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On some Limit Theorems Related to the Phase Separation Line in the Two-dimensional Ising Model

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Dedicated to Professor Leopold Schmetterer on the occasion of his 60th birthday

Summary. We prove that at low enough temperatures the phase separation line, when it is suitably normalized, converges almost surely in a suitable probability space to the path of a one-dimensional Brownian bridge. The convergence is in the sense of the distance between compact sets in $[0, 1] \times \mathbf{R}^1$.

§ 1. Introduction

Let $\mathbf{Z}^2$ be the square lattice and $\mathbf{L}$ its dual lattice, i.e. $\mathbf{Z}^2 = \{(x_1, x_2);\ x_1$ and x_2 are integers$\}$, and $\mathbf{L} = \{x + (1/2, 1/2);\ x \in \mathbf{Z}^2\}$. For each integer $N \geqq 1$ let

$$V_N = \{(x_1, x_2) + (1/2, 1/2);\ 0 \leqq x_1 \leqq N-1,\\ -[N/2] \leqq x_2 \leqq [(N-1)/2]\},$$

where $[\mathbf{u}]$ is the largest integer smaller than $\mathbf{u}$. We consider the following interacting system: the energy of the system on V_N is given by

$$(1.1)\quad U_N(\eta) = -\sum_{\substack{\langle x,y\rangle \\ x, y \in V_N}} \eta(x)\eta(y) - \sum_{\substack{\langle x,y\rangle \\ x \in V_N,\, y \in \partial V_N}} \eta(x)\omega(y)$$

for every $\eta \in \Omega_N = \{-1, +1\}^{V_N}$, where $\omega: \mathbf{L} \to \{-1, +1\}$ is defined by

$$\omega((x_1, x_2) + (1/2, 1/2)) = \begin{cases} +1 & \text{if } x_2 \geqq 0, \\ -1 & \text{if } x_2 < 0, \end{cases}$$

and where $\sum_{\langle x,y\rangle}$ is taken only over nearest neighbour pairs. For a given β, the Gibbs state on Ω_N for the interaction energy (1.1) is defined by

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$$P_N(\eta) = \exp\{-\beta U_N(\eta)\} / \sum_{\zeta \in \Omega_N} \exp\{-\beta U_N(\zeta)\} \qquad \eta \in \Omega_N$$

β is called the inverse temperature. Let

$$\tilde{V}_N = \{(x_1, x_2) \in \mathbf{Z}^2;\ 0 \leqq x_1 \leqq N,\ -[N/2] \leqq x_2 \leqq [(N+1)/2]\}.$$

Let $\eta \in \Omega_N$ be arbitrary; then for each bond in V_{N+1} connecting two points, say x and y, with $\eta(x)\eta(y) = -1$ or $\eta(x)\omega(y) = -1$ (if $y \in \partial V_{N+1}$), let us colour red the bond in $\tilde{V}_N$ which is perpendicular to $\langle x, y\rangle$, where $\langle x, y\rangle$ is used to denote the bond connecting x and y. After doing this we obtain a red subgraph of V_N, which can be partitioned into connected components. Among the components we are interested in the particular one which contains $(0,0)$ and $(N,0)$. Let us denote this component by $\lambda_N(\eta)$; $\lambda_N(\eta)$ is called the phase separation line or the interface. We are interested in the asymptotic behavior of λ_N as $N \to \infty$. Concerning this, G. Gallavotti has made a very essential contribution in [4]. He proved that the distribution of $\max\{k/\sqrt{N}; (N/2, k) \in \lambda_N\}$ converges to a Gaussian distribution. It is a consequence of [4] that we obtain the same limiting distribution when we take $\min\{k/\sqrt{N};\ (N/2, k) \in \lambda_N\}$ instead of the above maximum. Since λ_N is fixed at $(0,0)$ and at $(N,0)$, the fluctuation of λ_N will attain its maximum near $x_1 = N/2$; i.e. the above result seems to imply that the curve $\{(x_1/N, x_2/N);\ (x_1, x_2) \in \lambda_N\}$ will in some sense converge to the segment connecting $(0,0)$ and $(1,0)$. This is why we say that λ_N is a "line". Further, the above fact tells us the order of the fluctuation of λ_N. Let $\mathscr{A}_N$ be a mapping from $\mathbf{R}^2$ to $\mathbf{R}^2$ such that $\mathscr{A}_N(x_1, x_2) = (x_1/N,\ x_2/\sigma\sqrt{N})$, where $\sigma > 0$ will be defined later. Then Gallavotti's result seems to imply that the normalized curve $\mathscr{A}_N(\lambda_N(\cdot))$ will converge to some random curve in $\mathbf{R}^2$, more precisely to the path of a one-dimensional Brownian bridge. Two young mathematicians are working on this conjecture and have got some interesting results. C. Cammarota announced to the author that he had almost proved the convergence of the finite dimensional distribution of $\mathscr{A}_N(\lambda_N(\cdot))$ to the corresponding distribution of a one-dimensional Brownian bridge. Also R. Durrett recently sent the preprint of [3] to the author, in which he proved the weak convergence of measures, not for the interface itself but for a more probabilistic model.

Now we are going to state our result. For real t, let q_t be the vertical line passing through $(t, 0)$. Let $\mathscr{C} = \{c;$ connected compact set in $[0,1] \times \mathbf{R}^1$, $c \cap q_1 \neq \emptyset$, $c \cap q_0 \neq \emptyset\}$, and for any $c, c' \in \mathscr{C}$, let

$$\bar{\rho}(c, c') = \tfrac{1}{2}\big[\sup_{x \in c} \inf_{y \in c'} |x - y| + \sup_{y \in c'} \inf_{x \in c} |x - y|\big].$$

Then it is easy to verify that $(\mathscr{C}, \bar{\rho})$ is a complete separable metric space.

Theorem 1.2. *Assume that $\beta > 0$ is sufficiently large. Then there is a probability space $(\hat{\Omega}, \hat{\mathscr{B}}, \hat{P})$ and $\mathscr{C}$-valued random variables $\hat{\lambda}_1, \hat{\lambda}_2, \ldots, B^0$ such that* (i) *the law of $\hat{\lambda}_N$ is the same as the law of $(\mathscr{A}_N(\lambda_N), P_N)$,* (ii) *$B^0$ is a one-dimensional Brownian bridge, and*

$$\hat{P}(\bar{\rho}(\lambda_N, B^0) \to 0 \ \text{as}\ N \to \infty) = 1.$$

The statement of Theorem 1.2. is equivalent to the weak convergence of $P_N \circ \mathscr{A}_N^{-1}$ on $\mathscr{C}$, by the well-known theorem of Skorokhod (for example, see [7], p. 10, 11).

In § 2, we will exhibit some results and estimates due to Gallavotti [4, 5] and Del Grosso [2], which we will use later. In § 3, we give the proof of the convergence of the finite dimensional distribution of $\mathscr{A}_N(\lambda_N)$ to the corresponding distribution of a one-dimensional Brownian bridge. In Sect. 4, 5, 6, we prove the tightness of $\{P_N\}_{N=1}^{\infty}$, which together with the result of § 3, proves Theorem 1.2.

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§ 2. Some Auxiliary Results and Estimates

In this section, we introduce some expression of P_N and $\lambda_N(\eta)$, and some basic results which appeared in [2, 4]. Also we give a slite modification of estimates given in [4].

Let $I_N \equiv \{(x_1, x_2) \in \mathbf{Z}^2;\ 0 \leqq x_1 \leqq N\}$, and $\hat{\Lambda}_N \equiv \{$connected subgraph λ of I_N such that (i) the length $|\lambda|$ is finite, (ii) $\lambda \ni A = (0, 0)$, and (iii) there exists a point B' in $\{(N, j);\ j \in \mathbf{Z}^1\} \subset I_N$ such that λ can be regarded as a curve from A to $B'\}$. If we define Λ_N by $\Lambda_N \equiv \{\lambda_N(\eta);\ \eta \in \Omega_N\}$, then it is easy to see that $\Lambda_N \subset \hat{\Lambda}_N$. In the condition (iii), we said that λ can be considered to be a curve from A to B'. To be more precise, we must say that λ is unicursal. Then $\Lambda_N = \{\lambda \in \hat{\Lambda}_N;\ \lambda \subset \tilde{V}_N\}$.

Now we introduce very important concepts "shapes" and "clusters" which reduce $\hat{\Lambda}_N$ to a one-dimensional particle system. Let $\lambda \in \hat{\Lambda}_N$ be given arbitrarily. We define two sequences of integers $\{\tau_\nu^N\}_{\nu=1}^{\tilde{k}(\lambda)}$, $\{\bar{\tau}_\nu^N\}_{\nu=1}^{\tilde{k}(\lambda)}$ depending on λ by

$$\tau_1^N \equiv \begin{cases} \min\{0 \leqq j \leqq N,\ j \in \mathbf{Z}^1;\ \#\{\lambda \cap q_j\} \geqq 2\}, \\ \infty \quad \text{if the above set is } \emptyset, \end{cases}$$

$$\bar{\tau}_1^N \equiv \begin{cases} \min\{\tau_1^N \leqq j \in \mathbf{Z}^1;\ \#\{\lambda \cap q_{j+\frac{1}{2}}\} = 1\}, \\ N \quad \text{if } \tau_1^N < \infty \text{ and the above set is } \emptyset, \\ \infty \quad \text{if } \tau_1^N = \infty, \end{cases}$$

$$\vdots$$

$$\tau_\nu^N \equiv \begin{cases} \min\{\bar{\tau}_{\nu-1}^N < j \in \mathbf{Z}^1;\ \#\{\lambda \cap q_j\} \geqq 2\}, \\ \infty \quad \text{if the above set is } \emptyset, \end{cases}$$

$$\bar{\tau}_\nu^N \equiv \begin{cases} \min\{\tau_\nu^N \leqq j \in \mathbf{Z}^1;\ \#\{\lambda \cap q_{j+\frac{1}{2}}\} = 1\}, \\ N \quad \text{if } \tau_\nu^N < \infty \text{ and the above set is } \emptyset, \\ \infty \quad \text{if } \tau_\nu^N = \infty, \end{cases}$$

and $\tilde{k}(\lambda) \equiv \max\{\nu;\ \tau_\nu^N < \infty,\ \bar{\tau}_\nu^N < \infty\}$.

For each $1 \leqq \nu \leqq \tilde{k}(\lambda)$, we call $\xi_\nu \equiv \{j \in \mathbf{Z}^1;\ \tau_\nu^N \leqq j \leqq \bar{\tau}_\nu^N\}$ to be a cluster (of λ). "The shape $\mathscr{S}_{\xi_\nu}$ over ξ_ν" is the set $\lambda \cap \{\bigcup_{\alpha \in R';\, \tau_\nu^N \leqq \alpha \leqq \bar{\tau}_\nu^N} q_\alpha\}$. For a fixed cluster ξ, two shapes $\mathscr{S}_\xi$, $\mathscr{S}'_\xi$ are identified if one of them is merely a translation of the other.

Since we are given λ, we can define the "entrance" and the "exit" of each $\mathscr{S}_\xi$ uniquely (as we consider that $A=(0,0)$ is the starting point and B' is the end point). Conversely, if we are given a collection of non-intersecting clusters $(\beta_1, \xi_2, \ldots, \xi_s)$ in $\{0, 1, \ldots, N\}$ and shapes $(\mathscr{S}_{\xi_1}, \mathscr{S}_{\xi_2}, \ldots, \mathscr{S}_{\xi_s})$ over the clusters with their entrances and exits, there uniquely corresponds a curve $\lambda \in \hat{\Lambda}_N$. Hence we can represent each $\lambda \in \hat{\Lambda}_N$ by $(\xi_1, \xi_2, \ldots, \xi_{\tilde{k}(\lambda)};\ \mathscr{S}_{\xi_1}, \mathscr{S}_{\xi_2}, \ldots, \mathscr{S}_{\xi_{k(\lambda)}})$ with their entrances and exits. For each $(\xi_v, \mathscr{S}_{\xi_v})$, let (i) $\delta\mathscr{S}_{\xi_v} \equiv$(the height of the exit)$-$(the height of the entrance), (ii) $|\xi_v| \equiv \bar{\tau}_v^N - \tau_v^N$, (iii) $|\mathscr{S}_{\xi_v}| \equiv$(the total length of $\mathscr{S}_{\xi_v}$)$-|\xi_v|$. We call (i) the jump $\delta\mathscr{S}_{\xi_v}$, (ii) the cluster length $|\xi_v|$, (iii) the excess length $|\mathscr{S}_{\xi_v}|$ of $(\xi_v, \mathscr{S}_{\xi_v})$.

The first result by Gallavotti about the representation of P_N is the following theorem.

Theorem 2.1 (Gallavotti [4]). *Let $\beta > \beta_0 > 0$ be sufficiently large. Then there exists a function U_N: $\hat{\Lambda}_N \to \mathbf{R}^1$ such that*

(i) *for every bounded function f: $\hat{\Lambda}_N \to \mathbf{R}^1$,*

$$(2.2) \quad \sum_{\lambda \in \Lambda_N} f(\lambda) P_N(\lambda) = \sum_{\lambda \in \tilde{\Lambda}_N} f(\lambda) \hat{P}_N(\lambda) / \sum_{\lambda \in \tilde{\Lambda}_N} \hat{P}_N(\lambda) + o(1/N^\ell)$$

as $N \to \infty$ for any integer $\ell \geqq 1$, where $\tilde{\Lambda}_N \equiv \{\lambda \in \hat{\Lambda}_N;\ B' = B\}$,

$$(2.3) \quad \hat{P}_N(\lambda) = \exp\{-2\beta|\mathscr{S}_\mathbf{X}| - U_N(\mathscr{S}_\mathbf{X})\} / \sum_{\mathscr{S}_\mathbf{Y} \in \hat{\Lambda}_N} \exp\{-2\beta|\mathscr{S}_\mathbf{Y}| - U_N(\mathscr{S}_\mathbf{Y})\},$$

where $\mathscr{S}_\mathbf{X}$ is the "shape" representation of $\lambda \in \hat{\Lambda}_N$, i.e. $\mathbf{X} = (\xi_1, \xi_2, \ldots, \xi_{\tilde{k}(\lambda)})$, and $\mathscr{S}_\mathbf{X} = (\mathscr{S}_{\xi_1}, \mathscr{S}_{\xi_2}, \ldots, \mathscr{S}_{\xi_{\tilde{k}(\lambda)}})$,

(ii) *there exists a constant $k(\beta, \beta_0) > 0$ which converges exponentially to zero as $\beta - \beta_0 \to \infty$ or $\beta_0 \to \infty$, such that*

$$(2.4) \quad |U_N(\mathscr{S}_\mathbf{X})| \leqq k(\beta, \beta_0) |\mathscr{S}_\mathbf{X}|.$$

Naturally we define that $|\mathscr{S}_\mathbf{X}| = \sum_{i=1}^{s} |\mathscr{S}_{\xi_i}|$ if $\tilde{k}(\lambda) = s$.

Since each $\mathscr{S}_\mathbf{X} \in \hat{\Lambda}_N$ can be regarded as a configuration of "shape" particles $\mathscr{S}_{\xi_1}, \mathscr{S}_{\xi_2}, \ldots, \mathscr{S}_{\xi_s}$ in $[0, N]$, we can extend our notation to the space of all configurations of finite number of "shape" particles in $\mathbf{Z}^1$ (i.e. in the horizontal line).

