Z. Wahrscheinlichkeitstheorie verw. Gebiete
67, 1–62 (1984)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Naturality, Standardness, and Weak Duality for Markov Processes

R.K. Getoor and M.J. Sharpe*
Department of Mathematics, University of California, San Diego, CA 92093, USA

Contents

1. Introduction

Let X be a Borel right process on E with transition semigroup (P_t). (The precise hypotheses are set down in (1.1), but in essence this requires only that X be right continuous, strong Markov with Borel transition semigroup.) This paper's prime concern is a systematic study of weak duality of X with another Borel right process $\hat{X}$ relative to a σ-finite measure m. Part III, to be discussed later in this introduction,

* This research was supported, in part, by NSF Grant MCS 79-23922

contains the principal results and Part II is a collection of preliminaries under even weaker hypotheses. In the course of our study of weak duality, we encountered a number of issues not directly related to duality. The bulk of Part I is devoted to these questions, and we believe that this material is of interest in its own right.

Part I is organized around the notion of *natural* processes, the natural σ-algebra $\mathcal{N}$ being the trace of the predictable σ-algebra $\mathcal{P}$ on $]\!]0,\zeta[\![$, with ζ the lifetime of X. Developing some ideas introduced in [40], we discuss projections and dual projections on $\mathcal{N}$, criteria for a process to be natural, and so on, in §2. The reason for using the term natural is that in the special case where X is standard, an additive functional (AF) A will be natural in the sense above if and only if it is natural in the sense used in [4]. These are precisely the AF's in the class (U) of Meyer [25]. Another theme of Part I is that conditions should be stated in terms of *hitting times* rather than arbitrary stopping times wherever possible. This appears in §3 for the first time where we describe the representation of natural potentials by natural AF's, without standardness hypotheses. What is novel here is that the condition that a function on E be a natural potential is reduced to a condition involving hitting operators rather than arbitrary increasing sequences of stopping times. See Theorem 3.3 for the precise statement. For later applications it is necessary to extend such representations to not necessarily finite potentials, a step which requires the introduction of homogeneous random measures (HRM's). This is carried out in §4 which begins with the technicalities needed for extending the previous results on AF's to HRM's. The main extension is given in Theorem (4.11). The most important result in §5 is a characterization of standardness in terms of hitting operators. What we show in (5.5) is that a Borel right process X is standard provided $X_{T_n} \to X_T$ a.s. on $\{T<\zeta\}$ whenever $T_n \uparrow T$, the T_n being hitting times of finely closed sets. A second result on standard processes, which we consider interesting, is a characterization of *natural* stopping times T as those with $X_{T-}=X_T$ a.s. on $\{0<T<\zeta\}$. Here, T natural means $[\![T]\!] \in \mathcal{N}$. It has long been known that if $X_{T-}=X_T$ a.s. then T is accessible. The above result is much stronger, for $[\![T]\!] \in \mathcal{N}$ means that there exists an increasing sequence (T_n) of stopping times with $T_n \leqq T$, $T_n<T$ and $T_n \uparrow T$ on $\{0<T<\zeta\}$, and $\lim_n T_n \geqq \zeta$ on $\{T \geqq \zeta\}$ (see (5.4)). Another important fact about standard processes is given in Theorem 5.2 which states that the natural projection of $h(X)$ is just $h(X_-)1_{]\!]0,\zeta[\![}$ whenever h is a Borel function on the state space E.

In Part II we set down some facts about X relative to a fixed P^m, m being a countably finite excessive measure. In case m is the potential λU of a finite measure λ and the lifetime ζ is P^λ a.s. finite, Azéma [3] developed very powerful time reversal tools which have a bearing on some of the questions we investigate later, but we make no such specific assumptions here. Among the results of this part, which are preliminary to the study of weak duality, (7.4) is a typical sample stating essentially that a natural functional which is P^m additive may be perfected to a genuine natural AF.

In §8 we introduce the Revuz measure of a HRM, extending the definitions first given in [35], and discussed at length in, for example [36–38, 14, 15, 21].

The first section (§9) of Part III continues the discussion of Revuz measures under the hypotheses (9.1), (9.2) of weak duality of X, $\hat{X}$ relative to a σ-finite measure m, there being no absolute continuity hypotheses. The most important result here is the replacement (9.3) for the formula $u_A(x)\ (\equiv E^x A_\infty) = \int u(x,y)\, v_A(dy)$ of Revuz [35], in which A is a NAF with Revuz measure v_A, $u(x, y)$ the potential kernel density, it being assumed that X, $\hat{X}$ are in classical duality as in [4, VI]. The point here is that under weak duality, one does not necessarily have a potential kernel density. The replacement formula $v\hat{U}(dx) = u_A(x)\, m(dx)$ is valid however, and is the basis for much of the subsequent work. In particular, it yields (9.10) a uniqueness theorem for NAF's, or more generally, natural HRM's. Some of the results of this section have been obtained independently by Atkinson and Mitro in [2]. In §10 we bring in the stationary process Z associated with the dual pair X, $\hat{X}$. This process looks in the forward direction like X under P^m, and in the reverse direction like $\hat{X}_-$ under $\hat{P}^m$. In the case of symmetric processes, this process was discussed by Silverstein [42], though in a different framework. The general case was discussed by Mitro [29]. See also [23] for related work. Systematic use of Z makes intuitively clear a number of facts about weak duality, and arguments involving Z are freely used in subsequent sections. The first important example occurs in §11, where (11.3) gives a switching identity relating the hitting operator for X with the left hitting operator for $\hat{X}$. This result generalizes the original switching identity [4, VI-1.16] for hitting operators given by Hunt [22]. The related identities (11.7) and (11.8) contain important information for manipulating the general Revuz formula. An important application of these results is (11.11) characterizing those σ-finite measures which are the Revuz measure of some (necessarily unique) natural HRM as simply those not charging m-copolar sets. The proof of (11.11) also makes essential use of our earlier characterization of natural potentials in terms of hitting operators.

The connection between HRM's of X, $\hat{X}$ and Z is studied in §12, using methods of [30]. The main result here gives a very simple interpretation of the Revuz measure of a HRM in terms of the corresponding HRM over Z. In essence this identifies Revuz measure with Dynkin's characteristic measure [11, 12]. This connection has been discovered independently by Atkinson and Mitro [2] in a somewhat different setting.

The ideas of §12 are applied in §13 to give a probabilistic interpretation of capacity, extending the ideas of Chung [6] and the authors [17] (see also [28]). It turns out that by using Z we are able to give a probabilistic explanation of the equality of capacity and co-capacity of transient sets over standard processes. This equality, which is due to Hunt [22] (see also [4, VI-4.4]), does not appear to have a direct interpretation using X and $\hat{X}$ above. Actually, the discussion in §13 is more general, the specialization to standard processes being given in §15. The most interesting formulas are given in (13.16) through (13.19). A key observation is that if $\lambda_B = \sup\ \{t\colon Z_t \in B\}$ and $\sigma_B = \inf\{t\colon Z_{t-} \in B\}$, then under suitable transience hypotheses one has $P[\lambda_B \in dt] = c(B)\, dt$ and $P[\sigma_B \in dt] = \hat{c}(B)\, dt$ where P is the law of Z and $c(B)$ (resp. $\hat{c}(B)$) is the capacity (resp. co-capacity) of B.

In §14 we give the weak duality version of the time reversal theorem (for classical duality) of Nagasawa [32], modifying the proof of that theorem given in [41]. It turns out that this result remains valid provided the initial law does not charge m-polar sets.

In §15 a number of the foregoing results are specialized to standard processes. Actually the appropriate notion here is m-standard which only requires

$X(T_n) \to X(T)$ almost surely P^m on $\{T<\zeta\}$ when (T_n) is an increasing sequence of stopping times with limit T. See (5.1) for the precise definition. We also discuss the relationship between some of these results and Hunt's hypothesis (B) in this context. For example, if B is m-semipolar and $\hat{X}$ (the weak dual of X relative to m) is m-standard, then P^m almost surely $X_{t-}=X_t$ when $X_t \in B$ (see (15.12)). Theorem (16.15) shows that this condition is also sufficient for $\hat{X}$ to be m-standard.

Finally in §16 we give a number of structural results under the hypothesis of weak duality. The key technical fact (16.4) states that if Y is predictable and homogeneous on $\mathbb{R}^{++}$ (more precisely, $Y \in \mathscr{P} \cap \mathscr{H}^g$), then there exists a Borel function g such that Y and $g(X_-)$ are P^m indistinguishable on $]\!]0, \zeta[\![$. An important consequence (16.8) is that a σ-integrable natural HRM, κ, is P^m almost surely diffuse if and only if its Revuz measure ν_κ doesn't charge m-semipolars and is P^m almost surely purely discontinuous if and only if ν_κ is carried by an m-semipolar. Exact terminal times are characterized in (16.14). Theorem (16.21) gives a number of equivalent conditions each of which, in light of (16.19), implies that X is m-special standard.

It had been one of our original aims to give an elementary exposition of Revuz measures and the Revuz formula based on the formulas (8.4) and (8.8) as a separate part of this paper in order to make these important topics accessible to readers with limited background. This material comprises most of §8 and §9, and it is our hope that these two sections can be read almost independently of the rest and that their essential simplicity is not obscured too much. The reader of limited background might find it helpful to replace HRM by AF throughout these two sections, at least, the first time around.

Among the reasons why, in our opinion, processes in weak duality are worthy of a detailed study, are (i) once a result is proved for weak duality, a stronger result usually follows automatically in the classical duality case; (ii) weak duality survives passage to space-time processes, and this is usually not the case with classical duality (cf. [19]); (iii) every process with stationary independent increments in $\mathbb{R}^d$ (or more generally, a separable LCA group) has an obvious weak dual relative to Lebesgue (or Haar) measure – see [34] for a detailed discussion of the potential theory of such processes; (iv) the growing literature [11–13, 24, 42], to name just a few, on Dirichlet space methods in Markov processes fits in as the symmetric (i.e., self-dual) case in weak duality. Some aspects of the asymmetric case which are closely related to the latter works have been discussed recently by LeJan [24].

Finally, we set down the notation and minimal hypotheses which will remain in force throughout.

(1.1) The state space E is a Lusinian topological space – that is, E is homeomorphic to a Borel subset of some compact metric space.

(1.2) $\mathscr{E}$ is the Borel σ-algebra on E and $\mathscr{E}^*$ is the universal completion of $\mathscr{E}$.

(1.3) The semigroup (P_t) maps Borel functions to Borel functions.

(1.4) X is right continuous, strong Markov in E with transition semigroup (P_t), with lifetime ζ, death point Δ.

A process X satisfying (1.1), (1.3) and (1.4) is called a *Borel right process*. We have chosen to deal with Borel right processes rather than right processes in general in order to keep technicalities to a minimum. The essential assumption is (1.1): the fact that E is Lusinian. The assumption (1.3) may always be accomplished for an arbitrary right process by passing to the Ray topology (see [16] or [39]). However, we want to emphasize that the results of this paper, sometimes with minor changes, are valid for arbitrary right processes.

