only if, the last integral converges to $B<\infty$. Recalling now Theorem K 3 we see that $\mathscr{L}$ is transient if, and only if

$$\int_0^\infty l(t)\,dt=\lim_{\theta\to 0} r(\theta)<\infty.$$

This completes the proof.

It is clear that the classification of the last theorem should be related to the limiting properties of sample functions of the basic process. To see the connection we let

$$\sigma=\sup_{0\le t<\infty}\{X(t)\} \quad \text{and} \quad \sigma^*=\inf_{0\le t<\infty}\{X(t)\}.$$

It is known (Rogozin [11]) that $\sigma<\infty$ ($\sigma^*>-\infty$) a.s. if $B<\infty$ ($A<\infty$), and $\sigma=\infty$ ($\sigma^*=-\infty$) a.s. otherwise. Accordingly, using Theorem 5, we see that $\mathscr{L}$ is transient if, and only if $\sigma<\infty$ and $\sigma^*=-\infty$; $\mathscr{L}$ is null if, and only if $\sigma=\infty$ and $\sigma^*=-\infty$; and $\mathscr{L}$ is positive if, and only if $\sigma=\infty$ and $\sigma^*>-\infty$ (in making the last statement we used the fact that $A<\infty$ implies $B=\infty$). This proves the following proposition.

Proposition 7. *$\mathscr{L}$ is positive or transient according to whether sample functions of the basic process drift a.s. to $+\infty$ or $-\infty$. $\mathscr{L}$ is null if almost all sample functions oscillate between $+\infty$ and $-\infty$.*

6. Ladder Epochs

Consider the class of basic processes $X(t)$ for which at least one of the ladder phenomena $\mathscr{L}$ and $\mathscr{L}^*$ is standard. If $X(t)$ is compound Poisson then both $\mathscr{L}$ and $\mathscr{L}^*$ are standard, otherwise exactly one is standard and the other is degenerate (Proposition 5). Let us say that $\mathscr{L}$ is standard and $\mathscr{L}^*$ is degenerate. In view of the results of Sections 3 and 4, and the classification of Section 5, we can now apply the general theory of regenerative phenomena to describe the stochastic behavior of $\mathscr{L}$, and in particular the properties of the set S of all weak ascending ladder epochs of $X(t)$. This can be done in a straightforward manner and we shall not pursue it here. On the other hand, the only thing we know about the descending phenomenon $\mathscr{L}^*$ is that it is degenerate, and this of course is not very informative. It would therefore be only natural to investigate now the behavior of $\mathscr{L}^*$, and its relation to the standard phenomenon $\mathscr{L}$. To avoid trite complications we find it convenient to modify slightly the definition of $\mathscr{L}^*$ by changing the weak inequality in (24) to a strict one. Thus, in the sequel, $\mathscr{L}^* = \{L^*(t);\ t>0\}$ where

$$L^*(t) = \{\omega\colon X(t,\omega) < X(s,\omega)\ (0 \leqq s \leqq t)\}. \tag{37}$$

It is not hard to verify that with the exception of the compound Poisson processes the new $\mathscr{L}^*$ is the same as the old one. We can now state.

Theorem 6. *Assume that $\mathscr{L}$ is standard, and let $\mathscr{L}^*$ be defined by* (37). *We have*

(a) *$\mathscr{L}^*$ is a renewal process. The d.f. $F^*(x)$ of its lifetimes is absolutely continuous and its density $f^*(x)$ has the property that $f^*(0+) = M^-(0) \leqq \infty$. The Laplace transforms of $F^*(x)$ is given by $1-\theta r(\theta)$ $(\theta>0)$.*

(b) *The renewal process $\mathscr{L}^*$ is transient (terminating) when $\mathscr{L}$ is positive and is persistent otherwise. The expected value of the lifetimes is $1/\mu\{\infty\} < \infty$ if $\mathscr{L}$ is transient and infinite if $\mathscr{L}$ is null.*

(c) *The renewal function $U^*(x)$ of $\mathscr{L}^*$ has a density $u^*(x)$ and $u^*(x) = \mu\{(t,\infty]\}$* [2].