The next theorem is obtained by following the line of the proof of Theorem 2.1, but it is very labourious to carry out the whole calculation (see [4]). Here, we merely give the result.

Theorem 2.5. *There exists a function U on the space of configurations of finite number of shape particles in $\mathbf{Z}^1$ which is translationally invariant and*

$$(2.6) \quad |U_N(\mathscr{S}_\mathbf{X}) - U(\mathscr{S}_\mathbf{X})| \leqq C_1(|\mathscr{S}_\mathbf{X}| + C_2) \times 100^{-2(\log N)^2} \quad \text{if } \mathbf{X} \subset [(\log N)^2,\ N - (\log N)^2],$$

and U satisfies the estimate (2.4) for the same $k(\beta, \beta_0)$. The constants C_1 and $C_2 \downarrow 0$ exponentially as $\beta - \beta_0 \to \infty$ or $\beta_0 \to \infty$.

Let $\tilde{P}_N$ on $\tilde{\Lambda}_N$ be

(2.7) $\tilde{P}_N(\lambda)=\hat{P}_N(\lambda)/\hat{P}_N(\tilde{\Lambda}_N)$.

Then, as [6], Lemma 1, we have the following estimate.

Lemma 2.8. *Let* (Λ, P) *be* (Λ_N, P_N) *or* $(\tilde{\Lambda}_N, \tilde{P}_N)$ *or* $(\hat{\Lambda}_N, \hat{P}_N)$. *Then for any* $\varepsilon>0$, *there exists* $\beta(\varepsilon)>0$ *such that if* $\beta>\beta(\varepsilon)$,

(2.9) $P(\lambda\in\Lambda;\ |\lambda|\geqq(1+\varepsilon)N)=o(1/N^{\ell})$
as $N\to\infty$ *for any positive integer* ℓ, *or equivalently,*

(2.9)′ $P(\mathscr{S}_{\mathbf{X}}\in\Lambda;\ |\mathscr{S}_{\mathbf{X}}|\geqq\varepsilon N)=o(1/N^{\ell})$
as $N\to\infty$ *for any positive integer* ℓ.

The proof is much easier than [6], Lemma 1. Using (2.4), we have

$$\begin{aligned}\hat{P}_N(\mathscr{S}_{\mathbf{X}}\in\hat{\Lambda}_N;\ |\mathscr{S}_{\mathbf{X}}|\geqq\varepsilon N)&\leqq\sum_{|\mathscr{S}_{\mathbf{X}}|\geqq\varepsilon N}e^{-\beta|\mathscr{S}_{\mathbf{X}}|+k(\beta,\beta_0)|\mathscr{S}_{\mathbf{X}}|}/1\\&\leqq e^{\beta N}\cdot\sum_{\lambda\in\hat{\Lambda}_N;\ |\lambda|\geqq(1+\varepsilon)N}e^{-(\beta-k(\beta,\beta_0))|\lambda|}\end{aligned}$$

since $|\lambda|=N+|\mathscr{S}_{\mathbf{X}}|$ if $\mathscr{S}_{\mathbf{X}}$ corresponds to λ. The right hand side of the above inequality is smaller than

$$\begin{aligned}&e^{\beta N}\cdot\sum_{j=(1+\varepsilon)N}^{\infty}3^j e^{-(\beta-k(\beta,\beta_0))j}\\&=\frac{1}{1-3e^{-\beta+k(\beta,\beta_0)}}\times\exp\{-N\cdot(\beta\varepsilon-(1+\varepsilon)(k(\beta,\beta_0)+\log 3))\}\end{aligned}$$

which converges exponentially to zero as $N\to\infty$ if $\beta-\beta_0$, β_0 are sufficiently large. The same argument applies in the case of P_N or $\tilde{P}_N$. (q.e.d.)

Combining Theorems 2.1 and 2.5 with Lemma 2.8, we obtain the following result.

Theorem 2.10. *Let* $\tilde{U}_N$: $\hat{\Lambda}_N\to\mathbf{R}^1$ *be* $\tilde{U}_N(\mathscr{S}_{\mathbf{X}})=U_N(\mathscr{S}_{\mathbf{X}})-U_N(\mathscr{S}_{\tilde{\mathbf{X}}})+U(\mathscr{S}_{\tilde{\mathbf{X}}})$, *where* $\tilde{\mathbf{X}}=\{\xi\in\mathbf{X};\ \xi\subset[(\log N)^2, N-(\log N)^2]\}$, *and let*

(2.11) $$P_N^*(\mathscr{S}_{\mathbf{X}})=\exp\{-2\beta|\mathscr{S}_{\mathbf{X}}|-\tilde{U}_N(\mathscr{S}_{\mathbf{X}})\}/\sum_{\mathscr{S}_{\mathbf{Y}}\in\hat{\Lambda}_N}\exp\{-2\beta|\mathscr{S}_{\mathbf{Y}}|-\tilde{U}_N(\mathscr{S}_{\mathbf{Y}})\}.$$

Then we have

(2.12) $$\sum_{\mathscr{S}_{\mathbf{X}}\in\hat{\Lambda}_N}f(\mathscr{S}_{\mathbf{X}})\hat{P}_N(\mathscr{S}_{\mathbf{X}})=\sum_{\mathscr{S}_{\mathbf{X}}\in\hat{\Lambda}_N}f(\mathscr{S}_{\mathbf{X}})P_N^*(\mathscr{S}_{\mathbf{X}})+o(1/N^{\ell})$$
as $N\to\infty$ for every bounded f and positive integer ℓ.

To state the second representation theorem for P_N by Gallavotti, we need some more notations. Let $\mathfrak{S}\equiv\{\mathscr{S}_{\xi};$ shape over a cluster $\xi\subset\mathbf{Z}^1\}$, and $\mathscr{X}\equiv\{(\mathscr{S}_{\xi_1}, \mathscr{S}_{\xi_2}, \dots, \mathscr{S}_{\xi_s})$; family of shapes, $\mathscr{S}_{\xi_i}\in\mathfrak{S}$ for each $i\}$. Two shapes $\mathscr{S}_{\xi_i}$, $\mathscr{S}_{\xi_j}$

of $\mathscr{S}_{\mathbf{X}} \in \mathscr{X}$ can not only overlapp but also be just the same. Of course if $\mathbf{X} = (\xi_1, \xi_2, \ldots, \xi_s)$, then each pair ξ_i, ξ_j of X can intersect. $\mathbf{X}$ is merely a collection of clusters in $\mathbf{Z}^1$.

Theorem 2.13. *Let $P = \hat{P}_N$ or P_N^*, then there exist functions ϕ_N, ϕ_N^* respectively such that if $\chi: \mathscr{X} \to \mathbf{R}^1$ is multiplicative, i.e. $\chi(\mathscr{S}_{\mathbf{X} \cup \mathbf{Y}}) = \chi(\mathscr{S}_{\mathbf{X}})\chi(\mathscr{S}_{\mathbf{Y}})$, and if for large $\beta > \beta_0 > 0$, $\phi = \phi_N$ or ϕ_N^*,*

$$\sum_{\mathscr{S}_{\mathbf{X}} \in \mathscr{X}, \mathbf{X} \subset [0, N]} |\chi(\mathscr{S}_{\mathbf{X}})| \, |\phi(\mathscr{S}_{\mathbf{X}})| \cdot e^{-2(\beta - \beta_0)|\mathscr{S}_{\mathbf{X}}|} < \infty,$$

then

$$\sum_{\mathscr{S}_{\mathbf{X}} \in \Lambda_N} \chi(\mathscr{S}_{\mathbf{X}}) P(\mathscr{S}_{\mathbf{X}}) = \exp\{ \sum_{\mathscr{S}_{\mathbf{X}} \in \mathscr{X}, \mathbf{X} \subset [0, N]} \phi(\mathscr{S}_{\mathbf{X}}) e^{-2(\beta - \beta_0)|\mathscr{S}_{\mathbf{X}}|}(\chi(\mathscr{S}_{\mathbf{X}}) - 1)\},$$

where ϕ_N, ϕ_N^* have the following property.

(i) $\sum_{\mathscr{S}_{\mathbf{Y}} \in \mathscr{X}, \mathbf{Y} \subset [0, N]} |\phi(\mathscr{S}_{\xi \cup \mathbf{Y}})| \leq \frac{e^{-\beta_0(|\mathscr{S}_\xi| + 1)}}{1 - \bar{k}(\beta, \beta_0)}$ *for $\xi \subset [0, N]$, for sufficiently large β_0, $\beta - \beta_0$, and $\bar{k}(\beta, \beta_0) \to 0$ exponentially as $\beta_0 \to \infty$.*

(ii) *There exists a translationally invariant $\bar{\phi}$ independent of N, such that $\phi_N^*(\mathscr{S}_{\bar{\mathbf{X}}}) = \bar{\phi}(\mathscr{S}_{\bar{\mathbf{X}}})$ for any $\mathscr{S}_{\mathbf{X}} \in \mathscr{X}$.*

(iii) *Let $T: \mathfrak{S} \to \mathfrak{S}$ reverses every $\mathscr{S}_\xi \in \mathfrak{S}$ with upside down* (see Fig. 1 a and b). *Obviously T is bijective from $\mathfrak{S}$ to $\mathfrak{S}$, and it can be extended to $\mathscr{X}$ by $T\mathscr{S}_{\mathbf{X}} = (T\mathscr{S}_{\xi_1}, T\mathscr{S}_{\xi_2}, \ldots, T\mathscr{S}_{\xi_s})$ if $\mathscr{S}_{\mathbf{X}} = (\mathscr{S}_{\xi_1}, \mathscr{S}_{\xi_2}, \ldots, \mathscr{S}_{\xi_s})$. Then $\phi(T\mathscr{S}_{\mathbf{X}}) = \phi(\mathscr{S}_{\mathbf{X}})$ for every $\mathscr{S}_{\mathbf{X}} \in \mathscr{X}$, and for $\phi = \phi_N, \phi_N^*$.*

The idea of the proof is given by Gallavotti [4]. We can obtain the above result just by following his idea with a little technical change. But it takes too long time to get to the conclusion, so we omit the proof.

The third result by Gallavotti is about the correlation functions. Let $\mathscr{S}_{\mathbf{X}} \in \hat{\Lambda}_N$. We define correlation functions ρ_N, ρ_N^* by

$$\rho_N(\mathscr{S}_{\mathbf{X}}) = \sum_{\substack{\mathbf{Y} \cup \mathbf{X}; \text{N.O.} \\ \mathbf{Y} \subset [0, N]}} \sum_{\mathscr{S}_{\mathbf{Y}}} P_N(\mathscr{S}_{\mathbf{X} \cup \mathbf{Y}}), \tag{2.14a}$$

$$\rho_N^*(\mathscr{S}_{\mathbf{X}}) = \sum_{\substack{\mathbf{Y} \cup \mathbf{X}; \text{N.O.} \\ \mathbf{Y} \subset [0, N]}} \sum_{\mathscr{S}_{\mathbf{Y}}} P_N^*(\mathscr{S}_{\mathbf{X} \cup \mathbf{Y}}), \tag{2.14b}$$

where $\sum_{\substack{\mathbf{Y} \cup \mathbf{X}; \text{N.O.} \\ \mathbf{Y} \subset [0, N]}}$ is the summation over all collection $\mathbf{Y}$'s of clusters such that $\mathbf{X} \cup \mathbf{Y}$ consists of non-overlapping clusters in $[0, N]$.

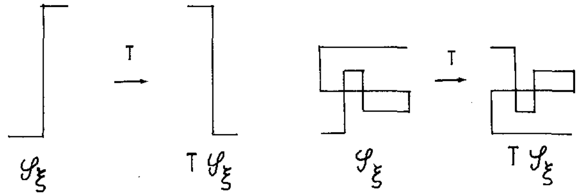

Fig. 1a and b

Theorem 2.15 (Gallavotti [4]). *Let $\bar{\rho}=\rho_N$ or ρ_N^*. Then we have*

$$|\bar{\rho}(\mathscr{S}_\xi)| \leqq \frac{1}{1-\bar{k}(\beta,\beta_0)} e^{-\beta|\mathscr{S}_\xi|} \quad \text{for any } \xi \subset [0,N]$$

and any shape $\mathscr{S}_\xi$ over ξ.

Corollary 2.16 ([4]). *Let $\bar{P}=P_N$ or P_N^*. Then we have*

$$\bar{P}(\lambda \in \hat{\Lambda}_N;\ \lambda \text{ contains a shape } \mathscr{S}_\xi \text{ such that } |\mathscr{S}_\xi| \geqq (\log N)^2)$$
$$= o(1/N^\ell) \quad \text{as } N\to\infty \text{ for any positive integer } \ell.$$

§ 3. Central Limit Theorems for the Jumps over Intervals

Let us fix an integer $k \geqq 1$ and $0=t_0<t_1<\ldots<t_{k+1}=1$, and real numbers $-\infty<T_j<T_j'<+\infty$, $j=1,2,\ldots,k$, arbitrarily. Let $a_j^{(N)} \equiv [N\cdot t_j]$, $j=1,2,\ldots,k+1$. Define functions $\chi_j^{(N)}: \mathfrak{S}\to\mathbf{R}^1$, $j=0,1,\ldots,k+1$ by

$$\chi_0^{(N)} \equiv 0 \quad \text{and} \quad \chi_j^{(N)}(\mathscr{S}_\xi) \equiv \begin{cases} 1 & \text{if } \xi \subset [0, a_j^{(N)}], \\ 0 & \text{otherwise.} \end{cases}$$

For $y_1, y_2, \ldots, y_{k+1} \in \mathbf{R}^1$, the function $f_N = f_N(y_1, y_2, \ldots, y_{k+1}): \mathscr{X}\to\mathbf{R}^1$ is defined by

$$f_N(\mathscr{S}_\mathbf{X}) = \sum_{j=1}^{k+1} y_j \sum_{\xi\in\mathbf{X}} \chi_j^{(N)}(\mathscr{S}_\xi)\cdot\delta\mathscr{S}_\xi.$$

Now let us consider the function

$$(3.1) \quad \langle e^{if_N(\mathscr{S}_\mathbf{X})/\sqrt{N}}\rangle_N^* \equiv \sum_{\mathscr{S}_\mathbf{X}\in\hat{\Lambda}_N} e^{if_N(\mathscr{S}_\mathbf{X})/\sqrt{N}} P_N^*(\mathscr{S}_\mathbf{X})$$
$$= \sum_{\substack{\mathbf{X};\,\text{N.O.},\\ \mathbf{X}\subset[0,N]}} \sum_{\mathscr{S}_\mathbf{X}} e^{if_N(\mathscr{S}_\mathbf{X})/\sqrt{N}} P_N^*(\mathscr{S}_\mathbf{X})$$

which is the characteristic function of the random vector

$$\Big(\sum_{\xi\in\mathbf{X}} \chi_j^{(N)}(\mathscr{S}_\xi)\cdot\delta\mathscr{S}_\xi;\ j=1,2,\ldots,k+1\Big).$$

For each j, $\sum_{\xi\in\mathbf{X}} \chi_j^{(N)}(\mathscr{S}_\xi)\cdot\delta\mathscr{S}_\xi$ denotes the total jump of $\mathscr{S}_\mathbf{X}$ between q_0 and $q_{a_j^{(N)}}$. Our first theorem is;

Theorem 3.1. *For sufficiently large $\beta>\beta_0>0$,*

$$P_N^*\Big(T_j\sigma\sqrt{N} \leqq \sum_{\xi\in\mathbf{X}} \chi_j^{(N)}(\mathscr{S}_\xi)\cdot\delta\mathscr{S}_\xi \leqq T_j'\sigma\sqrt{N},\ j=1,2,\ldots,k \,\Big|\, \delta\mathscr{S}_\mathbf{X}=0\Big)$$
$$\xrightarrow[N\to\infty]{} P_{0,0}^{1,0}(X(t_j)\in[T_j,T_j']\ j=1,2,\ldots,k)$$

where $\sigma=\sigma(\beta,\beta_0)>0$ is a constant, $\delta\mathscr{S}_\mathbf{X}=\sum_{\xi\in\mathbf{X}}\delta\mathscr{S}_\xi$, and $(X(t), P_{0\,0}^{1\,0})$ is a one-dimensional Brownian bridge.