We shall use the standard notation of Markov process theory [4, 16] and [26] without special mention. Thus $\mathscr{F}^0$, $\mathscr{F}_t^0$, $\mathscr{F}$, $\mathscr{F}_t$, $\mathscr{F}_t^\mu$, and $\mathscr{F}^\mu$ have their usual meanings. $\mathscr{F}^*$ will denote the σ-algebra of universally measurable sets over $(\Omega, \mathscr{F}^0)$. Given $q \geqq 0$, (P_t^q) will denote the semigroup $P_t^q \equiv e^{-qt} P_t$. Here and in the body of the paper "$\equiv$" is the symbol for "defined to be". The σ-algebra on E generated by those functions which are q-excessive for some $q \geqq 0$ is denoted by $\mathscr{E}^e$. Thus $\mathscr{E} \subset \mathscr{E}^e \subset \mathscr{E}^*$. We use $\mathbb{R}^+ = [0, \infty[$ and $\mathscr{R}^+$ for the Borel σ-algebra on $\mathbb{R}^+$. Any numerical function f on E is automatically extended to $E_\Delta \equiv E \cup \{\Delta\}$ by $f(\Delta) = 0$ unless explicitly stated otherwise. We shall use *rcll* (resp. *lcrl*) as abbreviations for right continuous left limits (resp. left continuous right limits). Standard references for the results from the general theory of processes that we need are [9] and [10]. More specialized results on projections and supermartingale theory over a Markov process X may be found in [8] and [39]. Some also appear in [27]. We also shall need some facts about Ray theory. These are all contained in [16, 39], or [45].

Part I: Naturality

2. The natural σ-algebra $\mathscr{N}$

Throughout this section it is supposed that X is a Borel right process with state space E as described in §1. We examine here a σ-algebra $\mathscr{N}$ of *natural* processes over X which is better suited than the predictable σ-algebra to describing the left limit behavior of X. For example, even if X_{t-} exists in E for all $t < \zeta$, the process $X_{t-} 1_{]0,\zeta[}(t)$ is not generally predictable unless ζ is predictable, but it will be the case (see (2.1)) that it is natural. The term natural here comes from the connection with natural additive functionals to be described later.

A rather cryptic explanation of the relationship between predictable and natural additive functionals was given in [40]. This section is a systematic development of some of the ideas introduced there.

Following [39], let $\mathscr{I}$ denote the class of processes over $\mathbb{R}^+ \times \Omega$ which are P^μ evanescent for every probability μ on E, and let $\mathscr{M} = (\mathscr{R}^+ \times \mathscr{F}) \vee \mathscr{I}$ be the class of *measurable* processes. Here $\mathscr{R}^+$ is the σ-algebra of Borel sets of $\mathbb{R}^+ = [0, \infty[$. Then $\mathscr{P}$ and $\mathscr{O}$ – the predictable and optional σ-algebras respectively – are obtained by adjoining $\mathscr{I}$ to the σ-algebras generated by processes adapted to $(\mathscr{F}_t)$ which are *lcrl*, *rcll* respectively.

Let $\mathscr{N}$ denote the trace of $\mathscr{P}$ on $]\!]0, \zeta[\![\equiv \{(t, \omega): 0 < t < \zeta(\omega)\}$, and call $\mathscr{N}$ the *natural* σ-algebra for X. Thus, a process Y defined on $]\!]0, \zeta[\![$ is natural in case Y extends to some predictable process on $\mathbb{R}^+ \times \Omega$. It is convenient to permit Y to be defined arbitrarily off $]\!]0, \zeta[\![$ so that Y being natural means $Y 1_{]\!]0,\zeta[\![} = W 1_{]\!]0,\zeta[\![}$ for some $W \in \mathscr{P}$. Since $]\!]0, \zeta]\!]$ is predictable one may suppose that W vanishes off $]\!]0, \zeta]\!]$.

(2.1) **Lemma.** *Let Y_t be adapted to $(\mathscr{F}_t)$ and suppose $t \to Y_t$ is lc on $]0, \zeta[$. Then $Y \in \mathscr{N}$.*

Proof. We may suppose that Y vanishes off $]\!]0, \zeta[\![$, without loss of generality. Then $Y \in \mathscr{M}$. Let $\underline{Y}_t \equiv \liminf_{r \uparrow\uparrow t} Y_r$ for $t > 0$. It is known, [9] or [10], that $\underline{Y}$ is P^μ predictable for every initial law μ, and so $\underline{Y} \in \mathscr{P}$ because of [39, 23.1]. Clearly $Y = \underline{Y}$ on $]\!]0, \zeta[\![$ which establishes (2.1).

Following [39] a stopping time T (over $(\mathscr{F}_t)$) is predictable provided it is P^μ predictable for every μ. Given a predictable T, there exists an increasing sequence (T_n) of $(\mathscr{F}_t)$ stopping times that announces T for all μ simultaneously; that is, almost surely $T_n < T$ on $\{T > 0\}$ for all n and $T_n \to T$ (see [39, 23.1]).

(2.2) *Definition.* A stopping time T is natural provided there exists a predictable stopping time R with $[\![R]\!] \cap]\!]0, \zeta[\![= [\![T]\!] \cap]\!]0, \zeta[\![$.

(2.3) *Definition.* (i) An increasing process A is a process in $\mathscr{M}$ with $A_0 = 0$, $t \to A_t$ right continuous and increasing, $A_t < \infty$ if $t < \zeta$, $A_t = A_\zeta$ if $t \geqq \zeta$, and $A_{\zeta-} \equiv \lim_{t \uparrow \zeta} A_t = \infty$ if $A_\zeta = \infty$.

(ii) An increasing process A is natural if $A_\zeta = A_{\zeta-}$ and $A \in \mathscr{N}$.

According to our definition an increasing process cannot have an infinite jump. Clearly a natural increasing process is optional.

(2.4) **Proposition.** *The following conditions on a stopping time T are equivalent.*

(i) *T is natural.*

(ii) *$[\![T]\!] \in \mathscr{N}$.*

(iii) *There exists an increasing sequence $\{T_n\}$ of stopping times with $\{T_n\}$ strictly increasing on $\{0 < T < \zeta\}$, $\lim_n T_n = T$ on $\{T < \zeta\}$, and $\lim_n T_n \geqq \zeta$ on $\{T \geqq \zeta\}$.*

(iv) *$A_t \equiv 1_{[T, \infty[}(t)\, 1_{\{0 < T < \zeta\}}$ is a natural increasing process.*

Proof. The implications (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iv) are immediate. So is (i) $\Rightarrow$ (iii) once one observes that if R satisfies (2.2) and $R' = R$ on $\{T > 0\}$, $R' = 0$ on $\{T = 0\}$, then R' is predictable and $[\![R']\!] \cap [\![0, \zeta[\![= [\![T]\!] \cap [\![0, \zeta[\![$. In other words one may replace $]\!]0, \zeta[\![$ by $[\![0, \zeta[\![$ in (2.2). If (ii) holds, then $1_{[\![T]\!]} 1_{]\!]0, \zeta[\![} = Y 1_{]\!]0, \zeta[\![}$ where Y is predictable and vanishes off $]\!]0, \zeta]\!]$. Thus $\{Y = 1\}$ is predictable and each of its ω sections contains most two points. Therefore its debut R is predictable [9, IV-T.16] and satisfies (2.2). It follows from (2.1) that $1_{]\!]S, \zeta[\![}$ is in $\mathscr{N}$ for any stopping time S. Hence, if (iii) holds $Y = 1_{[\![T, \zeta[\![} 1_{]\!]0, \zeta[\![}$ is in $\mathscr{N}$ because $1_{]\!]T_n, \zeta[\![} \to Y$. Then

$$1_{[\![T]\!]} 1_{]\!]0, \zeta[\![} = (1_{[\![T, \zeta[\![} - 1_{]\!]T, \zeta[\![})\, 1_{]\!]0, \zeta[\![}$$

and so (iii) $\Rightarrow$ (ii). This establishes (2.4).

In this and subsequent sections, given $M \in \mathscr{M}_+$ or $b\mathscr{M}$, ${}^pM$ and ${}^oM$ denote respectively the predictable and optional projections of M, defined independently of the initial law (see [8] or [39]). In what follows equalities or inequalities between processes or sets are understood to hold up to evanescence. The process l and set Λ defined in the next lemma will be of great importance in the remainder of this section.

(2.5) **Lemma.** *Define* $l \equiv {}^p 1_{]\!]0,\zeta[\![}$ *and* $\Lambda \equiv \{l>0\}$. *Then* $]\!]0,\zeta[\![\subset \Lambda \subset]\!]0,\zeta]\!]$.

Proof. Although a proof was given in [40], the result is important enough and the proof simple enough to warrant repeating. As $]\!]0,\zeta]\!] \in \mathscr{P}$, the second inclusion is obvious. In addition, $l = 1_{]\!]0,\zeta]\!]} - {}^p 1_{[\![\zeta]\!]}$ and so $]\!]0,\zeta]\!] \cap \{l=0\} \subset \{{}^p 1_{[\![\zeta]\!]} > 0\}$. Given an initial law μ choose P^μ predictable stopping times T_n such that the graph of the P^μ accessible part of ζ is contained in $\cup [\![T_n]\!]$. Then $\{{}^p 1_{[\![\zeta]\!]} > 0\} \subset \cup [\![T_n]\!]$, and so the predictable set $\{l=0\} \cap]\!]0,\zeta]\!]$ is contained in $\cup [\![T_n]\!]$. Therefore there exist P^μ predictable stopping times R_n so that $\{l=0\} \cap]\!]0,\zeta]\!] = \cup [\![R_n]\!]$ (see [9, IV-T.17]). Now

$$\begin{aligned} 0 = E^\mu [l_{R_n}; R_n < \infty] &= E^\mu [1_{]\!]0,\zeta[\![}(R_n); R_n < \infty] \\ &= P^\mu [0 < R_n < \zeta]. \end{aligned}$$

Since μ is arbitrary this proves that $\{l=0\} \cap]\!]0,\zeta]\!] \subset [\![\zeta,\infty[\![$, and so $]\!]0,\zeta[\![\subset \{l>0\}$ as claimed.

The natural σ-algebra has the following section theorem.

(2.6) **Theorem.** *Let* $Y^1, Y^2 \in \mathscr{N}$ *be bounded or positive and suppose that* $E^\mu\{Y^1_T; 0<T<\zeta\} = E^\mu\{Y^2_T; 0<T<\zeta\}$ *for every natural stopping time* T. *Then* Y^1 *and* Y^2 *are* P^μ*-indistinguishable.*

Proof. Let $Y^j 1_{]\!]0,\zeta[\![} = W^j 1_{]\!]0,\zeta[\![}$ with $W^j \in \mathscr{P}$ bounded if Y^j is bounded, positive if Y^j is positive. From the hypothesis it follows at once that

$$E^\mu\{W^1_T l_T; T<\infty\} = E^\mu\{W^2_T l_T; T<\infty\}$$

for every predictable stopping time T. If W^1 and W^2 are bounded, the result is immediate from the usual section theorem [10, IV-87]. If both W^1, W^2 are positive, the result follows applying [10, IV-85] to $\{W^1 \geqq W^2 + \varepsilon\} \cap \{W^1 \leqq k\}$ to show that this set is P^μ evanescent, then interchanging W^1 and W^2.

Given $M \in \mathscr{M}_+$, define the natural projection ${}^n M$ of M by (recall $l = {}^p 1_{]\!]0,\zeta[\![}$)

$$(2.7) \qquad {}^n M = {}^p(M 1_{]\!]0,\zeta[\![})\, l^{-1} 1_{]\!]0,\zeta[\![}.$$

The right side of (2.7) is unambiguous because of (2.5). Operationally, ${}^n M$ is determined as follows.

(2.8) **Proposition.** *Let* $M \in \mathscr{M}_+$. *Then* ${}^n M$ *is the unique member of* $\mathscr{N}$ *such that*

$$(2.9) \qquad E^\bullet({}^n M_T; 0<T<\zeta) = E^\bullet(M_T; 0<T<\zeta)$$

for every natural stopping time T. [*Uniqueness here means up to evanescence on* $]\!]0,\zeta[\![$].

