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Some Refinements in the Theory of Supercritical Multitype Markov Branching Processes

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§ 1. Introduction

Let $\{Z(t); t \geqq 0\}$ be a p-type $(2 \leqq p < \infty)$ continuous time Markov branching process. The problem of studying the limit behavior of $\eta \cdot Z(t)$ as $t \to \infty$ where η is an arbitrary complex vector was attacked by the author in [2]. Kesten and Stigum [3] studied the discrete time case. The objectives of this paper are i) to refine some of these works and ii) to fill a gap mentioned in [2]. These are important in themselves as well as for their methodology which we believe will have applications elsewhere.

§ 2. The Statement of Results

We shall use the set up in [2]. Briefly, we consider a supercritical, positively regular and nonsingular multitype Markov branching process $\{Z(t); t \geqq 0\}$ with mean matrix $M(t) = \exp(At)$ and finite second moments. Let the eigenvalues of A be arranged such that $\lambda_1 > \operatorname{Re} \lambda_2 \geqq \operatorname{Re} \lambda_3 \geqq \cdots \geqq \operatorname{Re} \lambda_r$ where $\lambda_1, \lambda_2, \ldots, \lambda_r$ are the distinct eigenvalues and let u and v denote respectively the left and right eigenvectors of A with eigenvalue λ_1 normalised such that u, v are both strictly positive and $u \cdot v = 1$. It is known that in the supercritical case, that is when $\lambda_1 > 0$, $Z(t) e^{-\lambda_1 t} \to u W$ almost surely as $t \to \infty$ where W is a nonnegative random variable such that $P(W > 0) > 0$ for any P with $P(Z(0) \neq 0) > 0$. If η is any complex vector such that $\eta \cdot u \neq 0$ then it follows that $\eta \cdot Z(t) e^{-\lambda_1 t} \to (\eta \cdot u) W$. Thus the problem is easily solved in this case. When $\eta \cdot u = 0$ we have to use different normalisation. In [2] we had established a trichotomy in the behavior of $\eta \cdot Z(t)$.

Let $\{v_{jk}, k = 1, 2, \ldots, d_j, j = 1, 2, \ldots, r\}$ be the generalized eigenvectors of A. Here d_j is the algebraic multiplicity of the eigenvalue λ_j and $\{v_{jk}\}$ satisfy

$$M(t)\, v_{jl} = e^{\lambda_j t} \sum_{k=1}^{l} v_{jk} \frac{t^{l-k}}{(l-k)!}, \qquad 1 \leqq l \leqq d_j. \tag{1}$$

Given any vector η we can find unique constants c_{jk} such that

$$\eta = \sum_{j=1}^{r} \sum_{k=1}^{d_j} c_{jk} v_{jk}. \tag{2}$$

Define four objects connected with η as follows:

$$\begin{aligned}
&a(\eta) = \sup\{\operatorname{Re} \lambda_j;\ c_{jk} \neq 0 \text{ for some } k\}\\
&\tilde{\gamma}(\eta) = \sup\{k\text{: there exists } j \text{ such that } \operatorname{Re} \lambda_j = a(\eta) \text{ and } c_{jk} \neq 0\}\\
&\gamma(\eta) = \tilde{\gamma}(\eta) - 1\\
&I(\eta) = \{j\text{: } \operatorname{Re} \lambda_j = a(\eta),\ c_{j\tilde{\gamma}} \neq 0,\ c_{jk} = 0 \text{ for } k > \tilde{\gamma}\}.
\end{aligned} \tag{3}$$

The trichotomy mentioned earlier is based on whether $2a(\eta)>$, $=$, or $<\lambda_1$. In this paper we shall refine the result proved in [2] for the case $2a(\eta)>\lambda_1$ and also fill a gap in the case $2a(\eta)=\lambda_1$. The gap in question is simply to establish the following

Theorem 1. *Let $2a(\eta)=\lambda_1$. Then for each $\varepsilon>0$*

$$\lim_{s\to\infty} E_r\{|\eta\cdot Z(s)|^2\, e^{-\lambda_1 s}\, s^{-(2\gamma+1)};\ |\eta\cdot Z(s)|>s^{2\gamma+1}\, e^{(\lambda_1+\varepsilon)s}\}=0 \tag{4}$$

where for any $1\leqq r\leqq p$, E_r stands for expectation when $Z(0)=e_r=(\delta_{r1},\delta_{r2},\dots,\delta_{rp})$ where δ_{rj}'s are Kronecker deltas.

We take this up in § 4. In [2] we established the following

Theorem 2. *Let $2a(\eta)>\lambda_1$. Then, there exist random variables Y_j for $j\in I(\eta)$ such that if*

$$X(t)=\gamma!\, t^{-\gamma}\, e^{-at}\, \eta\cdot Z(t)-\sum_{j\in I(\eta)} Y_j\, e^{ib_j t} \tag{5}$$

then

$$E\,|X(t)|^2=O(t^{-2}) \tag{6}$$

where $b_j=\operatorname{Im}(\lambda_j)$.