It is not a short way to get to the above result. We need several lemmas to carry out the proof.

Since for any $m_1, m_2, \dots, m_{k+1} \in \mathbf{Z}^1$,

$$(3.2)\quad P_N^*\Big(\sum_{\xi\in\mathbf{X}} \chi_j^{(N)}(\mathscr{S}_\xi)\cdot\delta\mathscr{S}_\xi = m_j,\ j=1,2,\dots,k+1\Big)$$
$$= \Big(\frac{1}{2\pi\sigma\sqrt{N}}\Big)^{k+1} \int_{-\pi\sigma\sqrt{N}}^{\pi\sigma\sqrt{N}} \cdots \int_{-\pi\sigma\sqrt{N}}^{\pi\sigma\sqrt{N}} \langle e^{if_N(\mathscr{S}_\mathbf{X})/\sigma\sqrt{N}}\rangle_N^* e^{-i\sum_{j=1}^{k+1} y_j m_j/\sigma\sqrt{N}} dy_1 \dots dy_{k+1},$$

we must first analyze $\langle e^{if_N(\mathscr{S}_\mathbf{X})/\sigma\sqrt{N}}\rangle_N^*$.

Lemma 3.3.

$$\langle e^{if_N(\mathscr{S}_\mathbf{X})/\sigma\sqrt{N}}\rangle_N^*$$
$$= \exp\Big\{ \sum_{\substack{\mathscr{S}_\mathbf{X}\in\mathscr{X} \\ \mathbf{X}\subset[0,N]}} \phi_N^*(\mathscr{S}_\mathbf{X})\, e^{-2(\beta-\beta_0)|\mathscr{S}_\mathbf{X}|} (e^{if_N(\mathscr{S}_\mathbf{X})/\sigma\sqrt{N}} - 1)\Big\}.$$

Proof. From Theorem 2.13, putting $\chi(\mathscr{S}_\mathbf{X}) \equiv e^{if_N(\ \mathbf{X})/\sigma\sqrt{N}}$, we obtain the above equality provided that

$$\sum_{\substack{\mathscr{S}_\mathbf{X}\in\mathscr{X}; \\ \mathbf{X}\subset[0,N]}} |\chi(\mathscr{S}_\mathbf{X})|\, |\phi_N^*(\mathscr{S}_\mathbf{X})|\, e^{-2(\beta-\beta_0)|\mathscr{S}_\mathbf{X}|} < \infty.$$

The above equality is true because

$$\sum_{\substack{\mathscr{S}_\mathbf{X}\in\mathscr{X}, \\ \mathbf{X}\subset[0,N]}} |\chi(\mathscr{S}_\mathbf{X})|\, |\phi_N^*(\mathscr{S}_\mathbf{X})|\, e^{-2(\beta-\beta_0)|\mathscr{S}_\mathbf{X}|}$$
$$\leqq \sum_{p\in[0,N]} \sum_{\mathscr{S}_\mathbf{X}\in\mathscr{X}, \mathbf{X}\ni p} |\phi_N^*(\mathscr{S}_\mathbf{X})|\, e^{-2(\beta-\beta_0)|\mathscr{S}_\mathbf{X}|}$$
$$\leqq \sum_{p\in[0,N]} \sum_{\xi\ni p} \sum_{\mathscr{S}_{\xi\cup\mathbf{X}}\in\mathscr{X}} |\phi_N^*(\mathscr{S}_{\xi\cup\mathbf{X}})|\, e^{-2(\beta-\beta_0)|\mathscr{S}_{\xi\cup\mathbf{X}}|}$$
$$\leqq (N+1) \sum_{\xi\ni p} \sum_{\mathscr{S}_\xi} \Big(\sum_{\mathscr{S}_\mathbf{X}\in\mathscr{X}} |\phi_N^*(\mathscr{S}_{\xi\cup\mathbf{X}})|\Big) e^{-2(\beta-\beta_0)|\mathscr{S}_\xi|}$$
$$\leqq (N+1) \sum_{r=0}^{\infty} \sum_{\xi\ni p; |\xi|=r} \sum_{\ell=r+1}^{\infty} \sum_{|\mathscr{S}_\xi|=\ell} \frac{e^{-(2\beta-\beta_0)\ell}}{1-\tilde{k}(\beta,\beta_0)}$$
$$\equiv (N+1) A(\beta,\beta_0) < \infty.$$

We used the fact that (1) $|\xi|+1 \leqq |\mathscr{S}_\xi|$, (2) $\#\{\mathscr{S}_\xi; |\mathscr{S}_\xi| = \ell\} \leqq 3^{2\ell}$ for given ξ, and the estimate (i) of Theorem 2.13. (q.e.d.)

Now we change the variables $y_1, y_2, \dots, y_{k+1}$.

Let $g_j^{(N)}(\mathscr{S}_\xi) \equiv \chi_j^{(N)}(\mathscr{S}_\xi) - \chi_{j-1}^{(N)}(\mathscr{S}_\xi)$, $j=1,2,\dots,k+1$, and $\tilde{y}_j \equiv y_j + y_{j+1} + \dots + y_{k+1}$, $j=1,2,\dots,k+1$. Then

$$f_N(\mathscr{S}_\mathbf{X}) = \sum_{j=1}^{k+1} \tilde{y}_j \sum_{\xi\in\mathbf{X}} g_j^{(N)}(\mathscr{S}_\xi)\cdot\delta\mathscr{S}_\xi,$$

and (3.2) is rewritten into

$$(3.2)' \quad P_N^*\Big(\sum_{\xi\in\mathbf{X}}\chi_j^{(N)}(\mathscr{S}_\xi)\cdot\delta\mathscr{S}_\xi=m_j, j=1,2,\dots,k+1\Big)$$

$$=\left(\frac{1}{2\pi\sigma\sqrt{N}}\right)^{k+1}\int_{-\pi\sigma\sqrt{N}}^{\pi\sigma\sqrt{N}}\dots\int_{\pi\sigma\sqrt{N}}^{\pi\sigma\sqrt{N}}\langle e^{if_N(\mathscr{S}_\mathbf{X})/\sigma\sqrt{N}}\rangle_N^*$$

$$\cdot e^{-i\sum_{j=1}^{k+1}\tilde{y}_j(m_j-m_{j-1})/\sigma\sqrt{N}}\,d\tilde{y}_1\dots d\tilde{y}_{k+1}$$

where $m_0=0$.

Let $\mathbf{X}=(\xi_1,\xi_2,\dots,\xi_s)$ be a collection of clusters in $\mathbf{Z}^1$ which may be overlapping, and $\xi_j=(v_j,v_j+1,\dots,v_j+|\xi_j|)$, $j=1,2,\dots,s$. Define $\overline{\mathbf{X}}\equiv\{m(\mathbf{X}),m(\mathbf{X})+1,\dots,\bar{m}(\mathbf{X})-1,\bar{m}(\mathbf{X})\}$, where $m(\mathbf{X})\equiv\min_{1\leqq j\leqq s}v_j$ and $\bar{m}(\mathbf{X})\equiv\max_{1\leqq j\leqq s}(v_j+|\xi_j|)$. The following lemma is the most important in this section.

Lemma 3.4. $\lim_{N\to\infty}\sup_{p\in[0,N]}\frac{1}{N}\sum_{\substack{\mathbf{X}\subset[0,N]\\ \overline{\mathbf{X}}\ni p}}\sum_{\mathscr{S}_\mathbf{X}}|\phi_N^*(\mathscr{S}_\mathbf{X})|=0.$

Proof. Let us fix N and $p\in[0,N]$ arbitrarily.

$$\sum_{\substack{\mathbf{X}\subset[0,N]\\ \overline{\mathbf{X}}\ni p}}\sum_{\mathscr{S}_\mathbf{X}}|\phi_N^*(\mathscr{S}_\mathbf{X})|$$

$$\leqq\sum_{\substack{\mathbf{X}\subset[0,N]\\ \mathbf{X}\not\subset[(\log N)^2,N-(\log N)^2]\\ \overline{\mathbf{X}}\ni p}}\sum_{\mathscr{S}_\mathbf{X}}|\phi_N^*(\mathscr{S}_\mathbf{X})|$$

$$+\sum_{\substack{\mathbf{X}\subset[(\log N)^2,N-(\log N)^2]\\ \overline{\mathbf{X}}\ni p}}\sum_{\mathscr{S}_\mathbf{X}}|\phi_N^*(\mathscr{S}_\mathbf{X})|\equiv\mathrm{I}+\mathrm{II}.$$

From Theorem 2.13, (i), as in the proof of Lemma 3.3,

$$(3.5)\quad \sum_{r\in[0,(\log N)^2]\cup[N-(\log N)^2,N]}\sum_{\overline{\mathbf{X}}\ni r}\sum_{\mathscr{S}_\mathbf{X}}|\phi_N^*(\mathscr{S}_\mathbf{X})|$$

$$\leqq 2(\log N)^2\sum_{\xi\ni p}\sum_{\mathscr{S}_\xi}\sum_{\mathscr{S}_\mathbf{X}}|\phi_N^*(\mathscr{S}_{\xi\cup\mathbf{X}})|\leqq 2(\log N)^2\cdot\tilde{A}(\beta_0),$$

where $\tilde{A}(\beta_0)\to 0$ exponentially as $\beta_0\to\infty$. Next, by Theorem 2.13(i),

$$\phi_N^*(\mathscr{S}_\mathbf{X})=\bar{\phi}(\mathscr{S}_\mathbf{X})\qquad\text{if } \mathbf{X}\subset[(\log N)^2,N-(\log N)^2],$$

and $\bar{\phi}$ is translationally invariant. For each integer $d\geqq 1$, let

$$\psi(d)\equiv\sum_{\substack{\mathbf{X}\subset\mathbf{Z}^1,\overline{\mathbf{X}}\ni 0,\\ \#(\overline{\mathbf{X}})=d}}\sum_{\mathscr{S}_\mathbf{X}}|\bar{\phi}(\mathscr{S}_\mathbf{X})|,$$

where $\#(\overline{\mathbf{X}})=\bar{m}(\mathbf{X})-m(\mathbf{X})+1$, i.e. the cardinality of $\overline{\mathbf{X}}$. Using the invariance of $\bar{\phi}$, we obtain

$$0 \leqq \psi(d) = d \cdot \sum_{\substack{\mathbf{X} \subset \mathbf{Z}^1, m(\mathbf{X}) = 0 \\ \#(\mathbf{X}) = d}} \sum_{\mathscr{S}_{\mathbf{X}}} |\bar{\phi}(\mathscr{S}_{\mathbf{X}})|,$$

which implies that

$$\sum_{d=1}^{\infty} \frac{\psi(d)}{d} = \sum_{\mathbf{X} \subset \mathbf{Z}^1, m(\mathbf{X}) = 0} \sum_{\mathscr{S}_{\mathbf{X}}} |\bar{\phi}(\mathscr{S}_{\mathbf{X}})| \leqq \sum_{\substack{\mathbf{X} \subset \mathbf{Z}^1 \\ \mathbf{X} \ni 0}} \sum_{\mathscr{S}_{\mathbf{X}}} |\bar{\phi}(\mathscr{S}_{\mathbf{X}})|.$$

By Theorem 2.13, (i), (iii), the right hand side is not larger than $\tilde{A}(\beta, \beta_0) < \infty$. Thus,

$$\text{(3.6)} \quad \mathrm{II} \leqq \sum_{d=1}^{N} \sum_{\substack{\mathbf{X} \subset \mathbf{Z}^1, \mathbf{X} \ni p, \\ \#(\mathbf{X}) = d}} \sum_{\mathscr{S}_{\mathbf{X}}} |\bar{\phi}(\mathscr{S}_{\mathbf{X}})| = \sum_{d=1}^{N} \psi(d).$$

Combining (3.5), (3.6) with the fact that $\sum_{d=1}^{\infty} \frac{\psi(d)}{d} < \infty$, we see that the assertion of the Lemma is true. (q.e.d.)

Before explaining how we make use of the above lemma, we claim the following fact which is also merely mentioned in [4].

Lemma 3.7. *All the following limits exist and positive if $\beta > \beta_0 > 0$ are sufficiently large.*

(i) $$\lim_{N \to \infty} \frac{1}{N} \sum_{\substack{\mathscr{S}_{\mathbf{X}} \in \mathscr{X}, \\ \mathbf{X} \subset [0, N]}} \phi_N^*(\mathscr{S}_{\mathbf{X}}) \cdot e^{-2(\beta - \beta_0)|\mathscr{S}_{\mathbf{X}}|} (\delta \mathscr{S}_{\mathbf{X}})^{2j}$$
$$\equiv B_j(\beta, \beta_0), \quad j = 0, 1, 2.$$

(ii) $$\lim_{N \to \infty} \frac{1}{N} \sum_{\substack{\mathscr{S}_{\mathbf{X}} \in \mathscr{X}, \\ \mathbf{X} \subset [0, N]}} |\phi_N^*(\mathscr{S}_{\mathbf{X}})| \cdot e^{-2(\beta - \beta_0)|\mathscr{S}_{\mathbf{X}}|} (\delta \mathscr{S}_{\mathbf{X}})^{2j}$$
$$\equiv \tilde{B}_j(\beta, \beta_0), \quad j = 0, 1, 2.$$

Moreover, if $\beta - \beta_0$ is large enough,

$$\text{(3.8)} \quad \frac{1}{2} \leqq \frac{B_1(\beta, \beta_0)}{\tilde{B}_2(\beta, \beta_0)} \leqq \frac{B_1(\beta, \beta_0)}{B_2(\beta, \beta_0)} \leqq \frac{\tilde{B}_1(\beta, \beta_0)}{B_2(\beta, \beta_0)} \leqq 2.$$

The proof of (i), (ii) is similar to the proof of the existence of the mean energy, for ϕ_N^* is asymptotically translationally invariant. (3.8) comes from the following estimate obtained by Gallavotti [4].

Lemma 3.9. (i) *Let $\xi \subset [0, N]$, $\xi \neq 0, N$, $|\xi| = 0$ and $|\mathscr{S}_\xi| = j$, for $j \geqq 1$. Then for sufficiently large $\beta - \beta_0 > 0$, $\beta_0 > 0$,*

$$\phi(\mathscr{S}_\xi) = \exp\{-2\beta_0 j - z e^{-8\beta} j + 3^5 e^{-10\beta} \theta(\beta) j\},$$

where $\phi = \phi_N$ or ϕ_N^* or $\bar{\phi}$, and $|\theta(\beta)| \leqq 1$.

(ii) $$\sum_{\substack{|\mathscr{S}_{\mathbf{X}}| \geqq 2, \mathscr{S}_{\mathbf{X}} \in \mathscr{X}, \\ p \in \mathbf{X} \subset [0, N]}} |\phi(\mathscr{S}_{\mathbf{X}})| \leqq e^{-2\beta_0} \times C(\beta, \beta_0)$$

for any $p \in [0, N]$, where $C(\beta, \beta_0) \to 0$ as $(\beta - \beta_0) \wedge \beta_0 \to \infty$, and $\phi = \phi_N$, ϕ_N^*, $\bar{\phi}$.