Proof. It is enough to check (2.9) in case T is predictable and M bounded. Using (2.7), properties of predictable projections, and setting $0/0 = 0$

$$\begin{aligned} E^\bullet\{{}^n M_T; 0<T<\zeta\} &= E^\bullet\left\{\frac{{}^p(M 1_{]\!]0,\zeta[\![})(T)}{l(T)}; 0<T<\zeta\right\} \\ &= E^\bullet\left\{\frac{{}^p(M 1_{]\!]0,\zeta[\![})(T)}{l(T)}\, l(T)\right\} \\ &= E^\bullet\{(M 1_{]\!]0,\zeta[\![})(T); T<\infty\} = E^\bullet\{M_T; 0<T<\zeta\}, \end{aligned}$$

where the third equality holds because ${}^{p}(M 1_{]\!]0,\zeta[\![})$ vanishes on $\{l=0\}$. This establishes (2.9). The uniqueness is an immediate consequence of (2.6).

(2.10) **Lemma.** *If $M \in b\mathcal{M}$ vanishes off $]\!]0,\zeta[\![$, then ${}^{p}M = {}^{p}({}^{n}M)$.*

Proof. If M vanishes off $]\!]0,\zeta[\![$, then ${}^{p}M$ vanishes off $\Lambda = \{l > 0\}$. Consequently the result follows from (2.5) and (2.7).

In defining the dual natural projection of an increasing process one needs to exercise a modicum of care. Recall the definition of an increasing process A given in (2.3). Let us say that A is carried by a random set Γ if the measure $dA_t(\omega)$ is a.s. carried by $\Gamma(\omega)$. We shall consider only those A for which the dual predictable projection A^p of A exists as an increasing predictable process. See [39, §28] for precise conditions under which A^p exists. Under the present assumptions A_t^p is carried by $]\!]0,\zeta]\!]$. We shall define A^n, the dual natural projection of A, only when A satisfies, in addition, the condition that A is carried by $\Lambda = \{l > 0\}$. Recall, again, that $l = {}^{p}1_{]\!]0,\zeta[\![}$.

(2.11) *Definition.* Let A be an increasing process carried by Λ for which A^p exists. Then one defines A^n by

(2.12) $$A_t^n = \int_{]0,t]} 1_{]0,\zeta[}(s)\,(l_s)^{-1}\, dA_s^p,$$

or, equivalently, in terms of measures

(2.13) $$dA_t^n = 1_{]0,\zeta[}(t)\, l_t^{-1}\, dA_t^p.$$

Since $t \to \int_{]0,t]} l_s^{-1}\, dA_s^p$ is a predictable process that agrees with A^n on $]0,\zeta[$, it is clear that A^n is a natural increasing process as defined in (2.3). Clearly $A^n = (A^p)^n$.

Remark. If A^p exists but A is not carried by Λ, one may define $(1_\Lambda * A)^n$ and one might call this the dual natural projection of A. [Here, $(Y * A)_t \equiv \int_{]0,t]} Y_s\, dA_s$.] However, the above definition is more convenient for us.

The next few propositions contain important properties of A^n.

(2.14) **Proposition.** *Let A be as in* (2.11).

(i) *If $M \in b\mathcal{M}$, then*

(2.15) $$E^\bullet \int {}^{n}M_t\, dA_t = E^\bullet \int M_t\, d(1_{]\!]0,\zeta[\![} * A)_t^n.$$

(ii) *$A^p = (A^n)^p$. In particular $A^n = (A^n)^n$, and $A^n = (A^p)^n$.*

Proof. Using standard properties of dual predictable projections, (2.7), and $l = {}^{p}1_{]\!]0,\zeta[\![}$ one has for $M \in b\mathcal{M}$

$$\begin{aligned} E^\bullet \int {}^{n}M_t\, dA_t &= E^\bullet \int {}^{p}(M 1_{]\!]0,\zeta[\![})(t)\, l_t^{-1}\, 1_{]0,\zeta[}(t)\, dA_t \\ &= E^\bullet \int {}^{p}(M 1_{]\!]0,\zeta[\![})(t)\, l_t^{-1}\, d(1_{]\!]0,\zeta[\![} * A)_t^p \\ &= E^\bullet \int M_t\, 1_{]0,\zeta[}(t)\, l_t^{-1}\, d(1_{]\!]0,\zeta[\![} * A)_t^p \\ &= E^\bullet \int M_t\, d(1_{]\!]0,\zeta[\![} * A)_t^n. \end{aligned}$$

Since A is carried by the predictable set Λ so is A^p. Thus by (2.13)

$$d(A^n)_t^p = l_t^{-1}\, d(1_{]\!]0,\zeta[\![} * A^p)_t^p$$
$$= l_t^{-1}\, l_t\, dA_t^p = 1_\Lambda(t)\, dA_t^p = dA_t^p .$$

The remaining assertions in (2.14-ii) are immediate.

(2.16) **Proposition.** *If A is a natural increasing process, then A^n exists and $A^n = A$.*

Proof. Because of (2.1), $A_- = (A_{t-})$ defined on $]0, \zeta[$ is in $\mathcal{N}$. Hence ΔA in in $\mathcal{N}$ and so $\Delta A = Y 1_{]\!]0,\zeta[\![}$ with $Y \in \mathcal{P}$ vanishing off $]\!]0, \zeta]\!]$. Let A^c be the continuous part of A, and define

(2.17) $$B_t = A_t^c + \sum_{s \leqq t} Y_s = A_t + Y_\zeta 1_{[\zeta]}(t) .$$

Since ΔA is finite one may choose Y finite and so B is an increasing process, and since $A^c \in \mathcal{P}$, $B \in \mathcal{P}$. From (2.17), $dA = 1_{]\!]0,\zeta[\![} * dB$ and so $dA^p = l * dB$ exists as an increasing process. Consequently

$$dA_t^n = 1_{]0,\zeta[}(t)\, l_t^{-1}\, dA_t^p$$
$$= 1_{]0,\zeta[}(t)\, 1_\Lambda(t)\, dB_t = dA_t ,$$

and (2.16) is established.

(2.18) **Proposition.** *Let A be an increasing process such that $E^x(A_\infty) < \infty$ for all x. Suppose, in addition, that for every $M \in b\mathcal{M}_+$ one has*

(2.19) $$E^\bullet \int_{]0,\zeta[} M_t\, dA_t = E^\bullet \int {}^n M_t\, dA_t .$$

Then $1_{]0,\zeta[}(t)\, dA_t$ is a natural increasing process.

Proof. Replacing A by $1_{]\!]0,\zeta[\![} * A$ one may suppose that A is carried by $]\!]0, \zeta[\![\subset \Lambda$. Also $E^x(A_\infty) < \infty$ for all x guarantees that A^p exists. See [39, §31] or [8, §3]. Then using (2.15), the condition (2.19) implies

$$E^\bullet \int M_t\, dA_t = E^\bullet \int M_t\, dA_t^n$$

for all $M \in b\mathcal{M}_+$. As these are finite quantities it follows that $A_t = A_t^n$ for all $t \geqq 0$ (see [39, §30]). This establishes (2.18).

(2.20) **Corollary.** *A stopping time T is natural if and only if for all $M \in b\mathcal{O}_+$*

(2.21) $$E^\bullet (M_T; 0 < T < \zeta) = E^\bullet ({}^n M_T; 0 < T < \zeta) .$$

Proof. Let $A_t = 1_{[T,\infty[}(t)\, 1_{\{0<T<\zeta\}}$. Because A is optional, (2.19) holds provided it holds for all $M \in b\mathcal{O}_+$. Note that ${}^n M = {}^n({}^o M)$ is immediate from (2.7). Thus (2.21) implies that A is natural, and (2.20) follows from (2.4).

If A is an increasing process, its potential Y is defined to be the optional projection of $t \to \int_{]t,\infty[} dA_s$. (This projection always exists, though it may take the value $+\infty$.) Given an initial law μ such that $E^\mu Y_0 (= E^\mu A_\infty) < \infty$, $t \to Y_t$ is a right continuous supermartingale relative to $(\Omega, \mathcal{F}_t^\mu, P^\mu)$. It follows from (2.14-ii) that, assuming A^p exists and that A is carried by Λ, A, A^p and A^n all have the same potential.

(2.22) **Proposition.** *If A^1, A^2 are natural increasing processes whose potentials Y^1, Y^2 are P^μ indistinguishable and $E^\mu Y_0^j < \infty$ then A^1, A^2 are P^μ indistinguishable.*

Proof. Combining (2.16), (2.14-ii) and the above remarks it follows that $(A^1)^p$, $(A^2)^p$ have the same potential relative to $(\Omega, \mathscr{F}_t^\mu, P^\mu)$ and they are therefore P^μ-indistinguishable. Appealing to (2.16) and (2.14-ii) once again, we establish (2.22).

We next describe how natural projections behave under shifts. We define Θ_t and $\hat{\Theta}_t$ as follows. If Y is a process and A an increasing process, then

$$\text{(2.23)}\quad \begin{aligned} &\text{(i)}\ (\Theta_t Y)(s,\omega) = 1_{[t,\infty[}(s)\, Y(s-t, \theta_t\omega).\\ &\text{(ii)}\ d_s(\hat{\Theta}_t A)(s,\omega) = 1_{]t,\infty[}(s)\, d_s A(s-t, \theta_t\omega).\end{aligned}$$

In terms of A rather than the measure dA, (2.23-ii) becomes (recall $A_0 = 0$)

$$\text{(2.24)}\quad \begin{aligned} (\hat{\Theta}_t A)(s,\omega) &= 1_{[t,\infty[}(s)\, A(s-t, \theta_t\omega)\\ &= (\Theta_t A)(s,\omega).\end{aligned}$$

The dual shift should really be viewed as a transformation of random measures and, hence, (2.23-ii). However, we postpone a discussion of random measures until §4. Of course, if T: $\Omega \to [0,\infty]$ one defines Θ_T and $\hat{\Theta}_T$ on $\{T<\infty\}$ in the obvious manner. It is shown in (22.11), (31.7), and (29.3) of [39], that if T is a stopping time (relative to $(\mathscr{F}_t)$), then up to evanescence

$$\text{(2.25)}\quad \begin{aligned} &\text{(i)}\ 1_{]\!]T,\infty[\![}\,{}^p(\Theta_T Y) = 1_{]\!]T,\infty[\![}\,\Theta_T({}^pY),\\ &\text{(ii)}\ 1_{]\!]T,\infty[\![} * (\hat{\Theta}_T A)^p = 1_{]\!]T,\infty[\![} * \hat{\Theta}_T(A^p),\\ &\text{(iii)}\ \hat{\Theta}_T(Y*A) = (\Theta_T Y)*(\hat{\Theta}_T A),\end{aligned}$$

provided $Y \in \mathscr{M}^+$ is finite and A is an increasing process (A^p must exist for (ii)). From (2.7), (2.13), and the fact that $1_{]\!]T,\infty[\![}$ is predictable one checks readily that up to evanescence

$$\text{(2.26)}\quad \begin{aligned} &\text{(i)}\ 1_{]\!]T,\infty[\![}\,{}^n(\Theta_T Y) = 1_{]\!]T,\infty[\![}\,\Theta_T({}^nY),\\ &\text{(ii)}\ 1_{]\!]T,\infty[\![} * (\hat{\Theta}_T A)^n = 1_{]\!]T,\infty[\![} * \hat{\Theta}_T(A^n),\end{aligned}$$

where in (2.26-ii) it is assumed that A^p exists as an increasing process and dA is carried by Λ. There are results analogous to (i) and (ii) of (2.25) for optional and dual optional projections. We refer the reader to [39].

The most important class of increasing processes in Markov process theory is the class of additive functionals.

(2.27) *Definition.* A raw additive functional (RAF), A, of X is an increasing process such that $A_{\zeta-} = A_\zeta$ and $A_{t+s} = A_t + A_s \circ \theta_t$ almost surely for each fixed t and s. An additive functional (AF) is a RAF that is adapted to $(\mathscr{F}_t)$.