It is immediate from this theorem and Borel-Cantelli that in the discrete time case

$$X(t)\to 0 \qquad \text{a.s. as } t\to\infty. \tag{7}$$

However, the almost sure convergence is far from obvious in the continuous time case. We can, of course, establish the following

Corollary 1. *For each $\delta>0$ there exists a set $A_\delta\in\mathbb{F}$ where $(\Omega,\mathbb{F},P)$ is our basic probability space such that $P(A_\delta)=1$ and*

$$\omega\in A_\delta \Rightarrow \lim_{n\to\infty} X(n\delta,\omega)=0.$$

In general, a result of the above type does not imply the a.s. convergence of $X(t)$. For a counterexample see Kingman [4]. For a related difficulty see Athreya [1]. Our approach to establish (7) in the continuous time case uses the representation (2) of η in terms of generalized eigenvectors v_{jl}. The assertion (7) is an easy corollary to the following complete breakdown of the asymptotic behavior of $\eta\cdot Z(t)$.

Theorem 3. *Let $\eta=\sum_{j=2}^{r}\sum_{k=1}^{d_j} v_{jk}\, c_{jk}$. We can always write*

$$\begin{aligned}\eta\cdot Z(t)=&\sum_{(2\operatorname{Re}\lambda_j>\lambda_1)}\ \sum_{k=1}^{d_j} c_{jk}\frac{e^{\lambda_j t}\, t^{(k-1)}}{(k-1)!}\, \overline{Y}_{jk}(t)\\ &+\sum_{(2\operatorname{Re}\lambda_j=\lambda_1)}\ \sum_{k=1}^{d_j} c_{jk}\, e^{\lambda_1 t/2}\, t^{(k-\frac{1}{2})}\, \overline{Y}_{jk}(t)\\ &+\sum_{(2\operatorname{Re}\lambda_j<\lambda_1)}\ \sum_{k=1}^{d_j} c_{jk}\, e^{\lambda_1 t/2}\, \overline{Y}_{jk}(t)\end{aligned} \tag{8}$$

where

$$\bar{Y}_{jk}(t)=\begin{cases}(k-1)!\, e^{-\lambda_j t}\, t^{-(k-1)}\, v_{jk}\cdot Z(t), & 2\operatorname{Re}\lambda_j>\lambda\\ v_{jk}\cdot Z(t)\, e^{-\lambda_1 t/2}\, t^{-(k-\frac{1}{2})}, & \text{if } 2\operatorname{Re}\lambda_j=\lambda_1\\ v_{jk}\cdot Z(t)\, e^{-\lambda_1 t/2}, & \text{if } 2\operatorname{Re}\lambda_j<\lambda_1.\end{cases}$$

Further, if $2\operatorname{Re}\lambda_j>\lambda_1$, *then for all* k, $\bar{Y}_{jk}(t)$ *converges in mean square and almost surely to a random variable* Y_j *independent of* k; *if* $2\operatorname{Re}\lambda_j\leqq\lambda_1$, *then* $\bar{Y}_{jk}(t)$ *converges in law to a mixture of normal distributions and* $\sup_t E|\bar{Y}_{jk}(t)|^2<\infty$. *Finally if* $2\alpha>\lambda_1$ *and* $2\operatorname{Re}\lambda_j\leqq\lambda_1$ *then* $e^{-\alpha t}\, v_{jk}\cdot Z(t)\to 0$ *almost surely and in mean square.*

The case $2a(\eta)>\lambda_1$ is discussed in Section 3.

§ 3. Two Basic Lemmas and Their Corollaries

Lemma 1. *Let*

$$w_{jl}(t)=e^{-\lambda_j t}\, t^{-(l-1)}\sum_{k=0}^{l-1}\frac{(-t)^k}{k!}\, v_{j(l-k)} \tag{9}$$

$$W_{jl}(t)=w_{jl}(t)\cdot Z(t).$$

Then for $n\geqq 1$

$$P_r\Big(\sup_{n-1\leqq t\leqq n}|W_{jl}(t)|>M\Big)\leqq K\frac{E_r|W_{jl}(n)|^2}{M^2} \tag{10}$$

where K *is a constant independent of* n *and* M.

Lemma 2.

$$E_r|W_{jl}(t)|^2=\begin{cases}O(t^{-2(l-1)}) & \text{if } 2\operatorname{Re}\lambda_j\geqq\lambda_1,\ l\geqq 2\\ O(t^{-1}) & \text{if } 2\operatorname{Re}\lambda_j=\lambda_1\\ O(e^{(\lambda_1-2\operatorname{Re}\lambda_j)t}) & \text{if } 2\operatorname{Re}\lambda_j<\lambda_1.\end{cases} \tag{11}$$

We now get a few corollaries as easy consequences of the above two lemmas.