Now we are going to estimate the right hand side of (3.2)′.

Lemma 3.10.

$$\langle e^{i f_N(\mathscr{S}_{\mathbf{X}})/\sigma\sqrt{N}}\rangle_N^* = \exp\left\{\sum_{j=1}^{k+1} \sum_{\mathbf{X}\subset[a_{j-1}^{(N)}+1,a_j^{(N)}]} \sum_{\mathscr{S}_{\mathbf{X}}} \phi_N^*(\mathscr{S}_{\mathbf{X}}) \cdot e^{-2(\beta-\beta_0)|\mathscr{S}_{\mathbf{X}}|}(e^{i\tilde{y}_j\cdot\delta\mathscr{S}_{\mathbf{X}}/\sigma\sqrt{N}}-1)+o(N)\right\}$$

as $N\to\infty$.

Proof. If $\mathbf{X}\not\subset[a_{j-1}^{(N)}+1,a_j^{(N)}]$, then for any $\xi\in\mathbf{X}$, $g_\ell^{(N)}(\mathscr{S}_\xi)=\delta_{j\ell}$, where $\delta_{j\ell}=0$ if $j\neq\ell$ and $\delta_{jj}=1$ for any j. Hence $f_N(\mathscr{S}_{\mathbf{X}})=\tilde{y}_j\cdot\delta\mathscr{S}_{\mathbf{X}}$ if $\mathbf{X}\subset[a_{j-1}^{(N)}+1,a_j^{(N)}]$ for some j. On the other hand, if $\mathbf{X}\not\subset[a_{j-1}^{(N)}+1,a_j^{(N)}]$ for any j, then $\bar{\mathbf{X}}\cap\{a_j^{(N)};j=1,2,\ldots,k+1\}=\emptyset$. Hence we have

$$\begin{aligned}
&\Big|\sum_{\substack{\mathbf{X}\not\subset[a_{j-1}^{(N)}+1,a_j^{(N)}]\\ \text{for any } j}} \sum_{\mathscr{S}_{\mathbf{X}}} \phi_N^*(\mathscr{S}_{\mathbf{X}})\, e^{-2(\beta-\beta_0)|\mathscr{S}_{\mathbf{X}}|}(e^{i f_N(\mathscr{S}_{\mathbf{X}})/\sigma\sqrt{N}}-1)\Big|\\
&\leqq 2\sum_{\mathbf{X}\not\subset[(\log N)^2,N-(\log N)^2]} \sum_{\mathscr{S}_{\mathbf{X}}} |\phi_N^*(\mathscr{S}_{\mathbf{X}})|\, e^{-2(\beta-\beta_0)|\mathscr{S}_{\mathbf{X}}|}\\
&\quad+2\sum_{j=1}^{k+1} \sum_{\substack{\bar{\mathbf{X}}\ni a_j^{(N)},\\ \mathbf{X}\subset[(\log N)^2,N-(\log N)^2]}} \sum_{\mathscr{S}_{\mathbf{X}}} |\bar{\phi}(\mathscr{S}_{\mathbf{X}})|\, e^{-2(\beta-\beta_0)|\mathscr{S}_{\mathbf{X}}|}\\
&=o(N)
\end{aligned}$$

as $N\to\infty$ from Lemmas 3.3 and 3.4. (q.e.d.)

Lemma 3.11. *For any* $1>\varepsilon>0$, *there exists* $\beta(\varepsilon)>0$ *such that if* $\beta-\beta_0>\beta(\varepsilon)$, *then*

$$\langle e^{i f_N(\mathscr{S}_{\mathbf{X}})/\sigma\sqrt{N}}\rangle_N^*=o(1/N^\ell)$$

as $N\to\infty$ *for any positive integer* ℓ *whenever there exists* j; $1\leqq j\leqq k+1$ *such that* $|\tilde{y}_j|\geqq\varepsilon\sigma\sqrt{N}$.

Remark. Of course we assume that $|\tilde{y}_j|\leqq\pi\sigma\sqrt{N}$ for any j because $\langle e^{i f_N(\mathscr{S}_{\mathbf{X}})/\sigma\sqrt{N}}\rangle_N^*$ is a periodic function of each $\tilde{y}_j$ with period $2\pi\sigma\sqrt{N}$.

Proof of Lemma 3.11. First note that

$$\begin{aligned}
&\frac{1}{N}\sum_{\mathbf{X}\subset[a_{j-1}^{(N)}+1,a_j^{(N)}]} \sum_{\mathscr{S}_{\mathbf{X}}} \phi_N^*(\mathscr{S}_{\mathbf{X}})\, e^{-2(\beta-\beta_0)|\mathscr{S}_{\mathbf{X}}|}(e^{i\tilde{y}_j\cdot\delta\mathscr{S}_{\mathbf{X}}/\sigma\sqrt{N}}-1)\\
&\leqq\frac{1}{N}\sum_{\substack{\xi\subset[a_{j-1}^{(N)}+1,a_j^{(N)}]\\ |\xi|=0}} \sum_{\delta\mathscr{S}_\xi=1} \phi_N^*(\mathscr{S}_\xi)\, e^{-2(\beta-\beta_0)}(e^{i\tilde{y}_j/\sigma\sqrt{N}}-1+e^{-i\tilde{y}_j/\sigma\sqrt{N}}-1)\\
&\quad+\frac{2}{N}\sum_{\mathbf{X}\subset[a_{j-1}^{(N)}+1,a_j^{(N)}]} \sum_{|\mathscr{S}_{\mathbf{X}}|\geqq 2} |\phi_N^*(\mathscr{S}_{\mathbf{X}})|\, e^{-2(\beta-\beta_0)|\mathscr{S}_{\mathbf{X}}|}
\end{aligned}$$

and the left hand side is real for all $\tilde{y}_j$. Using Lemma 3.9, we obtain

$$\frac{1}{N}\sum_{\mathbf{X}\subset[a_{j-1}^{(N)}+1,a_j^{(N)}]}\sum_{\mathscr{S}_{\mathbf{X}}}\phi_N^*(\mathscr{S}_{\mathbf{X}})\,e^{-2(\beta-\beta_0)|\mathscr{S}_{\mathbf{X}}|}(e^{i\tilde{y}_j\delta\mathscr{S}_{\mathbf{X}}/\sigma\sqrt{N}}-1)$$
$$\leqq(t_j-t_{j-1})\,e^{-2\beta-3e^{-8\beta}}((e^{i\tilde{y}_j/\sigma\sqrt{N}}-1)+(e^{-i\tilde{y}_j/\sigma\sqrt{N}}-1)$$
$$+2e^{-2(\beta-\beta_0)+3e^{-8\beta}}C(\beta,\beta_0))+o(1).$$

Note that $e^{i\tilde{y}_j/\sigma\sqrt{N}}-1+e^{-i\tilde{y}_j/\sigma\sqrt{N}}-1=-4\sin^2\left(\frac{\tilde{y}_j}{2\sigma\sqrt{N}}\right)\leqq-4\sin^2\frac{\varepsilon}{2}$ if $\pi\sigma\sqrt{N}\geqq|\tilde{y}_j|\geqq\varepsilon\sigma\sqrt{N}$. Hence if we take $\beta>\beta_0>0$ sufficiently large such that $2\sin^2\frac{\varepsilon}{2}>2e^{-2(\beta-\beta_0)+3e^{-8\beta}}C(\beta,\beta_0)$, we get

$$\sum_{\mathbf{X}\subset[a_{j-1}^{(N)}+1,a_j^{(N)}]}\sum_{\mathscr{S}_{\mathbf{X}}}\phi_N^*(\mathscr{S}_{\mathbf{X}})\,e^{-2(\beta-\beta_0)|\mathscr{S}_{\mathbf{X}}|}(e^{iy_j\delta\mathscr{S}_{\mathbf{X}}/\sigma\sqrt{N}}-1)$$
$$\leqq-2N\left\{(t_j-t_{j-1})\,e^{-2\beta-3e^{-8\beta}}\sin^2\frac{\varepsilon}{2}+o(1)\right\}.$$

For other j's such that $|\tilde{y}_j|\leqq\varepsilon\sigma\sqrt{N}$, from Taylor's theorem and Theorem 2.13, (iii),

$$\sum_{\mathbf{X}\subset[a_{j-1}^{(N)}+1,a_j^{(N)}]}\sum_{\mathscr{S}_{\mathbf{X}}}\phi_N^*(\mathscr{S}_{\mathbf{X}})\,e^{-2(\beta-\beta_0)|\mathscr{S}_{\mathbf{X}}|}(e^{i\tilde{y}_j\delta\mathscr{S}_{\mathbf{X}}/\sigma\sqrt{N}}-1)$$
$$=\sum_{\mathbf{X}\subset[a_{j-1}^{(N)}+1,a_j^{(N)}]}\sum_{\mathscr{S}_{\mathbf{X}}}\phi_N^*(\mathscr{S}_{\mathbf{X}})\,e^{-2(\beta-\beta_0)|\mathscr{S}_{\mathbf{X}}|}$$
$$\times\left(-\frac{1}{2}\frac{(\delta\mathscr{S}_{\mathbf{X}})^2}{\sigma^2N}\tilde{y}_j^2+\tilde{y}_j^4\frac{(\delta\mathscr{S}_{\mathbf{X}})^4}{12\sigma^4N^2}\theta(\tilde{y}_j,\mathscr{S}_{\mathbf{X}})\right),$$

where $|\theta|\leqq1$.

The right hand side is not larger than

$$-\frac{\tilde{y}_j^2}{2\sigma^2}\left[(t_j-t_{j-1})B_1(\beta,\beta_0)-\frac{\tilde{y}_j^2}{6\sigma^2N}\tilde{B}_2(\beta,\beta_0)(t_j-t_{j-1})+o(1)\right]$$
$$\leqq-\frac{\tilde{y}_j}{2\sigma^2}\left[(t_j-t_{j-1})\left\{B_1(\beta,\beta_0)-\frac{\varepsilon^2}{6}\tilde{B}_2(\beta,\beta_0)\right\}+o(1)\right].$$

From Lemma 3.7, the right hand side is non-positive for sufficiently large $\beta>\beta_0>0$ as $N\to\infty$.

Thus, we obtain from Lemma 3.10 and the above argument,

$$\langle e^{if_N(\mathscr{S}_{\mathbf{X}})/\sigma\sqrt{N}}\rangle_N^*$$
$$\leqq\exp\left\{-\sum_{j;|\tilde{y}_j|>\varepsilon\sigma\sqrt{N}}2N\cdot\left\{(t_j-t_{j-1})\,e^{-2\beta-3e^{-8\beta}}\sin^2\frac{\varepsilon}{2}+o(1)\right\}\right\}$$
$$=o(1/N^{\ell})\text{ as }N\to\infty\text{ for any positive integer }\ell.\quad\text{(q.e.d.)}$$

Hereafter we put $\sigma^2=\sigma^2(\beta,\beta_0)=B_1(\beta,\beta_0)$.

Lemma 3.12. *For sufficiently large $\beta > \beta_0 > 0$, and any given $m_1, m_2, \ldots, m_k \in \mathbf{Z}^1$,*

$$P_N^*\Big(\sum_{\xi\in\mathbf{X}} \chi_j^{(N)}(\mathscr{S}_\xi)\,\delta\mathscr{S}_\xi = m_j, j=1,2,\ldots,k, \delta\mathscr{S}_{\mathbf{X}}=0\Big)$$
$$= \left(\frac{1}{2\pi\sigma\sqrt{N}}\right)^{k+1} \underset{\substack{|\tilde{y}_j|\leqq \log N\\ \text{for all } j}}{\int\cdots\int} \left\{\langle e^{i f_N(\mathscr{S}_{\mathbf{X}})/\sigma\sqrt{N}}\rangle_N^* \prod_{j=1}^{k+1} e^{-\tilde{y}_j\frac{(m_j - m_{j-1})}{\sigma\sqrt{N}}}\right\} d\tilde{y}_1 \ldots d\tilde{y}_{k+1}$$
$$+ o(1/N^{\ell})$$

as $N\to\infty$ for any positive integer ℓ, where $m_0 = m_{k+1} = 0$.

Proof. From (3.2)′ and Lemma 3.11, we have

$$(3.13)\quad P_N^*\Big(\sum_{\xi\in\mathbf{X}} \chi_j^{(N)}(\mathscr{S}_\xi)\,\delta\mathscr{S}_\xi = m_j, j=1,2,\ldots,k, \delta\mathscr{S}_{\mathbf{X}}=0\Big)$$
$$= \left(\frac{1}{2\pi\sigma\sqrt{N}}\right)^{k+1} \underset{\substack{|\tilde{y}_j|\leqq \varepsilon\sigma\sqrt{N}\\ \text{for all } j}}{\int\cdots\int} \{\quad\} \, d\tilde{y}_1 \ldots d\tilde{y}_{k+1} + o(1/N^{\ell})$$

as $N\to\infty$ for any positive integer ℓ, where $\{\ \}$ is the same integrand as in the above equality. As in the proof of Lemma 3.11, if $|\tilde{y}_j| \leqq \varepsilon\sigma\sqrt{N}$, then we have

$$\sum_{\mathbf{X}\subset[a_{j-1}^{(N)}+1,\, a_j^{(N)}]} \sum_{\mathscr{S}_{\mathbf{X}}} \phi_N^*(\mathscr{S}_{\mathbf{X}})\, e^{-2(\beta-\beta_0)|\mathscr{S}_{\mathbf{X}}|}(e^{i\tilde{y}_j\delta\mathscr{S}_{\mathbf{X}}/\sigma\sqrt{N}} - 1)$$
$$\leqq -\frac{\tilde{y}_j^2}{2\sigma^2}\left[(t_j - t_{j-1})\left\{B_1(\beta,\beta_0) - \frac{\varepsilon^2}{6}\tilde{B}_2(\beta,\beta_0)\right\} + o(1)\right].$$

Since $\tilde{B}_2(\beta,\beta_0) < 2B_1(\beta,\beta_0)$, $\varepsilon < 1$ and $B_1(\beta,\beta_0) = \sigma^2$, the right hand side of the above inequality is smaller than

$$-\frac{\tilde{y}_j^2}{12\sigma^2}(t_j - t_{j-1})B_1(\beta,\beta_0) = -\frac{\tilde{y}_j^2}{12}(t_j - t_{j-1}).$$

Hence we have

$$\left| \underset{\substack{|\tilde{y}_1|\geqq \log N\\ |\tilde{y}_1|,|\tilde{y}_2|,\ldots,|\tilde{y}_{k+1}|\leqq \varepsilon\sigma\sqrt{N}}}{\int\cdots\int} \{\ \}\, d\tilde{y}_1 \ldots d\tilde{y}_{k+1}\right|$$
$$\leqq 2\left\{\prod_{j=2}^{k+1}\sqrt{\frac{t_j - t_{j-1}}{12}\cdot\pi}\right\}\cdot\int_{\log N}^{\infty} e^{-t^2\tilde{y}_1^2/12}\, d\tilde{y}_1 = o(1/N^{\ell})$$

as $N\to\infty$ for any positive integer ℓ, which proves the lemma. (q.e.d.)