Because of standard perfection theorems [46] or [39, §35] one may suppose that $A_{t+s} = A_t + A_s \circ \theta_t$ identically in t, s, and ω in (2.27) without loss of generality, and we shall do so in the sequel. If one drops the condition $A_{\zeta-} = A_\zeta$ in (2.27) one speaks of a RAF or AF possibly charging ζ. A *natural additive functional* (NAF) is a RAF that is a natural increasing process and, hence, an AF.

It is a routine matter to verify that an increasing process with $A_{\zeta-} = A_\zeta$ is a RAF if and only if $\hat{\Theta}_t A = 1_{]\!]t,\infty[\![} * A$ for each $t \geqq 0$. Consequently because of (2.26), and using (2.25 i) to see that $1_{]\!]t,\infty[\![}\Theta_t 1_\Lambda = 1_{]\!]t,\infty[\![}1_\Lambda$, one has the following important result.

(2.28) **Proposition.** *If A is a* RAF, *possibly charging ζ, and if $(1_\Lambda * A)^n$ exists, then $(1_\Lambda * A)^n$ is a* NAF.

If A is a RAF, its *q-potential function $u^q = u^q_A$* is defined by

(2.29) $$u^q(x) = u^q_A(x) = E^x \int_0^\infty e^{-qt}\, dA_t .$$

It is immediate that u^q is q-excessive. If $u^q(x) < \infty$, then the potential as defined above (2.22) of the increasing process $t \to \int_{]0,t[} e^{-qs}\, dA_s$ is given by $e^{-qt} u^q(X_t)$. Thus it follows from (2.22) that if A and B are NAF's with $u^q_A = u^q_B < \infty$, then $A = B$.

The following elementary fact will be used in the next section. We leave its proof to the reader.

(2.30) **Proposition.** *Let f be a q-excessive function and μ an initial measure with $\int f\, d\mu < \infty$. Let A be a* RAF. *Then*

$$e^{-qt} f(X_t) = E^\mu \left\{ \int_{]t,\infty[} e^{-qs}\, dA_s \,|\, \mathscr{F}_t \right\} P^\mu \text{ a.s.}$$

for each $t \geqq 0$, if and only if $f(X) = u^q_A(X)$ up to P^μ evanescence.

The following is an equivalent reformulation of the conclusion of (2.30): $(e^{-qt} f(X_t))$ is the potential of the increasing process $t \to \int_{]0,t[} e^{-qs}\, dA_s$ relative to P^μ in the sense of the general theory of processes if and only if $f = u^q$ a.e. μP_t for all $t \geqq 0$. In particular, under the assumptions in (2.30), it follows that if T is a stopping time, then

(2.31) $$f(X_T) = u^q_A(X_T) \quad \text{a.s. } P^\mu .$$

One additional fact concerning homogeneity will arise in subsequent sections. (The proof of (3.3) contains a typical example.) We need the fact that if g is a nearly Borel function on E, then ${}^p g(X)$ may be specified as $\bar{P}_0\, g(X^r_{t-})$ where (X^r_{t-}) is the Ray left limit of X in a Ray compactification $\bar{E}$ of E and $(\bar{P}_t)$ is the Ray semigroup constructed from (P_t) (see [16, 11.15] or [39, 42.1]). The point is that $Y = {}^p g(X)$ may be taken to be perfectly homogeneous on $]0,\infty[$; that is, $Y_{t+s} = Y_t \circ \theta_s$ for all $t > 0$ $s \geqq 0$. Thus the process $l = {}^p 1_{]\!]0,\zeta[\![} = 1_{]\!]0,\infty[\![}\, {}^p 1_E(X)$ may be assumed to be perfectly homogeneous on $]0,\infty[$.

3. Natural Potentials

As in section 2, X denotes a Borel right process with state space E. Let $\mathscr{S}^q$ denote the cone of q-excessive functions, $q \geqq 0$. It is well known that if $f \in \mathscr{S}^q$ is finite and satisfies the condition $E^x f(X_{T_n}) \to 0$ whenever $\{T_n\}$ is an increasing sequence of stopping times with $\lim T_n \geqq \zeta$ almost surely P^x, then f is the q-potential of a natural additive functional. This is proved in [4], p. 299–302 for standard processes. (The proof does not use standardness.) The purpose of this section is to show that it suffices to consider only increasing sequences of hitting times in this result. This will be crucial in the following sections and also is of interest in its own right.

We begin with the following lemma which is a variant of results in [4] and [7].

(3.1) **Lemma.** *Let T be an exact terminal time.*

(i) *Let $D_n = \{x: E^x(e^{-T}) \geqq 1 - 1/n\}$ and $T_n = T_{D_n}$ be the hitting time of D_n. If T is P^μ predictable, then almost surely P^μ, $T_n < T$ on $\{0 < T < \infty\}$, $\lim T_n = T$, and $\{t: X_t \in D_n\}$ contains an open interval $]U, T[$, $U < T$ if $T > 0$. If T is predictable the assertions in the preceding sentence hold almost surely (rather that almost surely P^μ).*

(ii) *Let $D_n = \{x: E^x(e^{-T\wedge\zeta}) \geqq 1 - 1/n\}$ and $T_n = T_{D_n}$. If T is natural, then almost surely $T_n < T$ on $\{0 < T < \zeta\}$, $\lim T_n = T$ on $\{T < \zeta\}$, $\lim T_n \geqq \zeta$ on $\{T \geqq \zeta\}$, and $\{t: X_t \in D_n)$ contains an open interval $]T-\varepsilon, T[$ if $0 < T < \zeta$.*

Proof. For (i) let $u(x) = E^x(1 - e^{-T}) = E^x \int_0^T e^{-t}\, dt$ and $Y_t = e^{-t} u(X_t)\, 1_{[0,T[}(t)$. Since T is exact, $Y = (Y_t)$ is almost surely right continuous. A simple calculation gives

$$Y_t = E^x \left\{ \int_{T\wedge t}^{T} e^{-s}\, ds \,|\, \mathcal{F}_t \right\},$$

and so Y is the potential of the increasing process $t \to \int_0^{T\wedge t} e^{-s}\, ds$ relative to each P^x. In particular Y is a supermartingale. Now suppose T is P^μ predictable and let (R_n) announce T relative to P^μ. The assertions in the remainder of this paragraph hold P^μ almost surely. Since

$$Y_{R_n} = E^\mu \left\{ \int_{R_n}^{T} e^{-s}\, ds \,|\, \mathcal{F}_{R_n} \right\},$$

Y_{R_n} is strictly positive on $\{T > 0\}$ and $Y_{R_n} \to 0$ as $n \to \infty$. It follows that $Y > 0$ on $]0, T[$ and $Y_{T-} = 0$ on $\{T > 0\}$. Since $D_n = \{u \leqq 1/n\}$ and $\operatorname{reg}(T) \equiv \{x: P^x(T=0) = 1\}$ is contained in D_n for every n, one easily checks that $T_n = T_{D_n}$ has the desired properties.

Obviously if T is predictable, the above argument is valid for every initial law μ and so (i) is established.

For (ii) let $u(x) = E^x \int_0^{T\wedge\zeta} e^{-t}\, dt$ and $Y_t = e^{-t} u(X_t)\, 1_{[0,T[}(t)$. As before Y is the potential of the increasing process

$$B_t = \int_0^t e^{-s}\, 1_{[0, T\wedge\zeta[}(s)\, ds.$$

Let $T_0 = \inf\{t: Y_t = 0\}$. Since $Y_{T_0} = 0$ if $T_0 < \infty$, B must be constant on $[T_0, \infty[$. Therefore $T \wedge \zeta \leqq T_0$. In particular $Y > 0$ on $[0, T\wedge\zeta[$. Using (2.4-iii) the remainder of the proof goes exactly as before.

(3.2) *Remark.* If in (ii), T is only μ-natural in the sense that there exists a P^μ predictable R with $[\![T]\!] \cap]\!]0, \zeta[\![= [\![R]\!] \cap]\!]0, \zeta[\![$ up to P^μ evanescence, then the conclusion of (ii) holds P^μ almost surely. Since each T_n is an exact terminal time without exceptional points, it follows from (i) that an exact predictable terminal time is equivalent to one without exceptional points. In other words (3.1-i) gives a simple proof of the perfection theorem for *predictable* exact terminal times. It also follows from (3.1-ii) that if $R = \lim T_n$ on $\{\lim T_n < \zeta\}$, $R = \infty$ on $\{\lim T_n \geqq \zeta\}$, then R is a predictable exact terminal time such that $[\![R]\!] \cap [\![0, \zeta[\![= [\![T]\!] \cap 0, \zeta[\![$.

We come now to the main result of this section. Recall from §2 that a NAF is an AF which is also a natural increasing process.

(3.3) **Theorem.** *Let f be a finite valued q-excessive function ($q \geqq 0$) and μ an initial measure with $\int f\,d\mu < \infty$. Then there exists a* NAF, *A, such that $f(X) = u_A^q(X)$ up to P^μ evanescence, with $u_A^q \leqq f$ everywhere, if and only if*

(i) $P_t^q f \to 0$ a.e. μ *as* $t \to \infty$.

(ii) *given a decreasing sequence (D_n) of finely closed sets in $\mathscr{E}^e$ with hitting times $T_n \equiv T_{D_n}$ satisfying* $\lim T_n \geqq \zeta$ *a.e. P^μ, one has $E^\mu[e^{-qT_n} f(X_{T_n})] \to 0$ as $n \to \infty$.*

Proof. The necessity follows from (2.31). For the converse, we construct a NAF, A, such that $Y_t \equiv e^{-qt} f(X_t)$ is the potential of $\int_{]0,t]} e^{-qs}\,dA_s$ relative to P^μ, in the sense of the general theory of processes. Then (2.31) gives the desired conclusion, except that $u_A^q \leqq f$ need not hold. However, $f \wedge u_A^q$ is also the q-potential of a NAF, and this NAF satisfies all requirements of the theorem.

First of all, we shall show that the right continuous supermartingale $Y \equiv (Y_t)$ is of class (D) relative to P^μ. Let $D_n \equiv \{f \geqq n\}$ and $T_n \equiv T_{D_n}$. Since f is q-excessive, D_n is finely closed, $D_n \in \mathscr{E}^e$, and

$$n\,E^\mu[e^{-qT_n}; T_n < \zeta] \leqq E^\mu[e^{-qT_n} f(X_{T_n})] \leqq \int f\,d\mu < \infty\,.$$

Therefore if $T = \lim T_n$

$$E^\mu[e^{-qT}; T < \zeta] \leqq \lim \frac{1}{n} \int f\,d\mu = 0\,,$$

and so (T_n) satisfies (ii). Given an arbitrary stopping time R, if $e^{-R} f(X_R) > n$, then $T_n \leqq R$, and so (recall $Y_t = e^{-t} f(X_t)$)

$$\begin{aligned} E^\mu[Y_R; Y_R > n] &\leqq E^\mu[Y_R; T_n \leqq R] \\ &\leqq E^\mu(Y_{T_n}) \to 0 \end{aligned}$$

as $n \to \infty$ because of (ii). Consequently the family (Y_R) as R ranges over all stopping times is P^μ uniformly integrable proving that Y is of class (D) relative to P^μ.

Because f is finite valued, $Y_t = e^{-qt} f(X_t)$ is a positive super-martingale relative to P^ν for every ν with $\int f\,d\nu < \infty$. Therefore Y has a Doob-Meyer decomposition of the form $Y_t = M_t - \int_0^t e^{-qs}\,dB_s$ where M is a local martingale relative to each P^ν with $\int f\,d\nu < \infty$ and B is a predictable additive functional of X possibly charging ζ (see the remarks following (2.27)). This decomposition is independent of ν (see [8, 3.12] or [39, VI]). But Y is a class (D) potential relative to P^μ and, hence, is the P^μ

potential of a predictable increasing process. From the uniqueness of the Doob-Meyer decomposition it follows that Y_t is the potential of $\int_0^t e^{-qs}\,dB_s$ relative to P^μ (in the general theory of processes sense).