Corollary 2. *Let* $2\operatorname{Re}\lambda_j>\lambda_1$ *and* $l\geqq 2$. *Then* $W_{jl}(t)\to 0$ *almost surely and in mean square as* $t\to\infty$.

Proof. Use (10), (11) and Borel-Cantelli. q.e.d.

Corollary 3. *Let* $2\operatorname{Re}\lambda_j>\lambda_1$ *and*

$$Y_{jl}(t)=(l-1)!\, e^{-\lambda_j t}\, t^{-(l-1)}\, v_{jl}\cdot Z(t). \tag{12}$$

Then $\lim_{t\to\infty} Y_{jl}(t)$ *exists in mean square and almost surely and is independent of* l.

Proof. For $l=1$ the result follows from martingale arguments since v_{j1} being an eigenvector with eigenvalue λ_j makes $Y_{j1}(t)$ a martingale and $2\operatorname{Re}\lambda_j>\lambda_1$ makes $\sup_t E|Y_{j1}(t)|^2<\infty$.

By Corollary 2

$$\lim_{t\to\infty} W_{jl}(t)=0 \quad \text{a.s. for } l\geqq 2.$$

Note that

$$W_{jl}(t)=\sum_{k=0}^{l-1}\frac{(-1)^k}{k!(l-1-k)!}\, Y_{j,l-k}(t). \tag{13}$$

The result now follows by an induction on l and an use of the identity

$$\sum_{k=1}^{l-1} \frac{(-1)^k}{k!(l-1-k)!} = \frac{1}{(l-1)!}\left[\sum_{k=0}^{l-1} \frac{(l-1)!(-1)^k 1^{l-k}}{k!(l-k-1)!} - 1\right]$$
$$= \frac{1}{(l-1)!}[(-1+1)^{l-1}-1]$$
$$= -\frac{1}{(l-1)!}. \quad \text{q.e.d.}$$

Corollary 4. *Let* $2\operatorname{Re}\lambda_j \leqq \lambda_1$. *Let* a *and* γ *be constants and*

$$\begin{aligned} \tilde{Y}_{jl}(t) &= e^{-(a-\lambda_j)t}\, t^{-(\gamma-l+1)}\, Y_{jl}(t) \\ \tilde{W}_{jl}(t) &= e^{-(a-\lambda_j)t}\, t^{-(\gamma-l+1)}\, W_{jl}(t). \end{aligned} \tag{14}$$

Then $\tilde{W}_{jl}(t)$ *and* $\tilde{Y}_{jl}(t)$ *both tend to zero in mean square and almost surely provided either* $2a > \lambda_1$ *or* $2a = \lambda_1$ *and* $\gamma \geqq l$.

Proof. From Lemma 2 we get

$$E|\tilde{W}_{jl}(t)|^2 \leqq \text{const } e^{-(2a-\lambda_1)t}\, t^{-2(\gamma-l+1)}$$

and the mean square convergence follows. Also v_{j1} being an eigenvector we know from [2] that

$$E|Y_{j1}(t)|^2 \leqq \text{const } e^{(\lambda_1 - 2\operatorname{Re}\lambda_j)t}$$

and this clearly implies the mean square convergence of $\tilde{Y}_{j1}(t)$. From (13) we get

$$\tilde{W}_{jl}(t) = \sum_{k=0}^{l-1} \frac{(-1)^k}{k!(l-1-k)!}\, \tilde{Y}_{j,l-k}(t). \tag{15}$$

The mean square convergence of $\tilde{Y}_{jl}(t)$ follows by induction.

Next we turn to almost sure convergence. Since $Y_{j1}(t)$ is a martingale in t

$$P\{\sup_{n-1 \leqq t \leqq n} |\tilde{Y}_{j1}(t)| > M\} \leqq P\{\sup_{n-1 \leqq t \leqq n} |Y_{j1}(t)| > M\, e^{(a-\operatorname{Re}\lambda_j)(n-1)}(n-1)^{(\gamma-l+1)}\}$$
$$\leqq \text{const} \frac{E|Y_{j1}(n)|^2}{M^2}\, e^{-2(a-\operatorname{Re}\lambda_j)n}\, n^{-2(\gamma-l+1)}.$$

Clearly, $\sum_{n=1}^{\infty} e^{-\theta n} n^{\delta} < \infty$ for $\theta > 0$, $|\delta| < \infty$ or $\theta = 0$, $\delta > 1$. The almost sure convergence of $\tilde{Y}_{j1}(t)$ follows by Borel-Cantelli. An exactly similar argument yields the almost sure convergence of $\tilde{W}_{jl}(t)$ for all l. The rest of the argument is the same as in Corollary 3 except that one uses (15) in place of (13). q.e.d.

It is easy to verify (7) from the definition of $a(\eta)$, $\gamma(\eta)$, $I(\eta)$ and Corollaries 3 and 4.