Proof. Let $S^* = S^*(\omega) = \{t\colon \omega \in L^*(t)\}$ and set

$$T_1^* = \inf\{t\colon t \in S^*\}. \tag{38}$$

It follows from (37) that T_1^* can be written as

$$T_1^* = \inf\{t\colon X(t) < 0\} \tag{39}$$

and therefore the d.f. of T_1^* is given by

$$\begin{aligned} F_1^*(x) = \Pr\{T_1^* \leqq x\} &= 1 - \Pr\{X(s) \geqq 0\ (0 \leqq s \leqq t)\} \\ &= 1 - l(t) \end{aligned} \tag{40}$$

according to (17), where $l(t)$ is the ascending ladder function. As $\mathscr{L}$ was assumed to be standard we find that $F^*(0+)=0$ and $T_1^*>0$ a.s. We can therefore define the sequence $\{T_n^*;\ n \geqq 1\}$, where T_1^* is given by (39) and

$$T_n^* = \inf\{t\colon X(t) < X(T_{n-1}^*)\} \qquad (n \geqq 2),$$

and it is clear that $S^* = \{T_n;\ n \geqq 1\}$. Let t_2, τ, and x be positive numbers and consider

$$\begin{aligned} &\Pr\{T_2^* - T_1^* > t_2 \mid T_1 = \tau;\ X(\tau) = x\} \\ &\quad = \Pr\{X(s) \geqq x\ (\tau \leqq s \leqq \tau + t_2) \mid T_1 = \tau;\ X(\tau) = x\}. \end{aligned}$$

Because of the Markov property of the basic process, this equals

$$\begin{aligned} &\Pr\{X(s) \geqq x\ (\tau \leqq s \leqq \tau + t_2) \mid X(\tau) = x\} \\ &\quad = \Pr\{X(s) \geqq 0\ (0 \leqq s \leqq t_2) \mid X(0) = 0\} \\ &\quad = \Pr\{T_1^* > t_2\}. \end{aligned}$$

[2] I owe this result to N. U. Prabhu.

Therefore, for $t_1 > 0$ and $t_2 > 0$ we have

$$\begin{aligned}\Pr\{T_2^* - T_1^* > t_2;\ T_1^* \leqq t_1\} &= \int_{\tau=0}^{t_1} \int_{x=0+}^{\infty} \Pr\{T_2^* - T_1^* > t_2 \mid T_1^* = \tau;\ X(\tau) = x\}\, d_\tau d_x \\ &\quad \Pr\{T_1^* \leqq \tau;\ X(\tau) \leqq x\} \\ &= \Pr\{T_1^* > t_2\} \Pr\{T_1^* \leqq t_1\}.\end{aligned}$$

This shows that T_1^* and $T_2^* - T_1^*$ are independent and have the same d.f. $F^*(x)$. It follows by induction that $T_1^*, T_2^* - T_1^*, \ldots$, are independent identically distributed random variables and therefore $\mathscr{L}^*$ is a renewal process. Using (40) we now find the Laplace transform of the d.f. $F^*(x)$ as

$$\int_0^\infty e^{-\theta x}\, dF^*(x) = \theta \int_0^\infty e^{-\theta x}[1 - l(x)]\, dx = 1 - \theta r(\theta) \qquad (\theta > 0).$$

(We note that this transform may also be derived from a general theorem of Rogozin [11].) Moreover according to Theorem K 1(b), $l(t)$ is absolutely continuous and so must be $F^*(x)$. If we let $f^*(x)$ be its density, then

$$f^*(0+) = F^{*\prime}(0+) = -l'(0+) = q,$$

and by Theorem 4, $q = \mu\{(0, \infty]\} = M^-(0)$ if $M^-(0) < \infty$ and $q = \infty$ if $M^-(0) = \infty$. This proves (a). To prove (b) we take the limit as $t \to \infty$ in (40) and obtain

$$\Pr\{T_1^* < \infty\} = \lim_{t \to \infty} F^*(x) = 1 - l(\infty),$$

and according to Theorem 4 and Proposition 6 this is < 1 if and only if $\mathscr{L}$ is positive. Also from (40),