As in section 2 let $l = {}^p 1_{]\!]0,\zeta[\![}$ and $\Lambda = \{l > 0\}$. We claim that almost surely P^μ, dB is carried by Λ. To this end let $C = 1_{\Lambda^c} * B$. As we pointed out at the end of §2, one may choose $l = {}^p 1_{]\!]0,\zeta[\![}$ so that $l_{t+s} = l_t \circ \theta_s$ identically for $t > 0$ and $s \geqq 0$. As a result C is a predictable AF possible charging ζ. According to (2.5), $\Lambda^c \cap]\!]0, \infty[\![\subset [\![\zeta, \infty[\![$, and so dC is carried by $[\![\zeta]\!]$. Let $R = \inf\{t: \Delta C_t > 0\}$. Then R is a predictable terminal time with $[\![R]\!] \subset [\![\zeta]\!]$. Since R is thin, it is exact and we may apply (3.1-i) to find a decreasing sequence (D_n) of finely closed sets in $\mathscr{E}^e$ whose hitting times T_n satisfy $\lim T_n = R$ with $T_n < R$ on $\{R < \infty\}$ almost surely. Since $R \geqq \zeta$, condition (ii) of (3.3) gives

$$E^\mu \int_{]T_n, \infty[} e^{-qs}\,dB_s = E^\mu\left[e^{-qT_n} f(X_{T_n})\right] \to 0$$

as $n \to \infty$. Hence $E^\mu \int_{[R, \infty[} e^{-qs}\,dB_s = 0$. But if $R < \infty$, then $0 < \Delta C_R \leqq \Delta B_R$. Consequently $P^\mu(R < \infty) = 0$ and so P^μ almost surely B is carried by Λ. Define now $A = (1_\Lambda * B)^n$. By the discussion above together with (2.14-i) and (2.28), A is a NAF and the P^μ potential of $\int_{]0,t]} e^{-qs}\,dA_s$ is Y. This completes the proof.

It is known that the condition that a function be a *regular* q-potential may be expressed in terms of hitting times. See (IV-3.6) and (IV-3.8) of [4], at least for standard processes. We shall see below that this is an easy consequence of (3.1) and (3.3).

(3.4) **Corollary.** *Let f be a finite q-excessive function satisfying the conditions of* (3.3). *If, in addition, given any decreasing sequence (D_n) of finely closed sets in $\mathscr{E}^e$ with hitting times T_n and $T = \lim T_n$ one has $E^\mu[e^{-qT_n} f(X_{T_n})] \to E^\mu(e^{-qT} f(X_T))$, then the A in* (3.3) *is continuous almost surely P^μ.*

Proof. Of course, the condition in (3.4) contains (ii) of (3.3). Given $\varepsilon > 0$ let $R = \inf\{t: \Delta A_t \geqq \varepsilon\}$. Then R is a natural thin terminal time, and by (3.1-ii) there exists a decreasing sequence (D_n) with $\lim T_n \geqq \zeta$ on $\{R \geqq \zeta\}$, $\lim T_n = R$ if $R < \zeta$ and $T_n < R$ on $\{0 < R < \xi\}$. But dA does not charge $[\zeta, \infty[$ and so if $T = \lim T_n$,

$$\begin{aligned} E^\mu \int_{]T_n, R]} e^{-qs}\,dA_s &= E^\mu \int_{]T_n, T]} e^{-qs}\,dA_s \\ &= E^\mu\left[e^{-qT_n} f(X_{T_n})\right] - E^\mu\left[e^{-qT} f(X_T)\right] \\ &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Consequently $P^\mu(R < \zeta) = 0$ completing the proof of (3.4).

(3.5) *Remark.* We have assumed in this paper that X is a Borel right process. However, (3.3) and (3.4) are valid for arbitrary right processes. This is clear from the proofs given above. We shall need (3.3) for a right process which is not necessarily Borel in the proof of (4.11).

4. Homogeneous Random Measures

In order to discuss results such as (3.3) for excessive functions which are not everywhere finite, it is necessary to extend the concept of an additive functional. The appropriate objects are the homogeneous random measures to be introduced in this section. A more detailed treatment is given in Chapter IV of [39].

Again in this section X is a Borel right process. A *random measure* κ is a kernel from $(\Omega, \mathscr{F})$ to $(\mathbb{R}^+, \mathscr{R}^+)$ which is carried by $]\!]0, \zeta]\!]$. That is, for each ω, $\kappa(\omega, \cdot)$ is a measure on the Borel sets $\mathscr{R}^+$ of $\mathbb{R}^+ = [0, \infty[$ which is carried by $]0, \zeta(\omega)]$ and for each $B \in \mathscr{R}^+$, $\omega \to \kappa(\omega, B)$ is $\mathscr{F}$ measurable. We shall *always suppose that for almost all* ω, $\kappa(\omega, \cdot)$ *is the countable sum of finite measures* (possibly depending on ω). This will permit the use of Fubini's theorem in most manipulations. Two random measures κ_1 and κ_2 are equal provided $\kappa_1(\omega, \cdot) = \kappa_2(\omega, \cdot)$ for almost all ω. Inequalities between random measures are defined similarly. Thus κ is *finite* if $\kappa(\omega, \mathbb{R}^+) < \infty$ for almost all ω. We shall adopt the usual convention of suppressing ω when convenient so that $\kappa(B)$ denotes the function $\omega \to \kappa(\omega, B)$. If A is an increasing process as defined in (2.3), then $\kappa(\omega, dt) = dA_t(\omega)$ is a random measure which is actually σ-finite for each ω. Clearly κ is finite if $A_\infty = A_\zeta < \infty$.

A random measure κ is σ-integrable on $\mathscr{O}$ (resp. $\mathscr{P}$, $\mathscr{N}$) if there exists a strictly positive process Y in $\mathscr{O}$ (resp. $\mathscr{P}$, $\mathscr{N}$) such that $E^x \int Y_t\, \kappa(dt) < \infty$ for all x. It suffices that $\{Y > 0\}$ carry κ, replacing Y by $Y + 1_{\{Y>0\}}$. Note that replacing Y_t by $Z_t = Y_t/(1 + f(X_0))$ where $f(x) = E^x \int Y_t\, \kappa(dt)$, one may suppose that $E^x \int Y_t\, \kappa(dt)$ is bounded in x in this definition. Since $Y \in \mathscr{N}$ is arbitrary off $]\!]0, \zeta[\![$, in the natural case $E^x \int Y_t\, \kappa(dt)$ is short for $E^x \int_{]0,\zeta[} Y_t\, \kappa(dt)$. In other words if $Y \in \mathscr{N}$ we suppose that $Y = 0$ off $]\!]0, \zeta[\![$. A random measure is optional (resp. predictable, natural) provided it may be represented in the form $\kappa(dt) = Y_t\, dA_t$ (written $\kappa = Y * dA$) where Y is a positive everywhere finite process in $\mathscr{O}$ (resp. $\mathscr{P}$, $\mathscr{N}$) and A is an optional (resp. predictable, natural) increasing process. The RM κ generated by an optional increasing process is σ-integrable on $\mathscr{O}$. See [39, §28]. A similar assertion holds in the predictable case, and the natural case is a simple consequence of this. Observe that κ is natural if and only if $\kappa = 1_{]\!]0,\zeta[\![} * \gamma$ where γ is a predictable random measure. In particular a natural random measure is carried by $]\!]0, \zeta[\![$. If κ is σ-integrable over $\mathscr{O}$ (resp. $\mathscr{P}$), then the dual optional (resp. predictable) projection of κ is defined and denoted by κ^0 (resp. κ^p). These are the unique optional (resp. predictable) random measures such that for all μ and $Z \in \mathscr{M}^+$

$$
(4.1) \qquad \begin{aligned} E^\mu \int Z_t\, \kappa^0(dt) &= E^\mu \int {}^0 Z_t\, \kappa(dt) \\ E^\mu \int Z_t\, \kappa^p(dt) &= E^\mu \int {}^p Z_t\, \kappa(dt). \end{aligned}
$$

In the representation $\kappa^p = Y * dA$ with $Y \in \mathscr{P}^+$, $Y < \infty$ and A a predictable increasing process one may suppose that $E^x(A_\infty)$ is bounded. A similar statement holds for κ^0.

Let κ be σ-integrable over $\mathscr{N}$. Then there exists $Y > 0$ in $\mathscr{N}$ (of course, this means $Y \in \mathscr{N}$ and $Y > 0$ on $]\!]0, \zeta[\![$) such that $A_t = \int_{]0,t]} Y_s 1_{]0,\zeta[}(s)\, \kappa(ds)$ is an increasing process with $\sup E^x(A_\infty) < \infty$. This certainly implies that A^p exists as an increasing process. Obviously dA is carried by $]\!]0, \zeta[\![\subset \Lambda$ where as in previous sections

$\Lambda = \{l > 0\}$, $l = {}^p 1_{]\!]0,\zeta[\![}$. Consequently A^n the dual natural projection of A exists and we define κ^n the dual natural projection of κ by

$$\kappa^n(dt) = Y_t^{-1}\, dA_t^n. \tag{4.2}$$

Since dA_t^n is carried by $]0, \zeta[$ it is clear that κ^n is a natural random measure as defined in the preceding paragraph. If $Z \in \mathscr{P}$ is such that $1_{]\!]0,\zeta[\![} Y = Z 1_{]\!]0,\zeta[\![}$, then using Z it is clear that $1_{]\!]0,\zeta[\![} * \kappa$ is σ-integrable over $\mathscr{P}$. Therefore from (2.12)

$$\begin{aligned}
\kappa^n(dt) &= Y_t^{-1} \frac{(1_{]\!]0,\zeta[\![} * dA)_t^p}{l(t)} 1_{]0,\zeta[}(t) \\
&= Y_t^{-1} \frac{(1_{]0,\zeta[}(t)\, Z_t\, \kappa(dt))^p}{l(t)} 1_{]0,\zeta[}(t) \\
&= Y_t^{-1} Z_t \frac{(1_{]0,\zeta[}(t)\, \kappa(dt))^p}{l(t)} 1_{]0,\zeta[}(t) \\
&= 1_{]0,\zeta[}(t) \frac{(1_{]0,\zeta[}(t)\, \kappa(dt))^p}{l},
\end{aligned}$$

in other words

$$\kappa^n = 1_{]\!]0,\zeta[\![} \frac{(1_{]\!]0,\zeta[\![} * \kappa)^p}{l}. \tag{4.3}$$

Hence κ^n does not depend on the choice of Y in (4.2). Moreover this shows the consistency of the definitions in (4.2) and (2.13) for κ^n and A^n. The results established in Sect. 2 for A^n extend without difficulty to κ^n and we shall use them in this situation. However, it should be remembered that κ^n has only been defined for κ which are σ-integrable over $\mathscr{N}$.

The shift $\hat{\Theta}_t$ is defined on random measures by

$$(\hat{\Theta}_t \kappa)(\omega, B) = \kappa(\theta_t \omega, (B - t) \cap]0, \infty[). \tag{4.4}$$

Since κ is carried by $]0, \infty[$ – more precisely $]0, \zeta] \cap]0, \infty[$) it is convenient to extend $\kappa(\omega, \cdot)$ to $\mathbb{R}$ by setting it equal to zero off $]0, \infty[$. Then (4.4) may be written $(\hat{\Theta}_t \kappa)(\omega, B) = \kappa(\theta_t \omega, B - t)$. The definition (4.4) agrees with (2.24) when $\kappa(dt) = dA_t$ with A an increasing process. Formulas (ii) and (iii) of (2.25) remain valid when A is replaced by a random measure κ which is σ-integrable over $\mathscr{P}$ – in (2.25-iii) κ may be any random measure. Finally if κ is σ-integrable over $\mathscr{N}$ one checks that the analog of (2.26-ii) holds; that is, for any stopping time T

$$1_{]\!]T,\infty[\![} * (\hat{\Theta}_T \kappa)^n = 1_{]\!]T,\infty[\![} * \hat{\Theta}_T(\kappa^n).$$

We define now the notion of a homogeneous random measure (HRM) which bears the same relationship to a random measure (RM) as a RAF does to an increasing process. However, there is no known general perfection theorem for a HRM so one must use some care.