Modulo a gap in the case $2\operatorname{Re}\lambda_j = \lambda_1$ it was established in [2] that $\overline{Y}_{jk}(t)$ as defined in Theorem 3 converges in law to a mixture of normal distributions when $2\operatorname{Re}\lambda_j \leqq \lambda_1$. Thus the proof of Theorem 3 is complete.

We shall finish this section with a proof of the basic lemmas.

Proof of Lemma 1. Let $T=\inf_n\{t\colon n-1\leqq t\leqq n, |W_{jl}(t)|>M\}$ and n if the set in braces is empty. Then T is a stopping time for our Markov process $\{Z(t);\, t\geqq 0\}$. Hence if $\mathbb{F}_T$ is the σ-field associated with T we have

$$E(W_{jl}(n)|\,\mathbb{F}_T)=e^{-\lambda_j n}\, n^{-(l-1)}\sum_{k=0}^{l-1}\frac{(-n)^k}{k!}\,M(n-T)\,v_{j,l-k}\cdot Z(T).$$

But on using (1) we get

$$\begin{aligned}\sum_{k=0}^{l-1}\frac{(-n)^k}{k!}\,M(n-T)\,v_{j,l-k}&=\sum_{k=0}^{l-1}\frac{(-n)^k}{k!}\,e^{\lambda_j(n-T)}\sum_{m=1}^{l-k}v_{j,m}\frac{(n-T)^{l-k-m}}{(l-k-m)!}\\&=e^{\lambda_j(n-T)}\sum_{m=1}^{l}v_{j,m}\sum_{k=0}^{l-m}\frac{(n-T)^{l-k-m}}{(l-k-m)!}\,\frac{(-n)^k}{k!}\\&=e^{\lambda_j(n-T)}\sum_{m=0}^{l-1}v_{j,l-m}\frac{(-T)^m}{m!}.\end{aligned}$$

Thus,

$$E(W_{jl}(n)|\,\mathbb{F}_T)=\left(\frac{T}{n}\right)^{l-1}W_{jl}(T). \tag{16}$$

Next

$$\begin{aligned}E\,|W_{jl}(n)|^2&=E\,|W_{jl}(n)-W_{jl}(T)+W_{jl}(T)|^2\\&=E\big(W_{jl}(n)-W_{jl}(T)\big)\,\overline{W}_{jl}(T)+E\big(\overline{W_{jl}(n)-W_{jl}(T)}\big)\,W_{jl}(T)\\&\quad+E\,|W_{jl}(n)-W_{jl}(T)|^2+E\,|W_{jl}(T)|^2.\end{aligned}$$

In view of (16)

$$\begin{aligned}E\big(W_{jl}(n)-W_{jl}(T)\big)\,\overline{W}_{jl}(T)&=E\big(\overline{W_{jl}(n)-W_{jl}(T)}\big)\,W_{jl}(T)\\&=E\left(\left(\frac{T}{n}\right)^{l-1}-1\right)|W_{jl}(T)|^2.\end{aligned}$$

Hence

$$E\,|W_{jl}(n)|^2\geqq E\,|W_{jl}(T)|^2\left[2\left\{\left(\frac{T}{n}\right)^{l-1}-1\right\}+1\right].$$

Since $n-1\leqq T\leqq n$,

$$\begin{aligned}2\left\{\left(\frac{T}{n}\right)^{l-1}-1\right\}+1&\geqq 2\left(\frac{n-1}{n}\right)^{l-1}-1\\&\geqq 2\left(1-\frac{1}{n}\right)^{p-1}-1\qquad (l\leqq p)\\&\geqq \tfrac{1}{2}\qquad\text{for } n \text{ large enough}.\end{aligned}$$

For n large enough

$$\begin{aligned}P\{\sup_{n-1\leqq t\leqq n}|W_{jl}(t)>M\}&\leqq P\{W_{jl}(T)|>M\}\\&\leqq\frac{E\,|W_{jl}(T)|^2}{M^2}\\&\leqq K\frac{E\,|W_{jl}(n)|^2}{M^2}\end{aligned}$$

where K is any constant larger than 2. q.e.d.

Proof of Lemma 2. We get from [1] that

$$E_r|W_{jl}(t)|^2 = \big(M(t)\overline{w_{jl}(t)}\big)^* D_r(0)\big(M(t)\,w_{jl}(t)\big) + \int_0^\tau \big(M(t-\tau)\overline{w_{jl}(t)}\big)^* \Big(\sum_{s=1}^p m_{rs}(\tau)\,B_s\Big)\big(M(t-\tau)\,w_{jl}(t)\big)\,d\tau. \tag{17}$$

The calculation made in Lemma 1 tells us that

$$M(t-x)\,w_{jl}(t) = \left(\frac{x}{t}\right)^{l-1} w_{jl}(x) \quad \text{for } 0 \leqq x \leqq t.$$