Lemma 3.14. *If $|\tilde{y}_j| < \log N$ for every j, then*

$$\langle e^{i f_N(\mathscr{S}_{\mathbf{X}})/\sigma\sqrt{N}}\rangle_N^* = \exp\left\{-\tfrac{1}{2}\sum_{i=1}^{k+1}\sum_{j=1}^{k+1} \gamma_{ij}^{(N)}\tilde{y}_i\tilde{y}_j + o(1/\sqrt{N})\right\}$$

as $N\to\infty$, where

$$\gamma_{ij}^{(N)}=\frac{1}{\sigma^2 N}\sum_{\substack{\mathbf{X}\subset[0,N]\\ \mathscr{S}_{\mathbf{X}}\in\mathscr{X}}}\phi_N^*(\mathscr{S}_{\mathbf{X}})\,e^{-2(\beta-\beta_0)|\mathscr{S}_{\mathbf{X}}|}\Big(\sum_{\xi\in\mathbf{X}}g_i^{(N)}(\mathscr{S}_\xi)\,\delta\mathscr{S}_\xi\Big)$$
$$\times\Big(\sum_{\xi\in\mathbf{X}}g_j^{(N)}(\mathscr{S}_\xi)\,\delta\mathscr{S}_\xi\Big),$$

for $1\leqq i,j\leqq k+1$.

Proof. By the same reason as in the proof of Lemma 3.11, using Taylor's theorem and Theorem 2.13, (iii), we obtain first;

$$\text{(3.15)}\quad \langle e^{if_N(\mathscr{S}_{\mathbf{X}})/\sigma\sqrt{N}}\rangle_N^*$$
$$=\exp\Bigg\{\frac{1}{2N}\sum_{\substack{\mathbf{X}\subset[0,N]\\ \mathscr{S}_{\mathbf{X}}\in\mathscr{X}}}\phi_N^*(\mathscr{S}_{\mathbf{X}})\,e^{-2(\beta-\beta_0)|\mathscr{S}_{\mathbf{X}}|}(f_N(\mathscr{S}_{\mathbf{X}}))^2$$
$$+\frac{1}{12N^2}\sum_{\substack{\mathbf{X}\subset[0,N]\\ \mathscr{S}_{\mathbf{X}}\in\mathscr{X}}}\phi_N^*(\mathscr{S}_{\mathbf{X}})\,e^{-2(\beta-\beta_0)|\mathscr{S}_{\mathbf{X}}|}(f_N(\mathscr{S}_{\mathbf{X}}))^1\cdot\theta(f_N(\mathscr{S}_{\mathbf{X}})/\sigma\sqrt{N})\Bigg\},$$

where $|\theta|\leqq 1$.

Since $|\tilde{y}_j|<\log N$ for every j,

$$\text{(3.16)}\quad \Bigg|\frac{1}{N^2}\sum_{\mathbf{X}\subset[0,N],\mathscr{S}_{\mathbf{X}}\in\mathscr{X}}\phi_N^*(\mathscr{S}_{\mathbf{X}})\,e^{-2(\beta-\beta_0)|\mathscr{S}_{\mathbf{X}}|}$$
$$\times\prod_{\nu=1}^{4}\Big(\tilde{y}_{j_\nu}\sum_{\xi\in\mathbf{X}}g_{j_\nu}^{(N)}(\mathscr{S}_\xi)\,\delta\mathscr{S}_\xi\Big)\cdot\theta(f_N(\mathscr{S}_{\mathbf{X}})/\sigma\sqrt{N})\Bigg|$$
$$\leqq\begin{cases}\dfrac{(\log N)^4}{N}\times[\tilde{B}_2(\beta,\beta_0)(t_j-t_{j-1})+o(1)] & \text{if } j_1=j_2=j_3=j_4=j,\\ \dfrac{(\log N)^4}{N}\times o(1) & \text{otherwise,}\end{cases}$$

by using Lemmas 3.4, 3.7, and the fact that

$$\Bigg|e^{-2(\beta-\beta_0)|\mathscr{S}_{\mathbf{X}}|}\prod_{\nu=1}^{4}\Big(\sum_{\xi\in\mathbf{X}}g_{j_\nu}^{(N)}(\mathscr{S}_\xi)\,\delta\mathscr{S}_\xi\Big)\Bigg|$$
$$\leqq\sup_{\ell_1,\ell_2,\ell_3,\ell_4\geqq 1}\prod_{\nu=1}^{4}(e^{-2(\beta-\beta_0)\ell_\nu}\ell_\nu)<1$$

if $\beta-\beta_0>0$ is sufficiently large. We also use the fact that for $i\neq j$,

$$\Big(\sum_{\xi\in\mathbf{X}}g_i^{(N)}(\mathscr{S}_\xi)\,\delta\mathscr{S}_\xi\Big)\Big(\sum_{\xi\in\mathbf{X}}g_j^{(N)}(\mathscr{S}_\xi)\,\delta\mathscr{S}_\xi\Big)=0\qquad\text{if }\mathbf{X}\subset[a_{\ell-1}^{(N)}+1,a_\ell^{(N)}]$$

for some ℓ. (3.16) implies that the contribution of the second term of the right hand side of (3.15) is smaller than

$$\frac{(\log N)^4}{N}(\tilde{B}_2(\beta,\beta_0)+o(1)),$$

which proves the lemma. (q.e.d.)

Lemma 3.17. *Let $\Gamma_N \equiv (\gamma_{ij}^{(N)})$. Then Γ_N is a strictly positive definite $(k+1)\times(k+1)$-matrix for sufficiently large N.*

Proof. By Lemmas 3.6 and 3.7,

$$\gamma_{ii}^{(N)} = \frac{t_i - t_{i-1}}{\sigma^2}[B_1(\beta,\beta_0)+o(1)] = t_i - t_{i-1} + o(1)$$

as $N\to\infty$, $i=1,2,\dots,k+1$, and

$$\gamma_{ij}^{(N)} = o(1) \quad \text{as} \quad N\to\infty \ \text{ if } i\neq j.$$

This implies that Γ_N is strictly positive definite for large enough N. (q.e.d.)

Lemma 3.18. *For $m_0=0$, $m_1, m_2, \dots, m_k \in \mathbf{Z}^1$, $m_{k+1}=0$,*

$$\left(\frac{1}{2\pi\sigma\sqrt{N}}\right)^{k+1} \underset{\substack{|\tilde{y}_j|<\log N \\ \text{for all } j}}{\int\dots\int} \langle e^{if_N(\mathscr{S}\mathbf{x})/\sigma\sqrt{N}}\rangle_N^* \prod_{j=1}^{k+1} e^{-i\tilde{y}_j\frac{(m_j-m_{j-1})}{\sigma\sqrt{N}}}\, d\tilde{y}_1\dots d\tilde{y}_{k+1}$$
$$= \frac{1}{\sqrt{(2\pi\sigma^2N)^{k+1}\det\Gamma_N}}\exp\left\{-\frac{1}{2\sigma^2N}{}^t\mathbf{m}(\Gamma_N)^{-1}\mathbf{m}\right\} + o(1/N^{(k+2)/2})$$

as $N\to\infty$, where ${}^t\mathbf{m}=(m_1, m_2-m_1,\dots,m_k-m_{k-1},-m_k)$.

Proof. Note that

$$\underset{\substack{|\tilde{y}_j|\geq\log N \\ \text{for some } j}}{\int\dots\int}\left|\exp\left\{-\tfrac{1}{2}{}^t\tilde{\mathbf{y}}(\Gamma_N)\tilde{\mathbf{y}} + o(1/\sqrt{N}) - i\cdot\frac{{}^t\mathbf{m}\cdot\tilde{\mathbf{y}}}{\sigma\sqrt{N}}\right\}\right| d\tilde{y}_1\dots d\tilde{y}_{k+1}$$
$$= o(1/N^{\ell}) \quad \text{as} \quad N\to\infty$$

for any positive integer ℓ, where ${}^t\tilde{\mathbf{y}}=(\tilde{y}_1,\dots,\tilde{y}_{k+1})$. This comes from the fact that if we define δ_N by

$$\delta_N \equiv \max_{1\leq i\leq k+1} \sum_{\substack{1\leq j\leq k+1 \\ j\neq i}} |\gamma_{ij}^{(N)}|,$$

then

$${}^t\tilde{\mathbf{y}}(\Gamma_N)\tilde{\mathbf{y}} \geq \sum_{j=1}^{k+1}(\gamma_{jj}-\delta_N)\tilde{y}_j^2, \quad \text{and} \quad \min_j(\gamma_{jj}-\delta_N)\geq\gamma>0$$

for some $\gamma>0$ if N is sufficiently large.

Since

$$\left(\frac{1}{2\pi\sigma\sqrt{N}}\right)^{k+1}\int_{-\infty}^{+\infty}\dots\int_{-\infty}^{+\infty}\exp\left\{-\tfrac{1}{2}{}^t\tilde{\mathbf{y}}(\Gamma_N)\tilde{\mathbf{y}}+o(1/\sqrt{N})-i\frac{{}^t\mathbf{m}\cdot\tilde{\mathbf{y}}}{\sigma\sqrt{N}}\right\}d\tilde{y}_1\dots d\tilde{y}_{k+1}$$
$$=\left(\frac{1}{2\pi\sigma\sqrt{N}}\right)^{k+1}\int_{-\infty}^{+\infty}\dots\int_{-\infty}^{+\infty}\exp\left\{-\tfrac{1}{2}{}^t\tilde{\mathbf{y}}(\Gamma_N)\tilde{\mathbf{y}}-i\frac{{}^t\mathbf{m}\cdot\tilde{\mathbf{y}}}{\sigma\sqrt{N}}\right\}d\tilde{y}_1\dots d\tilde{y}_{k+1}$$
$$+o(1/N^{(k+2)/2}) \quad \text{as} \quad N\to\infty,$$

we obtain the desired result using the fact that Γ_N has its inverse Γ_N^{-1} since Γ_N is strictly positive definite. (q.e.d.)

Corollary 3.19.

$$P_N^*(\delta\mathscr{S}_{\mathbf{X}}=0)=\frac{1}{2\pi\sigma\sqrt{N}}+o(1/N) \quad as \quad N\to\infty.$$

Now it would be obvious to see that Theorem 3.1 holds, since

$$\Gamma_N\to\begin{pmatrix} t_1 & & & & 0 \\ & t_2-t_1 & & & \\ & & \ddots & & \\ & & & t_k-t_{k-1} & \\ 0 & & & & 1-t_k \end{pmatrix}$$

as $N\to\infty$ and

$$P_{0,0}^{1,0}(X(t_1)\in[T_1,T_1'],\dots,X(t_k)\in[T_k,T_k'])$$
$$=\sqrt{2\pi}\int_{T_1}^{T_1'}dx\,p(t_1,x_1)\int_{T_2}^{T_2'}dx_2\,p(t_2-t_1,x_2-x_1)\times\dots\times$$
$$\times\int_{T_k}^{T_k'}dx_k\,p(t_k-t_{k-1},x_k-x_{k-1})\,p(1-t_k,-x_k)$$

where $p(t,x)=(2\pi t)^{-1/2}e^{-x^2/2t}$.

§4. The Maximum Process $Y_N(t)$

Throughout this section, we assume that $\beta>\beta_0>0$ are sufficiently large so that every result in §1~§3 holds.

Let us fix $N>0$. For any $\lambda\in\hat{\Lambda}_N$, let us define $Y_N(t;\lambda)$ by

$$Y_N(0;\lambda)\equiv Y_N(1;\lambda)\equiv 0,$$

$$Y_N(k/N;\lambda)\equiv\max\{\ell;(k,\ell)\in\lambda\}/\sigma\sqrt{N} \qquad k=1,2,\dots,N-1.$$

and

$$Y_N(t;\lambda)\equiv Y_N(k/N;\lambda)+(t-k/N)\left[Y_N\left(\frac{k+1}{N};\lambda\right)-Y_N(k/N;\lambda)\right]$$

for $k/N\leqq t<(k+1)/N$. Then it is easy to see that $Y_N(t;\lambda)$ is continuous in t. Moreover, Theorem 3.1 and Corollary 2.16 imply the following result.

Theorem 4.1. *Let* $0=t_0<t_1<t_2<\dots<t_k<t_{k+1}=1$, *and* $-\infty<T_j<T_j'<\infty$, $j=1,2,\dots,k$ *be given. Then for large enough* $\beta>\beta_0>0$,

$$P_N^*(Y_N(t_1)\in[T_1,T_1'],\ldots,Y_N(t_k)\in[T_k,T_k']\,|\,\delta\mathscr{S}_{\mathbf{X}}=0)$$
$$\to P_{0,0}^{1,0}(X(t_1)\in[T_1,T_1'],\ldots,X(t_k)\in[T_k,T_k']) \quad as \quad N\to\infty,$$

where $(P_{0,0}^{1,0},\{X(t)\}_{0\leqq t\leqq 1})$ *is a one-dimensional Brownian bridge.*

Proof. For simplicity we prove in the case $k=1$. Note that

$$\{Y_N(t_1)\in[T_1,T_1']\}\cap\{|\mathscr{S}_\xi|\leqq(\log N)^2 \text{ for any } \mathscr{S}_\xi\in\lambda\}$$
$$\subset\{\sum_{\xi\in\mathbf{X}}\chi_1^{(N)}(\mathscr{S}_\xi)\,\delta\mathscr{S}_\xi/\sigma\sqrt{N}\in[T_1-(\log N)^2/\sigma\sqrt{N},T_1'+(\log N)^2/\sigma\sqrt{N}]\},$$

where

$$\chi_1^{(N)}(\mathscr{S}_\xi)=\begin{cases}1 & \text{if } \xi\subset[0,a_1^{(N)}],\\ 0 & \text{otherwise,}\end{cases}$$

and $a_1^{(N)}=[t_1N]$. Combining this with Corollary 2.16, we obtain the above convergence. (q.e.d.)

In this section we will prove the following theorem which states a stronger fact than the above theorem.

Theorem 4.2. *Let* $\mathscr{C}_0\equiv C[0,1]=$ *the space of all continuous functions over* $[0,1]$, *with supremum norm as usual. Let* μ_N^* *be the distribution of* $\{Y_N(t;\lambda);\ 0\leqq t\leqq 1\}$ *derived from* P_N^*. *Then* μ_N^* *converges weakly to* $P_{0,0}^{1,0}$ *as* $N\to\infty$, *if* $\beta>\beta_0>0$ *are sufficiently large.*

Remark. Since we have proved the convergence of the finite-dimensional distribution of μ_N^*, we have only to show that the following conditions hold:

(4.3) there exist constants $M>0$ and $\gamma>0$ such that

$$E_N^*[|Y_N(0)|^\gamma\,|\,\delta\mathscr{S}_{\mathbf{X}}=0]\equiv\sum_{\lambda\in\tilde{\Lambda}_N}|Y_N(0;\lambda)|^\gamma P_N^*(\lambda\,|\,\tilde{\Lambda}_N)\leqq M$$

for sufficiently large N,

(4.4) there exist constants $L>0$, $\varepsilon>0$, $\delta>0$ such that

$$E_N^*[|Y_N(t)-Y_N(s)|^\varepsilon\,|\,\delta\mathscr{S}_{\mathbf{X}}=0]\leqq L|t-s|^{1+\delta}$$

for every $t,s\in[0,1]$, if N is sufficiently large.

(4.3) and (4.4) are well known conditions for the convergence of μ_N^* (see for example [1], p. 95).