(4.5) *Definition.* A homogeneous random measure (HRM) κ is a random measure carried by $]\!]0, \zeta[\![$ such that for each stopping time T, $\hat{\Theta}_T \kappa = 1_{]\!]T,\infty[\![} * \kappa$ a.s. on $\{T < \infty\}$.

Observe that since κ is carried by $]\!]0, \zeta[\![$, $\hat{\Theta}_T\kappa$ is carried by $]\!]T, \zeta[\![$. (The difference between (4.5) and the definition (35.3) of [39], is that in this paper we always suppose that a random measure does not charge $\{0\}$.) If one drops the condition that κ is carried by $]\!]0, \zeta[\![$, then one speaks of HRM possibly charging ζ. Of course, a random measure always is carried $]\!]0, \zeta]\!]$. The defining property of a HRM may also be written $\kappa(\theta_T\omega, B) = \kappa(\omega, B + T(\omega))$ for $B \in \mathscr{R}^+$. Now (2.28) takes the form:

(4.6) **Proposition.** *If κ is a* HRM, *possibly charging ζ, which is σ-integrable over $\mathscr{N}$, then $(1_\Delta * \kappa)^n$ is a natural* HRM.

Finally we shall need the notion of a *perfect* HRM: A random measure κ carried by $]\!]0, \zeta[\![$ is a perfect HRM provided there exists $\Omega^* \subset \Omega$ with $P^x(\Omega^*) = 0$ for all x such that $\hat{\Theta}_t\kappa(\omega, \cdot) = 1_{]t,\infty[}\kappa(\omega, \cdot)$ for all $t \geqq 0$ and $\omega \notin \Omega^*$. Clearly a perfect HRM is a HRM.

We now have in place the basic definitions and properties of random measures. The first result of this section is an extension theorem that is useful in constructing HRM's. For its statement recall that a nearly Borel set $E' \subset E$ is absorbing if $P^x(T_{E-E'} < \infty) = 0$ for all $x \in E'$. Since starting from $x \in E'$ the process almost surely never enters $E - E'$ one may define the restriction, X', of X to E'. It is convenient to suppose that X' is defined on

$$(4.7) \qquad \Omega' = \{\omega \in \Omega \colon X_t(\omega) \in E' \cup \{\Delta\} \quad \text{for all} \quad t \geqq 0\}.$$

Clearly $\Omega' \in \mathscr{F}$, $P^x(\Omega') = 1$ for $x \in E'$, $P^x(\Omega') = 0$ for $x \notin E'$, and X' defined on Ω' by $X'_t(\omega) = X_t(\omega)$ is a right process with state space E', although not Borel unless E' is Borel. Note $\theta_t\Omega' \subset \Omega'$ for all $t \geqq 0$. Also $\mathscr{F}' = \mathscr{F}(X')$ may be identified with the intersection over all initial laws μ carried by E' of the P^μ completions of $\mathscr{F}^0$.

Let κ' be a perfect HRM of X'; that is, there exists a set $\Omega^* \subset \Omega'$ with $P^x(\Omega^*) = 0$ for all $x \in E'$ such that $\hat{\Theta}_t\kappa'(\omega, \cdot) = 1_{]t,\infty[}\kappa'(\omega, \cdot)$ for all $t \geqq 0$ and $\omega \in \Omega' - \Omega^*$. Without loss of generality, we shall assume that Ω^* is empty. Also we suppose that κ' has been extended to $\mathbb{R}$ by setting it equal to zero off $]0, \infty[$. Finally we suppose κ' is a *countably Radon* kernel on $]0, \infty[$; that is, $\kappa' = \sum_n \kappa'_n$ where each κ'_n is a kernel from $(\Omega', \mathscr{F}')$ to $(\mathbb{R}^{++}, \mathscr{R}^{++})$ such that if K is a compact subset of $\mathbb{R}^{++} =]0, \infty[$, $\kappa'_n(\omega, K) < \infty$ for each n and $\omega \in \Omega'$. However, it is *not* assumed that the κ'_n are homogeneous. We may now state the extension result.

(4.8) **Theorem.** *Let E', Ω', X', and κ' be as above. Let $T = T_{E'}$ be the hitting time of E'. Then there exists a perfect* HRM, *κ, of X such that $\kappa(\omega, \cdot) = \kappa'(\omega, \cdot)$ if $\omega \in \Omega'$. Moreover, κ is countably Radon, carried by $]\!]T, \zeta[\![$, and $\kappa = \kappa'$ a.s. P^x for all $x \in E'$. If, in addition, κ' is optional (resp. predictable, natural) over X', then κ is optional (resp. predictable, natural) over X.*

Proof. We fix a strictly decreasing sequence (t_n) in $]0, \infty[$ with $t_n \downarrow 0$. Almost surely $X_t \in E' \cup \{\Delta\}$ on $]T, \infty[$. (This interval is empty if $T = \infty$.) Thus if $\Omega_0 = \{\omega \colon X_{T+t}(\omega) \in E' \cup \{\Delta\}$ for all $t > 0\}$, then $P^x(\Omega_0) = 1$ for all $x \in E$. Also if $t \geqq 0$, then $\theta_t\Omega_0 \subset \Omega_0$ and if $\omega \in \Omega_0$ with $T(\omega) < t$, then $\theta_t\omega \in \Omega'$ defined in (4.7). For $B \in \mathscr{R}^{++}$ define

$$(4.9) \qquad \begin{aligned} \kappa_n(\omega, B) &= \kappa'(\theta_{T+t_n}\omega, B - T(\omega) - t_n) \quad && \text{if } \omega \in \Omega_0 \cap \{T < \infty\} \\ &= 0 && \text{if } \omega \notin \Omega_0 \text{ or } T(\omega) = \infty. \end{aligned}$$

This is well defined since $\theta_{T+t_n}\omega \in \Omega'$ if $\omega \in \Omega_0$ and $T(\omega) < \infty$. Note also that $\kappa_n(\omega, \cdot)$ is carried by $]T(\omega) + t_n, \infty[$ for every ω.

We first claim that if $n < m$ so $t_m < t_n$ then κ_n and κ_m agree on $]T + t_n, \infty[$. For this if $\omega \in \Omega_0 \cap \{T < \infty\}$ and $B \subset]t_n + T(\omega), \infty[$, then $B - T(\omega) - t_m \subset]t_n - t_m, \infty[$, and so

$$\begin{aligned}\kappa_n(\omega, B) &= \kappa'(\theta_{T+t_n}\omega, B - T(\omega) - t_n)\\ &= \kappa'[\theta_{T+t_m+(t_n-t_m)}\omega, B - T(\omega) - t_m - (t_n - t_m)]\\ &= \kappa'[\theta_{t_n-t_m}\theta_{T+t_m}\omega, B - T(\omega) - t_m - (t_n - t_m)]\\ &= \kappa'(\theta_{T+t_m}\omega, B - T(\omega) - t_m)\\ &= \kappa_m(\omega, B)\end{aligned}$$

where the fourth equality uses the homogeneity of κ' and $\theta_{T+t_m}\omega \in \Omega'$. Consequently we may define $\kappa(\omega, \cdot)$ to be the consistent extension of the $\kappa_n(\omega, \cdot)$ to $]0, \infty[$. In particular, $\kappa(\omega, B) = \lim_n \kappa_n(\omega, B)$, and $\kappa(\omega, \cdot)$ is carried by $]T(\omega), \infty[$. More precisely, $\kappa_n(\omega, \cdot)$ is carried by $]T(\omega) + t_n, \zeta(\theta_{T+t_n}\omega) + t_n + T(\omega)[= [T(\omega) + t_n, \zeta(\omega)[$ since $\zeta \circ \theta_{T+t_n} = (\zeta - t_n - T)^+$. Therefore $\kappa(\omega, \cdot)$ is carried by $]T(\omega), \zeta(\omega)[$.

We next show that κ is homogeneous on Ω_0. Fix $t \geqq 0$ and $\omega \in \Omega_0 \cap \{T < \infty\}$. Then $\theta_t\omega \in \Omega_0$ and $T(\theta_t\omega) < \infty$. Suppose $B \subset]T(\omega), \infty[\cap [t, \infty[$, by considering the cases $t < T(\omega)$ and $t \geqq T(\omega)$ one sees that $B - t \subset]T(\theta_t\omega), \infty[$. (Note we are using $t + T(\theta_t\omega) = t$ if $t \geqq T(\omega)$ and $\omega \in \Omega_0$.) Then for some n, $B - t \subset]T(\theta_t\omega) + t_n, \infty[$ and so

$$\begin{aligned}\kappa(\theta_t\omega, B - t) &= \kappa'(\theta_{T+t_n}\theta_t\omega, B - t - T(\theta_t\omega) - t_n)\\ &= \kappa'(\theta_{t+T(\theta_t\omega)+t_n}\omega, B - (t + T(\theta_t\omega)) - t_n).\end{aligned}$$

If $t < T(\omega)$, this becomes

$$\kappa'(\theta_{T+t_n}\omega, B - T(\omega) - t_n) = \kappa(\omega, B).$$

If $t \geqq T(\omega)$ let $u = T(\omega)$. Then $t + T(\theta_t\omega) = t$ and so the above becomes

$$\begin{aligned}\kappa(\theta_t\omega, B - t) &= \kappa'(\theta_{t+t_n}\omega, B - t - t_n)\\ &= \kappa'(\theta_{t-u}\theta_{u+t_n}\omega, B - (u + t_n) - (t - u))\\ &= \kappa'(\theta_{u+t_n}\omega, B - u - t_n) = \kappa(\omega, B),\end{aligned}$$

where the third equality uses the homogeneity of κ' and $\theta_{u+t_n}\omega \in \Omega'$.

If $T(\omega) = \infty$ then $T(\theta_t\omega) = \infty$ and so both $\kappa(\omega, B)$ and $\kappa(\theta_t\omega, B - t)$ are zero. Because $\kappa(\omega, \cdot)$ is carried by $]T(\omega), \infty[$ we have shown that $\kappa(\omega, B) = \kappa(\theta_t\omega, B - t)$ for all $t \geqq 0$, $B \subset]t, \infty[$, and $\omega \in \Omega_0$ (actually $\omega \in \Omega_0 \cup \{T = \infty\}$). Since $P^x(\Omega_0) = 1$ for all x, κ is perfectly homogeneous.