$$E(T_1^*) = \int_0^\infty [1 - F^*(x)]\, dx = \int_0^\infty l(x)\, dx = [\mu\{\infty\}]^{-1} < \infty$$

if $\mathscr{L}$ is transient, and $E(T_1) = \infty$ if $\mathscr{L}$ is null (see Theorem K 3). To prove (c) let

$$U^*(x) = \sum_{n=1}^{\infty} F^{*(n)}(x) \tag{41}$$

be the renewal function of $\mathscr{L}^*$. Then $U^*(x)$ is the unique bounded solution of the renewal equation

$$U^*(x) = F^*(x) + \int_0^x U^*(x-s)\, dF^*(s), \tag{42}$$

and also, by (a) $U^*(x)$ is absolutely continuous. Substituting $F^*(x) = 1 - l(x)$ and rearranging terms we can write the last equation as

$$1 - l(x) = \int_0^x l(x-s)\, u^*(s)\, ds,$$

where $u^*(x)$ is the density of $U^*(x)$. On the other hand, since $l(x)\in\mathscr{P}$ it also satisfies the Volterra equation

$$1-l(x)=\int_0^x l(x-s)\,\mu\{(x,\infty]\}\,ds,$$

according to (8). We note that because of (6) $\mu\{(s,\infty]\}$ is integrable over $(0,a)$, for each $a>0$, and therefore the assersion follows by uniqueness. The theorem is completely proven.

In view of the last theorem, the established theory of renewal processes could be employed to completely describe the properties of $\mathscr{L}^*$.

Acknowledgment. This paper is based on parts of my doctoral dissertation at Cornell University, written under the guidance of Professor N.U. Prabhu. Professor Prabhu introduced me to this area of research, and I gratefully acknowledge his generous help throughout. I am also grateful to Professor J.F.C. Kingman, who kindly read this paper and made helpful comments. This research was partially supported by the Office of Naval Research under Contract N-onr-401 (53).

References

1. Bartlett, M.S.: Recurrence and first passage times. Proc. Cambridge philos. Soc. **49**, 263–275 (1953).
2. Blackwell, D.: Extension of a renewal theorem. Pacific J. Math. **3**, 315–320 (1958).
3. Doob, J.L.: Stochastic processes. New York: John Wiley 1953.
4. Feller, W.: On combinatorial methods in fluctuation theory. The Harald Cramêr Volume, p. 75–91. New York: John Wiley 1959.
5. – An introduction to probability theory and its applications, vol. II. New York: John Wiley 1966.
6. Kingman, J.F.C.: The stochastic theory of regenerative events. Z. Wahrscheinlichkeitstheorie verw. Geb. **2**, 180–224 (1964).
7. – Some further analytical results in the theory of regenerative events. J. math. Analysis Appl. **11**, 422–433 (1965).
8. – On measurable p-functions. Z. Wahrscheinlichkeitstheorie verw. Geb. **11**, 1–8 (1968).
9. Prabhu, N.U.: Ladder variables for a continuous time stochastic process. Z. Wahrscheinlichkeitstheorie verw. Geb. **16**, 157–164 (1970).
10. – Rubinovitch, M.: On a continuous time extension of Feller's lemma. Z. Wahrscheinlichkeitstheorie verw. Geb. **17**, 220–226 (1971).
11. Rogozin, B.A.: On distributions of functionals related to boundary problems for processes with independent increments. Theor. Probab. Appl. **11**, 580–591 (1966).
12. – Local behavior of processes with independent increments. Theor. Probab. Appl. **13**, 482–486 (1968).
13. Rudin, W.: Principles of mathematical analysis. New York: McGraw-Hill (1964).
14. Shtatland, E.S.: On local properties of processes with independent increments. Theor. Probab. Appl. **10**, 317–322 (1965).
15. Skorokhod, A.V.: Random processes with independent increments. M. Izd-vo "Nauka". [In Russian] 1964.
16. Worthington, F.M.: Some ladder processes in continuous time with applications to queues. Ph. D. Thesis, Cornell University (1967).

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(Received June 18, 1970)