Thus the first term on the right vanishes. The second term is of the form

$$t^{-2(l-1)} \int_0^t f(\tau)\,d\tau.$$

Now use Frobenius theory, the definition of $w_{jl}(\tau)$, and the relation between $2\operatorname{Re}\lambda_j$ and λ_1 to estimate the growth rate of

$$\int_0^t |f(\tau)|\,d\tau. \quad \text{q.e.d.}$$

§ 4. The Gap in the Case $2a(\eta) = \lambda_1$

We now tackle the gap mentioned in § 2 namely to establish Theorem 1. From the definition of $a(\eta)$ and $\gamma(\eta)$ it must be evident that

$$\big(M(s)\,\eta\big)\,e^{-as}\,s^{-(\gamma+\frac{1}{2})} \to 0 \quad \text{as } s \to \infty.$$

Thus (4) is equivalent to

$$\lim_{s\to\infty} E_r\{|Y_r(s,\eta)|^2;\ |Y_r(s,\eta)| > c\,e^{\delta s}\} = 0 \tag{18}$$

for $c > 0$ and $\delta > 0$, where

$$\begin{aligned} Y_r(s,\eta) &= \big[\eta\cdot Z(s) - E_r\big(\eta\cdot Z(s)\big)\big]\,e^{-as}\,s^{-(\gamma+\frac{1}{2})} \\ &= \big[\eta\cdot Z(s) - \big(M(s)\,\eta\big)_r\big]\,e^{-as}\,s^{-(\gamma+\frac{1}{2})}. \end{aligned} \tag{19}$$

Using the fundamental property of branching processes, namely additivity, we get the identity (dropping the sub index r)

$$Y(s+1,\eta) = (s+1)^{-(\gamma+\frac{1}{2})}\,R(s,\eta) + \left(\frac{s}{s+1}\right)^{\gamma+\frac{1}{2}} e^{-a}\,Y(s, M\eta) \tag{20}$$

where

$$R(s,\eta) = \sum_{i=1}^p \frac{1}{e^{as}} \sum_{j=1}^{Z_i(s)} \frac{\big(\eta\cdot Z^{(ij)}(1) - (M\eta)_i\big)}{e^a}, \qquad M = M(1),$$

and $Z^{(ij)}(1)$ (for $j = 1, 2, \ldots, Z_i(s)$) is the vector denoting the offspring population in one unit of time of the j-th particle among the $Z_i(s)$ particles of type i at time s.

Now set

$$\begin{aligned} \bar{F}(s,c,\eta) &= E\{|Y(s,\eta)|^2;\ |Y(s,\eta)| > c\,e^{\delta s}\} \\ F(s,c,\eta) &= s^{2\gamma+1}\,F(s,c,\eta). \end{aligned} \tag{21}$$

Clearly,

$$\begin{aligned}\bar{F}(s+1,c,\eta) &\leqq E\{|Y(s+1,\eta)|^2;\ |Y(s,M\eta)|>c e^a e^{\delta s}\}\\ &\quad + E\{|Y(s+1,\eta)|^2;\ |Y(s+1,\eta)|>c e^{\delta(s+1)},\ |Y(s,M\eta)|<c e^a e^{\delta s}\}\\ &= \mathrm{I}+\mathrm{II},\ \text{say}.\end{aligned}$$

By the orthogonality of $R(s,\eta)$ and the σ-field $\mathbb{F}_s$ we have

$$\begin{aligned}\mathrm{I} &= (s+1)^{-(2\gamma+1)} E\{|R(s,\eta)|^2;\ |Y(s,M\eta)|>c e^a e^{\delta s}\}\\ &\quad + \left(\frac{s}{s+1}\right)^{2\gamma+1} \frac{1}{e^{2a}} E\{|Y(s,M\eta)|^2;\ |Y(s,M\eta)|>c e^a e^{\delta s}\}\end{aligned}$$

and

$$\begin{aligned}\mathrm{II} \leqq 2E\Big\{&(s+1)^{-(2\gamma+1)} |R(s,\eta)|^2 + \left(\frac{s}{s+1}\right)^{2\gamma+1} \frac{1}{e^{2a}} |Y(s,\eta)|^2;\\ &|Y(s+1,\eta)|>c e^{\delta(s+1)},\ |Y(s,M\eta)|<c e^a e^{\delta s}\Big\}\end{aligned}$$

using the trivial inequality $(a+b)^2 \leqq 2(a^2+b^2)$.