It remains to check the measurability of κ. For later applications the most important cases are when κ' is predictable or natural. We shall give details in these cases. The other cases are handled similarly. Firstly, suppose κ' is predictable. Then $\kappa'(dt) = Y_t'\, dA_t'$ where Y' is a positive everywhere finite process and A' is an increasing process with both Y' and A' predictable over $\mathscr{F}_t' = \mathscr{F}_t(X')$. Clearly it suffices to show each κ_n defined in (4.9) is predictable. To simplify the notation let

$R = T + t_n$. Then R is a stopping time with $X_R \in E' \cup \{\Delta\}$ almost surely and $X_{t+R}(\omega) \in E' \cup \{\Delta\}$ for all $t \geqq 0$ if $\omega \in \Omega_0$. Then if $\omega \in \Omega_0$

$$\begin{aligned}\kappa_n(\omega, \cdot) &= \hat{\Theta}_R \kappa'(\omega, \cdot)\\ &= \Theta_R Y'(\omega) * d(\Theta_R A')(\omega),\end{aligned}$$

and so it suffices to show that if Z' is predictable over $(\mathscr{F}_t')$, then

$$Z_t(\omega) = 1_{]R(\omega), \infty[}(t)\, Z'_{t-R(\omega)}(\theta_R \omega)$$

is predictable over $(\mathscr{F}_t)$. Note Z is defined on the set of full measure Ω_0 since $\theta_R \Omega_0 \subset \Omega'$. For this it suffices to consider $Z' = 1_{]\!]U,V]\!]}$ where U and V are $(\mathscr{F}_t')$ stopping times. In this case Z is the indicator of $]\!]R + U \circ \theta_R, R + V \circ \theta_R]\!]$, and consequently it suffices to show that $S = R + U \circ \theta_R$ is an $(\mathscr{F}_t)$ stopping time for any $(\mathscr{F}_t')$ stopping time U. Now

$$\{S < t\} = \bigcup_{q \in \mathbb{Q}^+} \{R < t - q, U \circ \theta_R < q\},$$

and so finally it suffices to show $\{U \circ \theta_R < q,\ R < \infty\} \in \mathscr{F}_{q+R}$. Given an initial law μ, $\nu = \mu P_R$ is carried by E'. Since $\{U < q\} \in \mathscr{F}_q(X')$, it follows readily that there exist H_1', $H_2' \in \mathscr{F}_q^0$ with $H_1' \leqq 1_{\{U<q\}} \leqq H_2'$ on Ω' and $P^\nu(H_1' + H_2') = 0$. Let $H_j = 1_{\{R<\infty\}} H_j' \circ \theta_R, j = 1,2$. Then H_1 and H_2 are in $\mathscr{F}_{q+R}$, $H_1 \leqq 1_{\{U \circ \theta_R < q, R < \infty\}} \leqq H_2$ and $P^\mu(H_1 + H_2) \leqq P^\nu(H_1' + H_2') = 0$. This then shows that Z is $(\mathscr{F}_t)$ predictable and establishes the fact that κ is predictable if κ' is.

The natural case is an immediate consequence of this because $\kappa' = Y' * A'$ with Y' and A' natural over X'. Let Z' and B' be predictable over X' and agree with Y' and A' on $]0, \zeta'[$. Then $\kappa^n = 1_{]\!]0,\zeta[\![} * (\Theta_R Z' * d(\Theta_R B'))$ and hence κ^n is natural. The optional case is the came as the predictable except that $]\!]U, V]\!]$ must be replaced by $[\![U, V[\![$. In the general case one uses the fact that κ' is countably Radon to reduce to the case of an increasing process A'. This is handled exactly as the predictable case except that U and V are positive $\mathscr{F}'$ measurable random variables rather than $(\mathscr{F}_t')$ stopping times. This completes the proof of (4.8).

We shall now use (4.8) to extend some of the results of §3. First of all if κ is a HRM one defines its q-potential function $u = u^q = u_\kappa^q$ for $q \geqq 0$ by

$$u_\kappa^q(x) = E^x \int e^{-qt} \kappa(dt). \tag{4.10}$$

It is immediate that u^q is q-excessive. If μ is an initial law, then

$$E^\mu \left\{ \int_{]t,\infty[} e^{-qs} \kappa(ds) \,|\, \mathscr{F}_t \right\} = e^{-qt} u_\kappa^q(X_t),$$

and if $\int u_\kappa^q \, d\mu < \infty$, this expression is the potential in the general theory of processes sense of the increasing process $t \to \int_{]0,t]} e^{-qs} \kappa(ds)$ relative to P^μ. Since κ is carried by $]\!]0, \zeta[\![$ it follows from (2.14-ii) extended to random measures that if κ is σ-integrable over $\mathscr{N}$, then κ^n, the dual natural projection of κ, and κ have the same q-potential function for any $q \geqq 0$; that is, $u_{\kappa^n}^q = u_\kappa^q$. This remark will be used in the proof of the next result which extends (3.3).

(4.11) **Theorem.** *Let f be a q-excessive function and let μ be an initial law with $\int f\,d\mu < \infty$. Suppose that conditions* (i) *and* (ii) *of* (3.3) *hold. Then there exist a* NHRM, κ *such that $f(X)$ and $u^q_\kappa(X)$ are P^μ indistinguishable.*

Proof. Let $E' = \{f < \infty\}$. Since $\int f\,d\mu < \infty$, μ is carried by E'. Also E' is a nearly Borel absorbing set for X. Let X' be the restriction of X to E' as described above (4.8). If f' is the restriction of f to E', then f' is a finite q-excessive function of X' which satisfies the conditions of Theorem (3.3) relative to X' and μ. (Observe that if $D \subset E'$ is finely closed for X', then since μ is carried by E', $T_D = T_F$ a.s. P^μ where F is the fine closure relative to X of D.) Therefore there exists a NAF, A' of X' with $f'(X') = u^q_{A'}(X')$ up to P^μ evanescence relative to X'. Apply now (4.8) to obtain a NHRM, κ carried by $]\!]T, \zeta[\![$ where $T \equiv T_{E'}$, with κ extending A'. Clearly $f(X)$ and $u^q_\kappa(X)$ are P^μ-indistinguishable, establishing (4.11).

5. Standard Processes

Perhaps the most important subclass of Borel right processes is the class of standard processes. The reason for this is that often one can obtain much sharper results for standard processes as will be illustrated in the sections devoted to standard processes in what follows, and also that most familiar processes are standard – or even better. By a standard process we mean a Borel right process X which is quasi left continuous (*qlc*) on $[0, \zeta[$; that is, if (T_n) is an increasing sequence of stopping times with limit T, then $X(T_n) \to X(T)$ almost surely on $\{T < \zeta\}$. This is the definition used in [26, XIV], where it is shown that almost surely X then necessarily possesses a left limit X_{t-} in E for all $t \in]0, \zeta[$. *Therefore we shall take as part of our definition of a standard process that $X_{t-}(\omega)$ exists in E for all $t \in]0, \zeta(\omega)[$ and all $\omega \in \Omega$.*

This definition differs slightly from that in [4], where it is always assumed that E is locally compact with a countable base. However, it is possible to use most results from [4] directly in the present situation because of the following artifice. Let F be a compact metrizable space containing $E_\Delta = E \cup \Delta$ topologically. Then E is necessarily Borel in F by Lusin's theorem. Extend X on E to a Markov process $\bar{X}$ on F by setting $\bar{X}_t = X_t$ if $\bar{X}_0 \in E_\Delta$, $\bar{X}_t = \bar{X}_0$ if $\bar{X}_0 \in F - E_\Delta$. The process $\bar{X}$ is obviously standard in the sense of [4], and results about X are obtained by simple restriction to E_Δ. From now on we shall use results from [4] without further comment, leaving the technicalities to the reader.

In applications it is often necessary to consider a fixed initial measure μ.

(5.1) *Definition.* A Borel right process X is μ-standard provided

(i) P^μ almost surely X_{t-} exists in E on $]0, \zeta[$, and

(ii) given an increasing sequence (T_n) of stopping times with limit T, $X(T_n) \to X(T)$ almost surely P^μ on $\{T < \zeta\}$.

It is immediate from the main result of this section – Theorem 5.5 below – that condition (i) in the above definition is a consequence of (ii). Obviously if X is μ-standard for each initial measure μ, then X becomes a standard process after deleting the null subset of Ω in which $X_{t-}(\omega)$ fails to exist in E for some $t \in]0, \zeta(\omega)[$.

We also need to consider some of the results of section 2 for a single initial measure. To this end let $\mathscr{I}^\mu$ be the ideal of P^μ evanescent processes and let $\mathscr{M}^\mu \equiv (\mathscr{R}^+ \times \mathscr{F}^\mu) \vee \mathscr{I}^\mu$ be the *μ-measurable* processes on $\mathbb{R}^+ \times \Omega$. It is evident that $\mathscr{M}^\mu = (\mathscr{R}^+ \times \mathscr{F}^0) \vee \mathscr{I}^\mu$. The class $\mathscr{N}^\mu$ of μ-natural processes consists of those processes in $\mathscr{M}^\mu$ which are P^μ indistinguishable on $]\!]0, \zeta[\![$ from a natural process. A process in $\mathscr{N}^\mu$ is $(\mathscr{F}_t^\mu)$ adapted. It is easy to check that $Y \in \mathscr{N}^\mu$ if and only if there exists $W \in \mathscr{P}^\mu$ which is P^μ indistinguishable from Y on $]\!]0, \zeta[\![$. An $(\mathscr{F}_t^\mu)$ stopping time is μ-natural provided it is equal to a natural stopping time P^μ almost surely. A μ-natural increasing process A is an increasing process which is μ-natural and satisfies $A_{\zeta-} = A_\zeta$. We claim that such an A is P^μ indistinguishable from a natural increasing process B. To see this one may suppose A uniformly bounded. Since A is $(\mathscr{F}_t^\mu)$ adapted it is a standard fact that A is P^μ-indistinguishable from an (optional) increasing process $\bar{A}$ with $\bar{A}_t = \bar{A}_{\zeta-}$ for all $t > \zeta$. We may also assume that $\bar{A}$ is uniformly bounded. Then $\bar{A}^n$ exists as a natural increasing process, and in particular it is a dual natural projection of $\bar{A}$, hence of A, relative to P^μ. Therefore $\bar{A}^n$ and A are P^μ-indistinguishable, establishing the claim. If A is a μ-natural increasing process for every μ, then A being $(\mathscr{F}_t^\mu)$ adapted for every μ is $(\mathscr{F}_t)$ adapted. Consequently A is (indistinguishable from) a natural increasing process; namely A^n.

The next result shows that the natural projection of a function of X has a very simple form when X is standard.

(5.2) **Theorem.** *Let h be a positive (or bounded) nearly Borel function on E. If X is μ-standard, then ${}^n(h(X))$ is P^μ indistinguishable from $h(X_-)\,1_{]\!]0,\zeta[\![}$. Of course, $h(X_-)\,1_{]\!]0,\zeta[\![}$ is only defined a.s. P^μ.*

Proof. Clearly we may suppose h Borel in proving (5.2). By a monotone class argument it then suffices to assume h is a bounded continuous function. Let $Y_t = \liminf_{s \uparrow\uparrow t} h(X_s)$ for $t > 0$. It is known, [9] or [10], that Y is P^ν predictable for all ν, and hence $Y \in \mathscr{P}$ (see, for example [39, 23.1]). Now h being continuous, Y is P^μ indistinguishable from $h(X_-)$ on $]\!]0, \zeta[\![$. Let ${}^n h(X)\,1_{]\!]0,\zeta[\![} = W\,1_{]\!]0,\zeta[\![}$ with $W \in b\mathscr{P}$. If T is a predictable stopping time $X_T = X_{T-}$ a.s. P^μ on $\{T < \zeta\}$ because of (5.1-ii), and so

$$\begin{aligned} E^\mu[W_T; 0 < T < \zeta] &= E^\mu[h(X_T); 0 < T < \zeta] \\ &= E^\mu[h(X_{T-}); 0 < T < \zeta] \\ &= E^\mu[Y_T; 0 < T < \zeta]. \end{aligned}$$

Therefore ${}^p(1_{]\!]0,\zeta[\![} W) = {}^p(1_{]\!]0,\zeta[\![} Y)$ up to P^μ evanescence, and the result follows from (2.5).

We are now able to characterize natural increasing processes over a standard process.