Now we are going to check conditions (4.3) and (4.4). In our case (4.3) is trivial. Because we put $Y_N(0)\equiv 0$ for any λ and N. First we consider the case when $t=k/N$, $s=k'/N$ and for simplicity we assume that $k>k'$. Let $\Delta\equiv\{k',k'+1,\ldots,k\}$ and $\Delta^*\equiv[0,N]\backslash\Delta=\{0,1,\ldots,k'-1\}\cup\{k+1,\ldots,N\}$.

Lemma 4.5. *Assume that* $1\leqq|\Delta|\leqq N^{1/4}$. *Then* (4.4) *is true for* $\varepsilon=6$, $\delta=1/3$, *i.e.*

$$E_N^*[|Y_N(k/N)-Y_N(k'/N)|^6\,|\,\delta\mathscr{S}_{\mathbf{X}}=0]\leqq\text{Const.}\times(|\Delta|/N)^{4/3}$$

where $|\Delta|=k-k'$.

Proof. Assume first that $k=k'+1$. Then by Theorem 2.15,

$$\begin{aligned}
&P_N^*(|Y_N(k/N)-Y_N(k'/N)|=\ell/\sigma\sqrt{N} \mid \delta\mathscr{S}_{\mathbf{X}}=0)\\
&\quad\leqq P_N^*(\{\text{There exists } \mathscr{S}_\xi \text{ such that } \xi\ni k \text{ or } k' \text{ and } |\mathscr{S}_\xi|\geqq\ell\} \mid \delta\mathscr{S}_{\mathbf{X}}=0)\\
&\quad\leqq\{\sum_{\xi\ni k}\sum_{|\mathscr{S}_\xi|\geqq\ell}\rho_N^*(\mathscr{S}_\xi)+\sum_{\xi\ni k'}\sum_{|\mathscr{S}_\xi|\geqq\ell}\rho_N^*(\mathscr{S}_\xi)\}\times 2\sigma\sqrt{\pi N}
\end{aligned}$$

for sufficiently large N. (Use Corollary 3.18.)

Since for any $0\leqq k\leqq N$, by Theorem 2.15,

$$\begin{aligned}
\sum_{\xi\ni k}\sum_{|\mathscr{S}_\xi|\geqq\ell}\rho_N^*(\mathscr{S}_\xi)&\leqq\sum_{r=0}^{\infty}(r+1)\sum_{s\geqq(r+1)\vee\ell}3^{2s}e^{-\beta s}/[1-\tilde{k}(\beta,\beta_0)]\\
&\leqq\text{Const.}\times\ell(\ell+1)(9e^{-\beta})^\ell
\end{aligned}$$

we obtain that

$$\begin{aligned}
&E_N^*[|Y_N(k/N)-Y_N(k'/N)|^6 \mid \delta\mathscr{S}_{\mathbf{X}}=0]\\
&\quad\leqq\sum_{\ell=0}^{\infty}2\sqrt{\pi}/\sigma^5N^{5/2}\times 2\,\text{Const.}\,\ell(\ell+1)(9e^{-\beta})^\ell\\
&\quad\leqq\text{Const.}\times N^{-5/2}.
\end{aligned}$$

Hence if $0\leqq k-k'\leqq N^{1/4}$,

$$\begin{aligned}
&E_N^*[|X_N(k/N)-Y_N(k'/N)|^6 \mid \delta\mathscr{S}_{\mathbf{X}}=0]\\
&\quad\leqq E_N^*[(k-k')^5\cdot\sum_{j=1}^{k-k'}|Y_N((k+j-1)/N)-Y_N((k+j)/N)|^6 \mid \delta\mathscr{S}_{\mathbf{X}}=0]\\
&\quad\leqq|\Delta|^5\cdot|\Delta|\cdot\text{Const.}(1/N^{5/2})\\
&\quad=\text{Const.}\times(|\Delta|^6/N^{5/2})\leqq\text{Const.}\times(|\Delta|/N)^{4/3}. \quad \text{(q.e.d.)}
\end{aligned}$$

In the case when $|\Delta|\geqq N^{1/4}$, we must obtain better estimates for $P_N^*(|Y_N(k/N) - Y_N(k'/N)|=\ell/\sigma\sqrt{N} \mid \delta\mathscr{S}_{\mathbf{X}}=0)$. To do so, let us define functions J_Δ and G_Δ on $\mathscr{X}$ by

$$J_\Delta(\mathscr{S}_{\mathbf{X}})\equiv\sum_{\xi\in\mathbf{X}}\chi_\Delta(\mathscr{S}_\xi)\,\delta\mathscr{S}_\xi,\qquad G_\Delta(\mathscr{S}_{\mathbf{X}})\equiv\sum_{\xi\in\mathbf{X}}(1-\chi_\Delta(\mathscr{S}_\xi))\,\delta\mathscr{S}_\xi,$$

where

$$\chi_\Delta(\mathscr{S}_\xi)\equiv\begin{cases}1 & \text{if } \xi\subset\Delta\\ 0 & \text{otherwise.}\end{cases}$$

Lemma 4.6.

$$\begin{aligned}
&E_N^*[|Y_N(k/N)-Y_N(k'/N)|^6 \mid \delta\mathscr{S}_{\mathbf{X}}=0]\\
&\quad\leqq E_N^*[2^5|J_\Delta(\mathscr{S}_{\mathbf{X}})|^6/\sigma^6N^3 \mid \delta\mathscr{S}_{\mathbf{X}}=0]\\
&\qquad+\frac{2^{11}(\log N)^{12}}{\sigma^6N^2}\left(\frac{|\Delta|}{N}\right)^{4/3}+o\left(\frac{1}{N^\ell}\right)
\end{aligned}$$

as $N\to\infty$ for any positive integer ℓ.

Proof. Let $\tilde{P}_N^*(\cdot)$ be

$$\tilde{P}_N^*(\lambda) \equiv P_N^*(\lambda)/P_N^*(\tilde{\Lambda}_N) \quad \text{for every } \lambda \in \tilde{\Lambda}_N.$$

Then as Lemma 2.9, we can show that for any $\varepsilon > 0$,

$$\begin{aligned}
&\tilde{P}_N^*(\mathscr{S}_{\mathbf{X}} \in \tilde{\Lambda}_N;\ |\mathscr{S}_{\mathbf{X}}| \geqq \varepsilon N) \\
&\quad \leqq e^{\beta N} \sum_{n=(1+\varepsilon)N} 3^n e^{-(\beta - k(\beta,\beta_0))n} \\
&\quad = [1 - 3e^{-\beta + k(\beta,\beta_0)}]^{-1} \exp\{-\varepsilon\beta N + (1+\varepsilon)(k(\beta,\beta_0) + \log 3)\cdot N\}.
\end{aligned}$$

Now let $\varepsilon = 1 + \theta/N$. Then

$$\begin{aligned}
&\tilde{P}_N^*(\mathscr{S}_{\mathbf{X}} \in \tilde{\Lambda}_N;\ |\mathscr{S}_{\mathbf{X}}| \geqq N + \ell) \\
&\quad \leqq [1 - 3e^{-\beta + k(\beta,\beta_0)}]^{-1} \cdot \exp\{(N+\ell)(-\beta + 2k(\beta,\beta_0) + 2\log 3)\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\tilde{E}_N^*[|Y_N(k/N) - Y_N(k'/N)|^6;\ |\lambda| \geqq 2N] \\
&\quad \leqq \sum_{\ell=N}^{\infty} \frac{\ell^6}{\sigma^6 N^3} \tilde{P}_N^*(|Y_N(k/N) - Y_N(k'/N)| = \ell/\sigma\sqrt{N},\ |\lambda| \geqq 2N) \\
&\qquad + \frac{N^6}{\sigma^6 N^3} \tilde{P}_N^*(|Y_N(k/N) - Y_N(k'/N)| \leqq N/\sigma\sqrt{N},\ |\lambda| \geqq 2N) \\
&\quad \leqq 2\sigma\sqrt{\pi N} \sum_{\ell=N}^{\infty} \frac{\ell^6}{\sigma^6 N^3} [1 - 3e^{-\beta + k(\beta,\beta_0)}]^{-1} e^{\ell(-\beta + 2k(\beta,\beta_0) + 2\log 3)} \\
&\qquad + 2\sigma\sqrt{\pi N} \frac{N^6}{\sigma^6 N^3} [1 - 3e^{-\beta + k(\beta,\beta_0)}]^{-1} e^{N(-\beta + 2k(\beta,\beta_0) + 2\log 3)} \\
&\quad = o(1/N^{\ell}) \quad \text{as } N \to \infty \text{ for any positive integer } \ell.
\end{aligned}$$

Moreover, from Lemma 2.9, we obtain

$$\begin{aligned}
&\tilde{E}_N^*[|Y_N(k/N) - Y_N(k'/N)|^6;\ |\lambda| \leqq 2N] \\
&\quad \leqq \tilde{E}_N^*[|Y_N(k/N) - Y_N(k'/N)|^6;\ |\lambda| \leqq 2N,\ |\mathscr{S}_\xi| \leqq (\log N)^2 \\
&\qquad \text{for any } \mathscr{S}_\xi \in \lambda] + o(1/N^{\ell})
\end{aligned}$$

as $N \to \infty$ for any positive integer ℓ.

Finally if $|\lambda| \leqq 2N$ and $|\mathscr{S}_\xi| \leqq (\log N)^2$ for any $\mathscr{S}_\xi \in \lambda$, then

$$\begin{aligned}
|Y_N(k/N) - Y_N(k'/N)|^6 &\leqq \frac{1}{\sigma^6 N^3} [|J_\Delta(\mathscr{S}_{\mathbf{X}})| + 2(\log N)^2]^6 \\
&\leqq \frac{2^5}{\sigma^6 N^3} [|J_\Delta(\mathscr{S}_{\mathbf{X}})|^6 + 2^6(\log N)^{12}]
\end{aligned}$$

which proves the lemma. (q.e.d.)

Now it is sufficient to estimate the probability.

$$\tilde{P}_N^*(J_\Delta(\mathscr{S}_{\mathbf{X}}) = k) = P_N^*(J_\Delta(\mathscr{S}_{\mathbf{X}}) = k,\ G_\Delta(\mathscr{S}_{\mathbf{X}}) = -k)/P_N^*(\delta\mathscr{S}_{\mathbf{X}} = 0)$$

for every k.

Since

$$P_N^*(J_\Delta(\mathscr{S}_{\mathbf{X}})=k,\ G_\Delta(\mathscr{S}_{\mathbf{X}})=-k)$$
$$=(2\pi)^{-2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\langle e^{iy_1J_\Delta(\mathscr{S}_{\mathbf{X}})+iy_2G_\Delta(\mathscr{S}_{\mathbf{X}})}\rangle_N^* e^{-iy_1k+iy_2k}\,dy_1\,dy_2$$

we can estimate the above probability by the same argument as we used in Sect. 3. The only different part is; since $|\Delta|$ may be $o(N)$ as $N\to\infty$, we can not apply Lemma 3.4 in this case. But we can use Lemma 3.9 since we need only the upper estimate.

Hence, we obtain the following lemma.

Lemma 4.7. *If* $N/2\geqq|\Delta|\geqq N^{1/4}$, *then*

$$P_N^*(J_\Delta(\mathscr{S}_{\mathbf{X}})=k,\ G_\Delta(\mathscr{S}_{\mathbf{X}})=-k)$$
$$=\frac{1}{4\pi\sqrt{\alpha_N\cdot\delta_N}}e^{-\frac{k^2}{4}(1/\alpha_N+(1+\beta_N/\alpha_N)^2/\delta_N)}$$
$$+O\left(\frac{(\log N)^6}{\sqrt{(|\Delta|+1)(|\Delta^*|+1)}}\times((|\Delta|+1)^{-1}+(|\Delta^*|+1)^{-1})\right)$$

as $N\to\infty$, *where*

$$\alpha_N\equiv\sum_{\mathscr{S}_{\mathbf{X}}\in\mathscr{X};\,\mathbf{X}\cap\Delta\neq0}\phi_N^*(\mathscr{S}_{\mathbf{X}})e^{-2(\beta-\beta_0)|\mathscr{S}_{\mathbf{X}}|}|J_\Delta(\mathscr{S}_{\mathbf{X}})|^2,$$
$$\beta_N\equiv\sum_{\substack{\mathscr{S}_{\mathbf{X}}\in\mathscr{X};\,\mathbf{X}\cap\Delta\neq0\\ \mathbf{X}\cap\Delta^*\neq0}}\phi_N^*(\mathscr{S}_{\mathbf{X}})e^{-2(\beta-\beta_0)|\mathscr{S}_{\mathbf{X}}|}J_\Delta(\mathscr{S}_{\mathbf{X}})G_\Delta(\mathscr{S}_{\mathbf{X}}),$$
$$\gamma_N\equiv\sum_{\mathscr{S}_{\mathbf{X}}\in\mathscr{X};\,\mathbf{X}\cap\Delta^*\neq0}\phi_N^*(\mathscr{S}_{\mathbf{X}})e^{-2(\beta-\beta_0)|\mathscr{S}_{\mathbf{X}}|}|G_\Delta(\mathscr{S}_{\mathbf{X}})|^2,$$

and

$$\delta_N\equiv\gamma_N-\beta_N^2/\alpha_N\geqq M_1(\beta,\beta_0)(|\Delta^*|+1),\quad \alpha_N-\beta_N^2/\gamma_N\geqq M_2(\beta,\beta_0)(|\Delta|+1)$$

for some constants $M_1(\beta,\beta_0)$, $M_2(\beta,\beta_0)>0$.

Proof. The main idea is the same as in § 3, so we briefly sketch the proof.

Just in the same way as in the proof of Lemma 3.11, we obtain

$$(4.8)\quad \langle e^{iy_1J_\Delta(\mathscr{S}_{\mathbf{X}})+iy_2G_\Delta(\mathscr{S}_{\mathbf{X}})}\rangle_N^*=o(1/N^\ell)$$

as $N\to\infty$ for any positive integer ℓ, if $|y_1|$ or $|y_2|\geqq1$, and

$$(4.9)\quad \int_{\frac{\log N}{\sqrt{|\Delta|+1}}\leqq|y_1|\leqq1}\int_{|y_2|\leqq1}\langle e^{iy_1J_\Delta(\mathscr{S}_{\mathbf{X}})+iy_2G_\Delta(\mathscr{S}_{\mathbf{X}})}\rangle_N^* e^{-iy_1k+iy_2k}\,dy_1\,dy_2$$
$$=o(1/N^\ell),$$

$$(4.9)'\quad \int_{|y_1|\leqq1}\int_{\frac{\log N}{\sqrt{|\Delta^*|+1}}\leqq|y_2|\leqq1}\{\ \}\,dy_1\,dy_2=o(1/N^\ell)$$

as $N\to\infty$ for any positive integer ℓ, where $\{\ \}$ is the same integrand as in (4.9).