This leads to the recurrence relation

$$F(s+1,c,\eta) \leqq \rho F(s,c e^a, M\eta) + G(s,\eta,c) \tag{22}$$

where $\rho = e^{2a}$,

$$\begin{aligned}G(s,\eta,c) &= G_1(s,\eta,c) + G_2(s,\eta,c)\\ G_1(s,\eta,c) &= 2E\{|R(s,\eta)|^2; A(s,\eta,c)\}, \qquad A = A_1 \cup A_2,\\ A_1(s,\eta,c) &= \{|Y(s,M\eta)|>c e^a e^{\delta s}\}\\ A_2(s,\eta,c) &= \{|Y(s+1,\eta)|>c e^{\delta(s+1)},\ |Y(s,M\eta)| \leqq c e^a e^{\delta s}\}\\ G_2(s,\eta,c) &= 2E\{|Y(s,M\eta)|^2; A_2(s,\eta,c)\}.\end{aligned}$$

Iterating (22) yields

$$\begin{aligned}\frac{F(s+1,c,\eta)}{\rho^{s+1}} &\leqq \frac{F(s,c e^a, M\eta)}{\rho^s} + \frac{G(s,\eta,c)}{\rho^s}\\ &\leqq \sum_{r=0}^{s} \frac{G(s-r, M^r\eta, c e^{ar})}{\rho^{s-r}}\end{aligned}$$

which is the same as

$$\bar{F}(s+1,c,\eta) \leqq \frac{\rho}{(s+1)^{2\gamma+1}} \sum_{r=0}^{s} \rho^r G(s-r, c e^{ar}, M^r\eta). \tag{23}$$

Let us first look at

$$\sum_{r=0}^{s} \rho^r G_1(s-r, c e^{ar}, M^r\eta).$$

By definition

$$\rho^r G_1(s-r, c e^{ar}, M^r\eta) = 2E\left\{\frac{|R(s-r, M^r\eta)|^2}{e^{2ar}};\ A_{r,s}\right\}$$

where $A_{r,s} = A(s-r, M^r\eta, c e^{ar})$. We now make use of the definition of $a(\eta)$, $I(\eta)$ and $\gamma(\eta)$. If

$$l(r,\eta) = e^{-ar} r^\gamma \left[M^r\eta - \frac{r^\gamma}{\gamma!} \sum_{j\in I(\eta)} \xi_j e^{\lambda_j r}\right]$$

then

$$l(r,\eta)=O\left(\frac{1}{r}\right), \quad \text{where } \xi_j=c_{j1}v_{j1}. \tag{24}$$

Next, by the orthogonality of $R(s,\eta)$ and the σ-field $\mathbb{F}_s$

$$E\left|\frac{R(s-r,M^r\eta)}{e^{ar}r^\gamma}-\sum_{j\in I(\eta)} e^{ib_jr}R(s-r,\xi_j)\right|^2=E\{\sum_i Z_i(s-r)\,e^{-\lambda_1(s-r)}\,\overline{l(r,\eta)}^*\,\pi_i\,l(r,\eta)\}$$

where π_i is the covariance matrix of the vector $Z(1)$ with $Z(0)=e_i$ (i.e., we start with one particle of type i) and * denotes transpose,

$$\leqq\frac{\text{const}}{r^2} \quad \text{by (24)}.$$

For any set A

$$E\left\{\left|\frac{R(s-r,M^r\eta)}{e^{ar}r^\gamma}\right|^2;A\right\}\leqq 2E\{|\sum_{j\in I(\eta)} R(s-r,\xi_j)\,e^{ib_jr}|^2;A\}+\frac{\text{const}}{r^2}.$$

Thus

$$\begin{aligned}&\sum_{r=1}^{s}\rho^r G_1(s-r,ce^{ar},M^r\eta)\\ &\quad\leqq\text{const}\left(\sum_{r=1}^{s} r^{2\gamma}E\{|\sum_{j\in I(\eta)} R(s-r,\xi_j)\,e^{ib_jr}|^2;A_{r,s}\}\right)+\text{const}\sum_{r=1}^{s} r^{2(\gamma-1)}.\end{aligned} \tag{25}$$

Clearly,

$$\frac{1}{(s+1)^{2\gamma+1}}\sum_{r=1}^{s} r^{2(\gamma-1)}\to 0 \quad \text{as } s\to\infty.$$

The first term on the right side of (25) is majorized using Minkowski's inequality by

$$\text{const}\sum_{j\in I(\eta)}\sum_{r=1}^{s} r^{2\gamma}E(|R(s-r,\xi_j)|^2;A_{r,s}).$$

We shall now show that for each $j\in I(\eta)$

$$\frac{1}{(s+1)^{2\gamma+1}}\sum_{r=1}^{s} r^{2\gamma}E(|R(s-r,\xi_j)|^2;A_{r,s})\to 0 \quad \text{as } s\to\infty. \tag{26}$$

This is, of course, implied by

$$\frac{1}{s}\sum_{r=1}^{s} E\{|R(s-r,\xi_j)|^2;A_{r,s}\}\to 0 \quad \text{as } s\to\infty. \tag{27}$$

To establish (27) break up r into two regions; $r\leqq s(1-\varepsilon)$ and $r>s(1-\varepsilon)$. Thus

$$\begin{aligned}\frac{1}{s}\sum_{r=1}^{s} E\{|R(s-r,\xi_j)|^2;A_{r,s}\}&\leqq\frac{1}{s}\Big[\sum_{(1-\varepsilon)s<r\leqq s}+\sum_{r\leqq s(1-\varepsilon)}\Big]\\ &\leqq\text{const}\,\varepsilon+\frac{1}{s}\sum_{r\leqq s(1-\varepsilon)}\end{aligned}$$

since $\sup_s E|R(s, \xi_j)|^2 < \infty$ (use the fact $2a = \lambda_1$ and $\sup_s EZ_i(s)\, e^{-\lambda_1 s} < \infty$). Next,

$$\frac{1}{s} \sum_{r \leqq s(1-\varepsilon)} = \frac{1}{s} \sum_{k \geqq s\varepsilon} E\left(|R(k, \xi_j)|^2; A_{s-k,s}\right).$$

Since we can write

$$R(s, \xi_j) = \sum_{i=1}^{p} \frac{\sqrt{Z_i(s)}}{e^{as}} \frac{1}{\sqrt{Z_i(s)}} \sum_{j=1}^{Z_i(s)} \frac{\left(\xi \cdot Z^{(ij)}(1) - \xi_i e^{\lambda}\right)}{e^a}.$$

Now, by Renyi's generalization of the classical central limit theorem [5], we see that for each j, $R(s, \eta)$ converges in law to a random variable R (which is a mixture of normal distributions). Further $E|R(s, \xi_j)|^2 \to ER^2$. Thus the sequence $|R(k, \xi_j)|^2$ for $k = 1, 2, \ldots$ is uniformly integrable. Thus, if we show

$$\sup_{k \geqq s\varepsilon} P(A_{s-k,s}) \to 0 \quad \text{as } s \to \infty. \tag{28}$$

(27) will follow since ε is arbitrary. Since $A_{s-k,s} = A(k, M^{s-k}\eta, c\, e^{as-k})$ and $A(s, \eta, c) = A_1 \cup A_2$,

$$\begin{aligned} P(A_{s-k,s}) &\leqq P\{|Y(k, M^{s-k+1}\eta)| > c\, e^{a(s-k)}\, e^{\delta k}\} + P\{A_2(k, M^{s-k}\eta, c\, e^{a(s-k)}\} \\ &= \mathrm{I}' + \mathrm{II}'. \end{aligned} \tag{29}$$

By Chebychev's inequality

$$\begin{aligned} \mathrm{I}' &\leqq \frac{\text{const}}{e^{2\delta k}} E\left\{\left|Y\left(k, \frac{M^{s-k+1}\eta}{e^{a(s-k)}}\right|^2\right\} \\ &\leqq \frac{\text{const}}{e^{2\delta s\varepsilon}} \frac{1}{s} \left(\frac{1}{\varepsilon}\right)^{2\gamma+1} + \frac{(2s-k)^{2\gamma+1} - (s-k)^{2\gamma+1}}{s^{2\gamma+1}} \end{aligned}$$

(the last step can be established using (17) in much the same way as Lemma 2)

$$\to 0 \quad \text{as } s \to \infty.$$

Turning to II′ we notice first that

$$A_2(k, M^{s-k}\eta, c\, e^{a(s-k)}) \subset \{|R(k, M^{s-k}\eta)| > c\, e^{a(s-k)}\, e^{\delta k}[e^{\delta}(k+1)^{\gamma+\frac{1}{2}} - k^{\gamma+\frac{1}{2}}]\}.$$

Thus

$$\begin{aligned} \mathrm{II} &\leqq \text{const} \cdot E\{|R(k, M^{s-k}\eta)|^2\}\, \{e^{2a(s-k)}\, e^{2\delta k}[e^{\delta}(k+1)^{\gamma+\frac{1}{2}} - k^{\gamma+\frac{1}{2}}]\}^{-2} \\ &\leqq \text{const} \cdot \frac{(s-k)^{2\gamma}}{e^{2\delta k}\, k^{2\gamma+1}\left[e^{\delta}\left(1+\frac{1}{k}\right)^{\gamma+\frac{1}{2}} - 1\right]^{\frac{1}{2}}} \end{aligned}$$

again using the fact

$$\begin{aligned} &\sup_{s(1-\varepsilon) \leqq k \leqq s} \left(E \frac{|R(k, M^{s-k}\eta|^2}{e^{2a(s-k)}(s-k)^{2\gamma}}\right) < \infty \\ &\leqq \text{const}\left(\frac{1}{1-\varepsilon} - 1\right)^{2\gamma} \frac{1}{e^{2\delta s(1-\varepsilon)}} \frac{1}{(s(1-\varepsilon))^2} \quad (\text{for } s(1-\varepsilon) \leqq k \leqq s) \\ &\to 0 \quad \text{as } s \to \infty. \end{aligned}$$

This $\Rightarrow(29)\Rightarrow(28)\Rightarrow(26)$. From (26) and (25) we conclude that

$$\frac{1}{(s+1)^{2\gamma+1}}\sum_{r=0}^{s}\rho^r G_1(s-r, c e^{ar}, M^r\eta)\to 0 \quad \text{as } s\to\infty. \tag{30}$$