Now let

$$A \equiv \int_{|y_1| \leqq \frac{\log N}{\sqrt{|\Delta|+1}}} \int_{|y_2| \leqq \frac{\log N}{\sqrt{|\Delta^*|+1}}} \{\ \} \, dy_1 \, dy_2,$$

and

$$A' \equiv \iint \exp\{-\sum_{\mathscr{S}_{\mathbf{X}} \in \mathscr{X};\, \mathbf{X} \subset [0,N]} \phi_N^*(\mathscr{S}_{\mathbf{X}}) \, e^{-2(\beta-\beta_0)|\mathscr{S}_{\mathbf{X}}|} (f_N(\mathscr{S}_{\mathbf{X}}))^2/2$$

$$-iy_1 k + iy_2 k\} \, dy_1 \, dy_2,$$

where the integration range is the same as in the definition of A. Taylor's theorem and Theorem 2.13, (iii) together with the estimate;

$$\begin{aligned}
&|\sum_{\mathscr{S}_{\mathbf{X}} \in \mathscr{X},\, \mathbf{X} \subset [0,N]} \phi_N^*(\mathscr{S}_{\mathbf{X}}) \cdot e^{-2(\beta-\beta_0)|\mathscr{S}_{\mathbf{X}}|} (f_N(\mathscr{S}_{\mathbf{X}}))^4| \\
&\quad \leqq 2^3 \sum_{\mathscr{S}_{\mathbf{X}} \in \mathscr{X},\, \mathbf{X} \subset [0,N]} |\phi_N^*(\mathscr{S}_{\mathbf{X}})| \, e^{-2(\beta-\beta_0)|\mathscr{S}_{\mathbf{X}}|} \{y_1^4 J_\Delta(\mathscr{S}_{\mathbf{X}})^4 + y_2^4 G_\Delta(\mathscr{S}_{\mathbf{X}})^4\} \\
&\quad \leqq 2^3 \{y_1^4(|\Delta|+1) + y_2^4(|\Delta^*|+1)\} \, e^{-2\beta}(2 + 2^4 e^{-2(\beta-\beta_0)} C(\beta,\beta_0)) \\
&\quad \leqq 2^4 e^{-2\beta}(1 + 8e^{-2(\beta-\beta_0)} C(\beta,\beta_0)) \left[\frac{(\log N)^4}{|\Delta|+1} + \frac{(\log N)^4}{|\Delta^*|+1}\right]
\end{aligned}$$

for $|y_1| \leqq (\log N)/\sqrt{|\Delta|+1}$, $|y_2| \leqq (\log N)/\sqrt{|\Delta^*|+1}$, imply that

$$|A - A'| \leqq \text{Const.} \times \frac{(\log N)^6}{\sqrt{(|\Delta|+1)(|\Delta^*|+1)}} \left[\frac{1}{|\Delta|+1} + \frac{1}{|\Delta^*|+1}\right]$$

for sufficiently large N. Obviously the constant depends only on β, β_0.

Since

$$\begin{aligned}
A' = \frac{1}{\sqrt{(|\Delta|+1)(|\Delta^*|+1)}} \int_{|y_1|,\, |y_2| \leqq \log N} \exp\Bigg\{ &-\frac{\alpha_N}{|\Delta|+1} y_1^2 \\
&-2\frac{\beta_N}{\sqrt{(|\Delta|+1)(|\Delta^*|+1)}} y_1 y_2 - \frac{\gamma_N}{|\Delta^*|+1} y_2^2 \\
&-\frac{iky_1}{\sqrt{|\Delta|+1}} + \frac{iky_2}{\sqrt{|\Delta^*|+1}}\Bigg\} \, dy_1 \, dy_2,
\end{aligned}$$

if we take

$$A'' \equiv \frac{1}{\sqrt{(|\Delta|+1)(|\Delta^*|+1)}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(y_1, y_2) \, dy_1 \, dy_2,$$

where $F(y_1, y_2)$ is the same integrand function as in A', it would be easy to see that $|A' - A''| = o(1/N^{\ell})$ as $N \to \infty$ for any positive integer ℓ provided that $\alpha_N - \beta_N^2/\gamma_N \geqq M_1(|\Delta|+1)$ and $\gamma_N - \beta_N^2/\alpha_N \geqq M_2(|\Delta^*|+1)$ for some positive constants M_1, M_2.

Since

$$A'' = \frac{\pi}{\sqrt{\alpha_N \cdot \delta_N}} e^{-\frac{k^2}{4}(\alpha_N^{-1} + (1+\beta_N/\alpha_N)^2/\delta_N)},$$

it suffices to show the existence of above M_1 and M_2. By using Lemma 3.9, it is easy to see that

$$(4.10)\quad 2e^{-2\beta-3e^{-8\beta}}(1-2e^{3e^{-8\beta}-2(\beta-\beta_0)}) \leqq \alpha_N/(|\Delta|+1),\ \gamma_N/(|\Delta^*|+1) \leqq 2e^{-2\beta}(1+2e^{-2(\beta-\beta_0)}),$$

and

$$(4.11)\quad |\beta_N| \leqq \sum_{\substack{\mathbf{X}\cap\Delta\neq 0\\ \mathbf{X}\cap\Delta^*\neq 0}} \sum_{|\mathscr{S}_\mathbf{X}|\geqq 2} |\phi_N^*(\mathscr{S}_\mathbf{X})| \cdot \sup_{\substack{k+j\geqq 2\\ k\geqq 0, j\geqq 0}} kje^{-2(k+j)(\beta-\beta_0)} \leqq \{(|\Delta|+1)\wedge(|\Delta^*|+1)\}\, e^{-2\beta-2(\beta-\beta_0)}.$$

From (4.10) and (4.11) we obtain $M_1(\beta,\beta_0)>0$ and $M_2(\beta,\beta_0)>0$. (q.e.d.)

Corollary 4.12. *Assume that* $N/2\geqq|\Delta|\geqq N^{1/4}$. *Then*

$$\sum_{k=0}^{|\Delta|^{3/5}} \frac{k^6}{\sigma^6 N^3} \tilde{P}_N^*(|J_\Delta(\mathscr{S}_\mathbf{X})|=k) \leqq \text{Const.}\times(|\Delta|/\mathrm{N})^3.$$

The proof would be obvious from Lemma 4.7.

Corollary 4.13. *Assume that* $|\Delta|\geqq|\Delta^*|\geqq N^{1/4}$. *Then*

$$\sum_{k=0}^{|\Delta^*|^{3/5}} \frac{k^6}{\sigma^6 N^3} \tilde{P}_N^*(|J_\Delta(\mathscr{S}_\mathbf{X})|=k) \leqq \text{Const.}(|\Delta|/N)^3.$$

Proof. Since we can change the roles of J_Δ and G_Δ, we obtain

$$\sum_{k=0}^{|\Delta^*|^{3/5}} \frac{k^6}{\sigma^6 N^3} \tilde{P}_N^*(|J_\Delta(\mathscr{S}_\mathbf{X})|=k) \leqq \text{Const.}\times(|\Delta^*|/N)^3 \leqq \text{Const.}\times(|\Delta|/N)^3. \quad \text{(q.e.d.)}$$

Now we have to estimate $\tilde{P}_N^*(|J_\Delta(\mathscr{S}_\mathbf{X})|=k)$ for $k\geqq N^{3/5}$. This leads us to the calculation of the probability of large deviation. We will carry out this calculation in the next section.

§ 5. Probability of Large Deviation for $|J_\Delta(\mathscr{S}_\mathbf{X})|$

Since

$$\tilde{P}_N^*(J_\Delta(\mathscr{S}_\mathbf{X})=k) \leqq \frac{P_N^*(J_\Delta(\mathscr{S}_\mathbf{X})=k)}{P_N^*(\delta\mathscr{S}_\mathbf{X}=0)} = \text{const.}\times\sqrt{N}\cdot P_N^*(J_\Delta(\mathscr{S}_\mathbf{X})=k),$$

we estimate $P_N^*(J_\Delta(\mathscr{S}_\mathbf{X})=k)$ for $k\geqq|\Delta|^{3/5}$.

Let

$$P_{N,k} \equiv P_N^*(J_\Delta(\mathscr{S}_\mathbf{X})=|\Delta|^{3/5}+k) \qquad \text{for } k\geqq 0.$$

Then

$$(5.1)\quad P_{N,k}=\frac{1}{2\pi}\int_{-\pi}^{\pi}\langle e^{iyJ_\Delta(\mathscr{S}_\mathbf{X})}\rangle_N^* e^{-iy(|\Delta|^{3/5}+k)}\,dy$$
$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp\{|\Delta|\,h(iy)-k\cdot(iy)\}\,dy,$$

where

$$(5.2)\quad h(z)=\frac{1}{|\Delta|}\sum_{\mathscr{S}_\mathbf{X}\in\mathscr{X},\,\mathbf{X}\subset[0,N]}\phi_N^*(\mathscr{S}_\mathbf{X})\,e^{-2(\beta-\beta_0)|\mathscr{S}_\mathbf{X}|}(e^{zJ_\Delta(\mathscr{S}_\mathbf{X})}-1)$$
$$-|\Delta|^{-2/5}z.$$

Since $h(z)$ is analytic if $|z|<2(\beta-\beta_0)$, we have

$$(5.3)\quad h'(0)=-|\Delta|^{-2/5}<0,\qquad h'''(0)=0,$$
$$h'(z)=-|\Delta|^{-2/5}+h''(0)\cdot z+\frac{h^{(4)}(0)}{3!}z^3+O(z^4)\quad\text{as}\quad z\to 0.$$

From Lemma 3.9, we obtain that

$$(5.4)\quad h''(0)=\frac{1}{|\Delta|}\sum_{\mathscr{S}_\mathbf{X}\in\mathscr{X},\,\mathbf{X}\subset[0,N]}\phi_N^*(\mathscr{S}_\mathbf{X})\,e^{-2(\beta-\beta_0)|\mathscr{S}_\mathbf{X}|}(J_\Delta(\mathscr{S}_\mathbf{X}))^2$$
$$\geqq e^{-2\beta}(e^{-3e^{-8\beta}}-4e^{-2(\beta-\beta_0)})>0,$$

and $h^{(4)}(0)\geqq e^{-2\beta}(e^{-3e^{-8\beta}}-16e^{-2(\beta-\beta_0)})>0.$

Let $z_0\leqq\min\{z>0;\,h'(z)=0\}$. Then from (5.3) and (5.4),

$$(5.5)\quad z_0=\frac{1}{h''(0)}|\Delta|^{-2/5}(1+o(1))\quad\text{as}\quad N\to\infty.$$

Using the analyticity of $h(z)$, we have

$$P_{N,k}=\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp\{|\Delta|\,h(iy)-k\cdot(iy)\}\,dy$$
$$=\frac{1}{2\pi}\int_{z_0-i\pi}^{z_0+i\pi}\exp\{|\Delta|\,h(z)-kz\}\,dz$$
$$=\frac{1}{2\pi i}\int_{-\pi}^{\pi}\exp\{|\Delta|\,h(z_0+it)-k\cdot(z_0+it)\}\,dt$$
$$\geqq\exp\{|\Delta|\,h(z_0)-kz_0\}\times I.$$

We decompose I into two parts I_1, I_2;

$$2\pi|I|=\Big|\int_{1/4\leqq|t|\leqq\pi}\Big|+\Big|\int_{|t|\leqq 1/4}\Big|=I_1+I_2.$$

Lemma 5.6. *If $\beta-\beta_0$ is sufficiently large, then*

$$|I_1|\leqq 2(\pi-1/4).$$

Proof. Since

$$I_1 = \int_{\pi \geqq |t| \geqq 1/4} dt \exp\{ \sum_{\mathscr{S}_{\mathbf{X}} \in \mathscr{X}, \mathbf{X} \subset [0,N]} \phi_N^*(\mathscr{S}_{\mathbf{X}}) \mathrm{e}^{-2(\beta-\beta_0)|\mathscr{S}_{\mathbf{X}}|} \times e^{z_0 J_\Delta(\mathscr{S}_{\mathbf{X}})}(e^{itJ_\Delta(\mathscr{S}_{\mathbf{X}})} - 1) - ikt\},$$

it suffices to show that $|\text{integrand}| \leqq 1$.

$$\begin{aligned}
&|\text{integrand}| \\
&= \exp\{\mathrm{Re} \sum_{\mathscr{S}_{\mathbf{X}} \in \mathscr{X}, \mathbf{X} \cap \Delta = 0} \phi_N^*(\mathscr{S}_{\mathbf{X}}) e^{-2(\beta-\beta_0)|\mathscr{S}_{\mathbf{X}}|} e^{z_0 J_\Delta(\mathscr{S}_{\mathbf{X}})}(e^{itJ_\Delta(\mathscr{S}_{\mathbf{X}})} - 1)\} \\
&= \exp\{ \sum_{\mathscr{S}_{\mathbf{X}} \in \mathscr{X}, \mathbf{X} \cap \Delta = 0} \phi_N^*(\mathscr{S}_{\mathbf{X}}) e^{-2(\beta-\beta_0)|\mathscr{S}_{\mathbf{X}}|} e^{z_0 J_\Delta(\mathscr{S}_{\mathbf{X}})}(\cos J_\Delta(\mathscr{S}_{\mathbf{X}})\, y - 1)\} \\
&\leqq \exp\{-2(|\Delta|+1) e^{-2\beta}[e^{-3e^{-8\beta}} \sin^2(1/8) - e^{2-2(\beta-\beta_0)}]\}
\end{aligned}$$

from Lemma 3.8 and (5.5). Obviously, (5.5) depends on $|\Delta| \geqq N^{1/4}$, so the above inequality holds for sufficiently large N. (q.e.d.)

Lemma 5.7. *If $\beta - \beta_0$ is sufficiently large, then*

$$|I_2| \leqq 1/2 \quad \text{as} \quad N \to \infty.$$

Proof. This time we need the Taylor Theorem for $e^{iyJ_\Delta(\mathscr{S}_{\mathbf{X}})}$, and use Lemma 3.9 to obtain

$$\begin{aligned}
&|\text{integrand}| \\
&\leqq \exp\{-2e^{-2\beta}(|\Delta|+1)[e^{-3e^{-8\beta}} - \frac{1}{6 \cdot 4^2}\{e + 8e^{2-2(\beta-\beta_0)}\} - 2e^{2-2(\beta-\beta_0)}]\} \\
&< 1. \quad \text{(q.e.d.)}
\end{aligned}$$

Theorem 5.8. *Assume that $|\Delta| \geqq |\Delta^*| \geqq N^{1/4}$. If $\beta - \beta_0$, β_0 are sufficiently large, then for large enough N,*

$$P_{N,k} \leqq \exp\left\{ -\frac{|\Delta|^{1/5}}{4h''(0)} - k\frac{|\Delta|^{-2/5}}{2h''(0)} \right\}$$

Proof. From Lemmas 5.6, 5.7

$$P_{N,k} \leqq \exp\{-|\Delta| h(z_0) - kz_0\}.$$

But from (5.5), $z_0 \geqq \frac{1}{2h''(0)} |\Delta|^{-2/5}$ for large enough N. Substituting this into

$$h(z) = -|\Delta|^{-2/5} z + \frac{h''(0)}{2} z^2 + O(z^4) \quad \text{as} \quad z \to 0,$$

and noting that $h(z_0) < h\left(\frac{1}{2h''(0)} |\Delta|^{-2/5}\right)$ since $h(z)$ is decreasing for $0 \leqq z \leqq z_0$, we obtain the desired result. (q.e.d.)

Corollary 5.9. *Assume that* $|\Delta^*| \geqq |\Delta| \geqq N^{1/4}$. *Then*

$$\sum_{k=|\Delta|^{3/5}}^{\infty} \left(\frac{k}{\sigma\sqrt{N}}\right)^6 \tilde{P}_N^*(J_\Delta(\mathscr{S}_{\mathbf{X}})=k)=o\left(\frac{1}{N^\ell}\right)$$

as $N \to \infty$ *for any positive integer* ℓ.