To finish the proof we need to show

$$\frac{1}{(s+1)^{2\gamma+1}}\sum_{r=0}^{s}\rho^r G_2(s-r, c e^{ar}, M^r\eta)\to 0 \quad \text{as } s\to\infty. \tag{31}$$

Recall that

$$\begin{aligned} G_2(s,\eta,c) &= 2E\{|Y(s,M\eta)|^2; |Y(s+1,\eta)|>c e^{\delta(s+1)}, |Y(s,M\eta)|\leqq c e^a e^{\delta s}\}\\ &= 2E\{|Y(s,M\eta)|^2\,\chi_{A_2(s,\eta,c)}\},\end{aligned}$$

where χ_A stands for the indicator function of the set A

$$\leqq 2E\{\tilde{Y}(s,M\eta)|^2\,\chi_{\bar{A}_2(s,\eta,c)}\}$$

where $\bar{A}_2(s,\eta,c)=\{|R(s,\eta)|>c e^{\delta s}[e^{\delta}(s+1)^{\gamma+\frac{1}{2}}-s^{\gamma+\frac{1}{2}}]\}$, $\tilde{Y}=Y$ on $\{|Y(s,M\eta)|\leqq c e^a e^{\delta s}\}$ and 0 otherwise

$$\leqq 2E\{|\tilde{Y}(s,M\eta)|^2\,E(\chi_{\bar{A}_2(s,\eta,c)}|\,\mathbb{F}_s)$$

(by conditioning on the σ-field $\mathbb{F}_s$)

$$\leqq \text{const } E\{|\tilde{Y}(s,M\eta)|^2\,E(|R(s,\eta)|^2|\mathbb{F}_s)\,(f(s,c))^{-2}\}$$

where $f(s,c)=c e^{\delta s}[e^{\delta}(s+1)^{\gamma+\frac{1}{2}}-s^{\gamma+\frac{1}{2}}]$,

$$\leqq \text{const}(f(s))^{-2}\,E\left\{|\tilde{Y}(s,M\eta)|^2\sum_{i=1}^{p}Z_i(s)\,e^{-\lambda_1 s}\,\bar{\eta}^*\,\Pi_i\,\eta\right\}.$$

Thus,

$$\begin{aligned}
&\rho^r G_2(s-r, c e^{ar}, M^r\eta)\\
&\quad\leqq \text{const } E\left\{|\tilde{Y}(s-r, M^{r+1}\eta)|^2\sum_{i=1}^{p}Z_i(s-r)\,e^{-\lambda_1(s-r)}\left(\frac{M^r\bar{\eta}}{e^{ar}r^{\gamma}}\right)^*\Pi_i\,\frac{M^r\eta}{e^{ar}r^{\gamma}}\right\}\\
&\qquad\cdot r^{2\gamma}(f(s-r, c e^{ar}))^{-2}\\
&\quad\leqq \text{const } E\left\{\left|\frac{\tilde{Y}(s-r, M^{r+1}\eta)}{c e^{ar}e^{\delta(s-r)}}\right|^2\sum_{i=1}^{p}Z_i(s-r)\,e^{-\lambda_1(s-r)}\right\}\times(c e^{ar}e^{\delta(s-r)})^2\,r^{2\gamma}\\
&\qquad\times(c e^{ar}e^{\delta(s-r)})^2\times[e^{\delta}(s-r+1)^{\gamma+\frac{1}{2}}-(s-r)^{\gamma+\frac{1}{2}}]^2\\
&\quad\leqq \text{const}\left(\frac{r}{s-r}\right)^{2\gamma} \quad \text{if } (s-r)\geqq 1\\
&\quad= r^{2\gamma} \quad \text{if } r=s.
\end{aligned}$$

Again break up r into three regions $r=s$, $s(1-\varepsilon)\leqq r\leqq s-1$, and $r\leqq s(1-\varepsilon)$.

$$\frac{1}{(s+1)^{2\gamma+1}}\sum_{r=0}^{s}\rho^r G_2(s-r, c e^{ar}, M^r\eta)$$

$$\leqq \frac{1}{(s+1)^{2\gamma+1}}\Big[\sum_{r<s(1-\varepsilon)}+\sum_{s(1-\varepsilon)\leqq r\leqq s-1}+\sum_{r=s}\Big]$$

$$\leqq \text{const}\left[\left(\frac{1}{\varepsilon}-1\right)^{2\gamma}\frac{s(1-\varepsilon)}{s^{2\gamma+1}}+\frac{1}{s}\sum_{s(1-\varepsilon)\leqq r\leqq s-1}\frac{1}{s^{2\gamma}}\left(\frac{r}{s-r}\right)^{2\gamma}+\frac{1}{s}\right]$$

$$\to 0 \quad \text{as } s\to\infty. \quad \text{q.e.d.}$$

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