Corollary 5.10. *Assume that* $|\Delta| \geqq |\Delta^*| \geqq N^{1/4}$. *Then*

$$\sum_{k=|\Delta^*|^{3/5}}^{\infty} \left(\frac{k}{\sigma\sqrt{N}}\right)^6 \tilde{P}_N^*(|J_\Delta(\mathscr{S}_{\mathbf{X}})|=k)=o\left(\frac{1}{N^\ell}\right)$$

as $N \to \infty$ *for any positive integer* ℓ.

Proof. Since $\tilde{P}_N^*(J_\Delta(\mathscr{S}_{\mathbf{X}})=k)=\tilde{P}_N^*(G_\Delta(\mathscr{S}_{\mathbf{X}})=-k)$, we consider $G_\Delta(\mathscr{S}_{\mathbf{X}})$ instead of $J_\Delta(\mathscr{S}_{\mathbf{X}})$ and

$$h^*(z) \equiv \frac{1}{|\Delta^*|} \sum_{\mathscr{S}_{\mathbf{X}} \in \mathscr{X}, \mathbf{X} \subset [0,N]} \phi_N^*(\mathscr{S}_{\mathbf{X}}) \times e^{-2(\beta-\beta_0)|\mathscr{S}_{\mathbf{X}}|}(e^{zG_\Delta(\mathscr{S}_{\mathbf{X}})}-1)-|\Delta^*|^{-2/5} z$$

instead of $h(z)$. Then by the same argument as we did in this section, we obtain the desired result. (q.e.d.)

Lemma 5.11. *Assume that* $\beta > \beta_0 > 0$ *are sufficiently large. If* $0 \leqq k \leqq N^{1/4}$, $N - N^{1/4} \leqq k' \leqq N$, *then*

$$\tilde{E}_N^*\left[\left|Y_N\left(\frac{k}{N}\right)-Y_N\left(\frac{k'}{N}\right)\right|^6\right]=\left(\frac{|\Delta|}{N}\right)^{4/3} \times O\left(\frac{1}{N}\right)$$

as $N \to \infty$.

Proof. $\tilde{E}_N^*\left[\left|Y_N\left(\frac{k}{N}\right)-Y_N\left(\frac{k'}{N}\right)\right|^6\right]$

$$2^5 \cdot \tilde{E}_N^*\left[\left|Y_N\left(\frac{k}{N}\right)\right|^6+\left|Y_N\left(\frac{k'}{N}\right)\right|^6\right].$$

Since $\tilde{P}_N^*[Y_N(0)=Y_N(1)=0]=1$, we obtain by the same argument as in the proof of Lemma 4.5,

$$\tilde{E}_N^*\left[\left|Y_N\left(\frac{k}{N}\right)\right|^6\right] \leqq \text{Const.} \times \left(\frac{k}{N}\right)^{4/3},$$

$$\tilde{E}_N^*\left[\left|Y_N\left(\frac{k'}{N}\right)\right|^6\right] \leqq \text{Const.} \times \left(\frac{N-k'}{N}\right)^{4/3}.$$

Hence

$$\tilde{E}_N^*\left[\left|Y_N\left(\frac{k}{N}\right)-Y_N\left(\frac{k'}{N}\right)\right|^6\right]\leqq \text{Const.}\times\left(\frac{k+(N-k')}{N}\right)^{4/3}$$
$$=\text{Const.}\times\left(1-\frac{|\Delta|}{N}\right)^{4/3}=\left(\frac{|\Delta|}{N}\right)^{4/3}\cdot o\left(\frac{1}{N}\right)$$

as $N\to\infty$. (q.e.d.)

Combining Lemmas 4.5, 4.6, 5.11 and Corollaries 4.12, 4.13, 5.9, 5.10, we obtain the estimate (4.4) for $t=\frac{k}{N}$, $s=\frac{k'}{N}$, $k, k'\in\{0,1,2,\dots,N\}$. From this, it is very easy to get (4.4) for every $s,t\in[0,1]$. Thus, Theorem 4.2 is proved.

§6. Proof of Theorem 1.2

Note that the relative topology of $\mathscr{C}_0$ induced from $\mathscr{C}$ coincides with the topology of the supremum norm in $\mathscr{C}_0$.

Let f be a bounded uniformly continuous function on $\mathscr{C}$. Then $f|_{\mathscr{C}_0}$ is also uniformly continuous.

Define $\pi_N\colon \mathscr{C}\to\mathscr{C}_0$ by

$$[\pi_N(c)]\left(\frac{k}{N}\right)=\begin{cases}0 & \text{if } k=0,N\\ \max\left\{y\in\mathbf{R}^1;\left(\frac{k}{N},y\right)\in c\right\} & \text{if } k=1,2,\dots,N-1\end{cases}$$

$$[\pi_N(c)](t)=[\pi_N(c)]\left(\frac{k}{N}\right)+\left(t-\frac{k}{N}\right)\left\{[\pi_N(c)]\left(\frac{k+1}{N}\right)-[\pi_N(c)]\left(\frac{k}{N}\right)\right\}$$
$$\text{for } k/N\leqq t\leqq(k+1)/N.$$

Then Theorem 4.2 states that

$$(6.1)\qquad \int_{\mathscr{C}} f(\pi_N(c))(\tilde{P}_N^*\circ\mathscr{A}_N^{-1})(dc)\xrightarrow[N\to\infty]{}\int_{\mathscr{C}_0} f(c)\,P_{0,0}^{1;0}(dc)$$

because the left hand side is equal to

$$\int_{\mathscr{A}_N\hat{\Lambda}_N} f(\pi_N(c))(\tilde{P}_N^*\circ\mathscr{A}_N^{-1})(dc)=\int_{\mathscr{C}_0} f(c)\,\mu_N^*(dc)$$

by the definition of μ_N^*.

From Corollary 2.16, we obtain that

$$\left|\int_{\mathscr{C}} f(\pi_N(c))(\tilde{P}_N^*\circ\mathscr{A}_N^{-1})(dc)-\int_{\mathscr{C}} f(c)(\tilde{P}_N^*\circ\mathscr{A}_N^{-1})(dc)\right|$$
$$\leqq\int_{\mathscr{A}_N\hat{\Lambda}_N}|f(\pi_N(c))-f(c)|(\tilde{P}_N^*\circ\mathscr{A}_N^{-1})(dc)$$
$$\leqq\int_{\{c\in\mathscr{A}_N\hat{\Lambda}_N;\,\bar{\rho}(\pi_N(c),c)\leqq\varepsilon\}}+\int_{\{c\in\mathscr{A}_N\hat{\Lambda}_N;\,\bar{\rho}(\pi_N(c),c)>\varepsilon\}}$$

$$\leqq \sup_{\bar{\rho}(c,c')\leqq\varepsilon} |f(c)-f(c')|$$
$$+2\{\sup_{c\in\mathscr{C}} |f(c)|\}\times(\tilde{P}_N^*\circ\mathscr{A}_N^{-1})(\{c\in\mathscr{C};\bar{\rho}(\pi_N(c),c)>\varepsilon\})$$
$$\to \sup_{\bar{\rho}(c,c')\leqq\varepsilon} |f(c)-f(c')| \quad \text{as} \quad N\to\infty.$$

Letting $\varepsilon\to 0$, we obtain that

$$(6.2) \quad \int_{\mathscr{C}} f(c)(\tilde{P}_N^*\circ\mathscr{A}_N^{-1})(dc)\to\int_{\mathscr{C}_0} f(c)P_{0,0}^{1,0}(dc) \quad \text{as} \quad N\to\infty.$$

We used the fact that if $|\mathscr{S}_\xi|<(\log N)^2$ for any $\mathscr{S}_\xi\in\lambda$, then

$$\bar{\rho}(\mathscr{A}_N\lambda,\pi_N(\mathscr{A}_N\lambda))=\bar{\rho}(\mathscr{A}_N\lambda,Y_N(\cdot,\lambda))\leqq(\log N)^2/\sigma\sqrt{N}.$$

From (6.2), we can find the unique probability measure μ on $\mathscr{C}$ such that

$$\tilde{P}_N^*\circ\mathscr{A}_N^{-1}\to\mu \qquad \text{weakly as } N\to\infty.$$

If $\mathscr{C}_0$ is measurable in $(\mathscr{C},\bar{\rho})$, then $\mu=P_{0,0}^{1,0}$. We will give the proof of the measurability of $\mathscr{C}_0$ in the appendix. Here, we assume the measurability.

From (2.2) and (2.12) we obtain for any bounded uniformly continuous f on $\mathscr{C}$,

$$\left|\int_{\mathscr{C}} f(c)(\tilde{P}_N^*\circ\mathscr{A}_N^{-1})(dc)-\int_{\mathscr{C}} f(c)(P_N\circ\mathscr{A}_N^{-1})(dc)\right|$$
$$=\left|\sum_{\lambda\in\Lambda_N} f(\mathscr{A}_N\lambda)(\tilde{P}_N^*(\lambda)-P_N(\lambda))\right|=o(1/N^\ell)$$

as $N\to\infty$, for any positive integer ℓ. Hence we have

$$\int_{\mathscr{C}} f(c)(P_N\circ\mathscr{A}_N^{-1})(dc)\to\int_{\mathscr{C}_0} f(c)P_{0,0}^{1,0}(dc)=\int_{\mathscr{C}} f(c)P_{0,0}^{1,0}(dc)$$

as $N\to\infty$. Thus, using Skorokhod's theorem [7], we proved Theorem 1.2. The idea appeared in this section is due to T. Shiga.

Appendix. Measurability of $\mathscr{C}_0$ in $(\mathscr{C},\bar{\rho})$

Lemma A.1. *Let G be an open set in $\mathbf{R}^1$. For any $c\in\mathscr{C}$, $t\in[0,1]$, let*

$$c_t\equiv\{x\in\mathbf{R}^1;(t,x)\in c\}\subset\mathbf{R}^1.$$

Then $\Lambda(t;G)\equiv\{c\in\mathscr{C};c_t\subset G\}$ is an open set in $(\mathscr{C},\bar{\rho})$.

Proof. For any $c\in\Lambda(t;G)$, $c\cap\{(t,x)\in\mathbf{R}^2;x\in G^c\}=\emptyset$. Since c is compact

$$\sup_{p\in c,}\ \inf_{q\in\{(t,x)\in\mathbf{R}^2;x\in G^c\}} |p-q|\geqq\delta_1>0$$

and

$$\sup_{q\in\{(t,x)\in\mathbf{R}^2;x\in G^c\},\ p\in c} \inf |p-q| \geqq \delta_2 > 0$$

Hence if $\delta \equiv \delta_1 \wedge \delta_2$,

$$U(c, \delta/2) \equiv \{c' \in \mathscr{C}; \bar{\rho}(c, c') < \delta/2\} \subset \Lambda(t; G). \quad \text{(q.e.d.)}$$

Lemma A.2. *For any* $t \in [0, 1]$, *let*

$$S_t \equiv \{c \in \mathscr{C};\ c_t \text{ consists of one point}\}$$

and

$$S \equiv \bigcap_{t\in\mathbf{Q}\cap[0,1]} S_t$$

where $\mathbf{Q}$ *is the set of all rational numbers. Then* S_t *and* S *are measurable, i.e.*

$$S_t, S \in \sigma\{\text{open sets in } (\mathscr{C}, \bar{\rho})\} \equiv \mathscr{B}(\mathscr{C})$$

Proof. Note that

$$S_t = \bigcap_{n=1}^{\infty} \bigcup_{k\in\mathbf{Z}^1} \{c \in \mathscr{C}; c_t \subset ((k-\tfrac{2}{3})\cdot 2^{-n}, (k+\tfrac{2}{3})\cdot 2^{-n})\} \quad \text{(q.e.d.)}$$

Let us fix $n \geqq 1$, $0 \leqq k \leqq 2^n - 1$, $\varepsilon > 0$ and $\delta > 0$, and define $r_{n,k} \equiv \dfrac{k}{2^n}$,

$$\Delta_{n,k}(\varepsilon) \equiv \{c \in \mathscr{C}; \bar{\rho}_1(c_{r_{n,k}}, c_{r_{n,k+1}}) > \varepsilon\},$$

where $\bar{\rho}_1(A, B)$ is a distance between compact sets A, B in $\mathbf{R}^1$,

$$\bar{\rho}_1(A, B) \equiv \tfrac{1}{2}\{\sup_{p\in A} \inf_{q\in B} |p-q| + \sup_{q\in B} \inf_{p\in A} |p-q|\},$$

and

$$\bar{\Delta}^{\delta}_{n,k}(\varepsilon) \equiv [\bigcup_{c\in\Delta_{n,k}(\varepsilon)} U(c, \delta)] \cap S.$$

$\bar{\Delta}^{\delta}_{n,k}(\varepsilon)$ is measurable for any n, k, ε, δ.

Lemma A.3. *Let* $\delta_n \equiv 2^{-n}$. *Then*

$$S \cap \{c \in \mathscr{C};\ \text{there exists some } t \in [0,1] \text{ such that } c_t \text{ is an interval of } \mathbf{R}^1\}$$
$$= \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{k=0}^{2^n-1} \bar{\Delta}^{\delta_n}_{n,k}\left(\frac{1}{m}\right) \in \mathscr{B}(\mathscr{C}).$$

Proof. Let $c \in \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{k=0}^{2^n-1} \bar{\Delta}^{\delta_n}_{n,k}\left(\frac{1}{m}\right)$. Then there exists $m > 0$ such that for any $n \geqq 1$, we can find $c^{(n)} \in S$ and $r_{n,k_n} \in [0, 1]$ such that

$$\bar{\rho}_1(c^{(n)}_{r_{n,k_n}}, c^{(n)}_{r_{n,k_n+1}}) > \frac{1}{m} \quad \text{and} \quad \bar{\rho}(c, c^{(n)}) < \delta_n.$$

Hence we can find r_n, $r'_n \in \mathbf{Q}$ such that

$$r_{n,k_n-1} \leqq r_n \leqq r_{n,k_n+1}, r_{n,k_n} \leqq r'_n \leqq r_{n,k_n+2}$$

and

$$\bar{\rho}_1(c^{(n)}_{r_{n,k_n}}, c_{r_n}) < \delta_n, \qquad \bar{\rho}_1(c^{(n)}_{r_{n,k_n+1}}, c_{r'_n}) < \delta_n.$$

From this, we have

$$\text{(A.4)} \quad \bar{\rho}_1(c_{r_n}, c_{r'_n}) > \frac{1}{m} - 2\delta_n.$$

Since $0 \leqq r_{n,k_n} \leqq 1$, $\{r_{n,k_n}\}_{n=1}^{\infty}$ has an accumulating point, say r. For simplicity let us assume that $r_{n,k_n} \to r$ as $n \to \infty$. Then r_n, $r'_n \to r$ as $n \to \infty$. Note that c is compact in $[0,1] \times \mathbf{R}^1$. So any accumulating point of $\{c_{r_n}\}_{n=1}^{\infty}$ and $\{c_{r'_n}\}_{n=1}^{\infty}$ belongs to c_r. From (A.4), we can deduce that $\operatorname{diam}(c_r) \geqq \frac{1}{m}$. Noting that $c \in S$ and c is connected, we can find that c_r is an interval. Hence

$$c \in S \cap \{c \in \mathscr{C};\ c_t \text{ is an interval of } \mathbf{R}^1 \text{ for some } t \in [0,1]\}$$

The other inclusion is trivial. (q.e.d.)

At last, noting that $\mathscr{C}_0 = S \cap \left[\bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{k=0}^{2^n-1} \bar{\Delta}^{\delta_n}_{n,k}\left(\frac{1}{m}\right)\right]^c$, we find the measurability of $\mathscr{C}_0$.

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