(5.3) **Theorem.** *Let X be μ-standard. Then an optional right continuous increasing process A is μ-natural if and only if A is carried by $]\!]0, \zeta[\![$ and ΔA vanishes P^μ almost surely on $\{X \neq X_-\}$. Of course, $\{X \neq X_-\}$ is the set of (t, ω) with $0 < t < \zeta(\omega)$ such that either $X_{t-}(\omega)$ does not exist in E or $X_{t-}(\omega)$ exists in E but $X_{t-}(\omega) \neq X_t(\omega)$.*

Proof. If A is μ-natural, then dA is carried by $]\!]0, \zeta[\![$ and by the discussion preceding (5.2) one may suppose A is natural in showing that ΔA vanishes P^μ almost surely on

$\{X \neq X_-\}$. Let $\delta > 0$ and let $R = \inf\{t: \Delta A_t > \delta\}$. Then R is a natural stopping time. It follows from (2.4-iii) and (5.1-ii) that $X_{R-} = X_R$ almost surely P^μ on $\{R < \zeta\}$. The same argument applies to the successive times at which $\Delta A > \delta$ and by varying δ one sees that ΔA vanishes on $\{X \neq X_-\}$ almost surely P^μ. For the converse let $H = \{t > 0: X_{t-}$ exists in E and $X_{t-} = X_t\}$ and $B = 1_H * A$. Then $B = A$ up to P^μ evanescence and B is an optional increasing process with dB carried by $]\!]0, \zeta[\![$ and ΔB vanishing off $\{X = X_-\} \equiv H$. It suffices to show that B is natural. In other words to complete the proof of (5.3) it is enough to show that an optional increasing process A with dA carried by $]\!]0, \zeta[\![$ and ΔA vanishing off $\{X = X_-\}$ is natural. For this we may assume A bounded. Let $W = Yh(X)$ with $Y \in b\mathscr{P}$ and $h \in b\mathscr{E}$. Then from (5.2) and (2.7), ${}^n(Yh(X)) = Yh(X_-)1_{]\!]0,\zeta[\![}$, and hence

$$E^\bullet \int Yh(X)\, dA = E^\bullet \int Yh(X_-)\, dA = E^\bullet \int {}^n(Yh(X))\, dA.$$

But processes of the form W generate $\mathcal{O}$ and consequently $E^\bullet \int Z\, dA = E^\bullet \int {}^nZ\, dA$ for all $Z \in b\mathcal{O}$. Since A is optional this last equality holds for all $Z \in b\mathscr{M}$. Therefore (2.18) implies that A is natural, completing the proof.

(5.4) **Corollary.** *Let T be a stopping time. If X is μ-standard, then T is μ-natural if and only if $X_{T-} = X_T$ almost surely P^μ on $\{0 < T < \zeta\}$. Under this last condition there exists an increasing sequence of stopping times (T_n) such that P^μ almost surely the following hold: (T_n) is strictly increasing on $\{0 < T < \zeta\}$, $\lim T_n = T$ on $\{T < \zeta\}$, and $\lim T_n \geqq \zeta$ on $\{T \geqq \zeta\}$.*

Proof. Applying (5.3) to the increasing process $A_t = 1_{[T,\infty[}(t)\, 1_{\{0<T<\zeta\}}$ and using the fact proved above (5.2) that a μ-natural increasing process is P^μ indistinguishable from a natural increasing process, the desired conclusions follow from (2.4).

Remark. The best previous result along these lines is that if X is a right process and T a stopping time with $X_T = X_{T-}$ on $\{0 < T < \zeta\}$, then T is accessible (see [45] or [16]). Corollary (5.4) greatly strengthens the conclusion when X is standard. Of course, if X is a Hunt process and $X_T = X_{T-}$ on $\{0 < T < \infty\}$, then T is predictable.

We come now to the main result of this section. It characterizes standardness in terms of hitting times.

(5.5) **Theorem.** *Let X be a Borel right process and μ an initial law. Then the following are equivalent:*

(i) *If (D_n) is a decreasing sequence of finely closed sets in $\mathscr{E}^e$ and $T = \lim T_{D_n}$, then $X(T_{D_n}) \to X(T)$ a.s. P^μ on $\{T < \zeta\}$.*

(ii) *X is μ-standard.*

(iii) *P^μ a.s. X_{t-} exists in E on $]0, \zeta[$ and if X^r_{t-} denotes the Ray left limit in a Ray compactification $\bar{E}$ of $E \cup \{\Delta\}$, then up to P^μ evanescence*

$$(\{X^r_- \notin E_\Delta\} \cup \{X^r_- = X\}) \cap]\!]0, \zeta[\![\subset \{X_- = X\} \cap]\!]0, \zeta[\![.$$

Proof. In proving (5.5) we may suppose that Δ is isolated in $E_\Delta \equiv E \cup \{\Delta\}$. Obviously (ii) $\Rightarrow$ (i). The majority of the argument is to show that (i) $\Rightarrow$ (iii), so suppose that (i) holds. Let F be a compact metric space containing $E \cup \{\Delta\}$ topologically as a Borel subset. (We are not using the Ray topology here.) The first step is to show the existence of X_{t-} in F. To this end if g is a continuous function on F and $a < b$, then the set

$$\Gamma \equiv \left\{\liminf_{s \uparrow\uparrow t} g(X_s) < a,\ \limsup_{s \uparrow\uparrow t} g(X_s) > b\right\}$$

is predictable (see [9, VI-(T3)] or [39, §41]). Since $t \to g(X_t)$ is right continuous, Γ does not contain any infinite strictly decreasing sequence. It follows that the debut T of Γ is a predictable thin terminal time with $[\![T]\!] \subset \Gamma$. We shall show that $P^\mu(T<\zeta) = 0$ and by letting $a<b$ run over the rationals and g over a countable dense subset of $C(F)$, this implies the existence of X_{t-} in F on $]0,\zeta[$ almost surely P^μ. Let (D_n) be the sets in (3.1-i) and $T_n = T_{D_n}$. Because $X_t \in D_n$ on an interval $]T-\varepsilon, T[$ if $0<T<\infty$ and $[\![T]\!] \subset \Gamma$, the hitting times of $D_n \cap \{g \leqq a\}$ and $D_n \cap \{g \geqq b\}$ respectively both increase to T on $\{T<\infty\}$. Using (i) for these sequences we obtain $g(X_T) \leqq a$ and $g(X_T) \geqq b$ almost surely P^μ on $\{T<\zeta\}$. Consequently $P^\mu(T<\zeta)=0$.

Let $X_- \equiv (X_{t-})$ be the left limit of X in F which exists P^μ almost surely on $]0,\zeta[$. We next claim that up to P^μ evanescence $\{X_- \neq X\} \cap]\!]0,\zeta[\![$ cannot intersect the graph of any thin terminal time T which is P^μ predictable. For, given such a T, let $D_n = \{x\colon E^x(e^{-T}) \geqq 1-1/n\}$ and $T_n = T_{D_n}$. Then according to (3.1-i) (T_n) increases to T strictly from below almost surely P^μ on $\{T<\zeta\}$, and so it follows from (i) that $X_{T-} = X_T$ a.s. P^μ on $\{T<\zeta\}$. This establishes the claim in the second sentence of this paragraph.

Recall that $X^r_- = (X^r_{t-})_{t>0}$ denotes the Ray left limit of X in a Ray compactification $\bar{E}$ of E_Δ. Here E_Δ plays the role of E in [16], [39], or [45]. It is known [16, 13.1] or [45, §1] that any stopping time with graph contained in $\{X^r_- = X\}$ up to P^μ evanescence is P^μ predictable. Now let d be a metric for F and $\varepsilon>0$. Set $H = \{(t,\omega)\colon X_{t-}(\omega)$ exists in F and $d(X_{t-}(\omega), X_t(\omega)) \geqq \varepsilon\}$. Then $W = \{X^r_- = X\} \cap H \cap]\!]0,\zeta[\![$ is a discrete optional set, and its debut T is a thin terminal time which is P^μ predictable. Consequently $P^\mu(T<\zeta)=0$. Hence up to P^μ evanescence

$$\{X^r_- = X\} \cap]\!]0,\zeta[\![\subset \{X_- = X\} \cap]\!]0,\zeta[\![. \tag{5.6}$$

At this point we require the following result from [39, §45], but for the convenience of the reader, a proof will be sketched in (5.9) at the end of this section.

(5.7) *There exists a predictable AF, A, possibly charging ζ, with bounded one potential such that $\{\Delta A>0\} = \{X^r_- \notin E_\Delta\}$ up to evanescence.*

Let T be the debut of $\{\Delta A \geqq \varepsilon\}$. Then T is a thin predictable terminal and, by the argument given two paragraphs above, the intersection of $\{X \neq X_-\} \cap]\!]0,\zeta[\![$ and $[\![T]\!]$ is P^μ evanescent. Varying $\varepsilon>0$ it follows that $\{X^r_- \notin E_\Delta\} \cap]\!]0,\zeta[\![\cap \{X \neq X_-\}$ is P^μ evanescent. Writing this in the form

$$\{X^r_- \notin E_\Delta\} \cap]\!]0,\zeta[\![\subset \{X_- = X\} \cap]\!]0,\zeta[\![\tag{5.8}$$

up to P^μ evanescence, and bringing in (5.6), the fact ([16, 13.4] or [39, §46]) that $X_- = X^r_-$ P^μ almost surely on $\{X^r_- \in E_\Delta,\ X^r_- \neq X\}$, and the fact that Δ is isolated in E_Δ, we see that $X_- \in E$ almost surely P^μ on $]\!]0,\zeta[\![$ and that (i) $\Rightarrow$ (iii).

To complete the proof we show that (iii) $\Rightarrow$ (ii). Let (T_n) be an increasing sequence of stopping times with limit T. We must show that $X(T_n) \to X(T)$ a.s. P^μ on $\{T<\zeta\}$. From [16, 13.1] or [45, §1], T is totally inaccessible on $\{X^r_{T-} \in E_\Delta, X^r_{T-} \neq X_T, T<\infty\}$ and so $T_n = T$ and $X(T_n) = X(T)$ for large enough n on this set. On $\{X_{T-} = X_T, T<\zeta\}$ one obviously has $X(T_n) \to X(T)$. In view of (iii) this covers all possibilities.

(5.9) *Proof of* (5.7). In this argument the state space for X is taken to be E_Δ, so that X has infinite lifetime and there is no difference between natural and predictable. Because the minimum of two natural q-potentials is again a natural q-potential, it is easy to see from the construction of the Ray cone R – see [16, §10] or [39, §39] – that each $g \in R$ is a bounded q-potential for some $q > 0$. Let $g \in R$. Then for some $q > 0$, $g = u^q_B$ with B a predictable AF of X. Using the fact ([16, 11.15] or [39, §39]) that the predictable projection of $g(X)$ is $\bar{P}_0 g(X^r_-)$ where $(\bar{P}_t)$ is the Ray semigroup constructed from (P_t), it is not difficult to check that $\Delta B_t = g(X_t)_- - \bar{P}_0 g(X^r_-)$ (see [39, 42.7]). Let $\bar{g}$ denote the continuous extension of g to $\bar{E}$. Then $\Delta B_t = \bar{g}(X^r_{t-}) - \bar{P}_0 g(X^r_{t-})$ and so $A_t = \sum_{0<s\leqq t} (\bar{g} - \bar{P}_0 g)(X^r_{s-})$ defines a predictable AF of X with q-potential dominated by $u^q_B = g$. If $g \leqq c$ then $u^1_B = u^q_B + (q-1)\, U^1 u^q_B \leqq (q+1)\, c$, and so A has a bounded one potential. Next let (g_n) be a uniformly dense sequence in R with g_n being a q_n-potential. Define

$$A^n_t = \sum_{0<s\leqq t} (\bar{g}_n - \bar{P}_0 g_n)(X^r_{s-}),$$

$$A_t = \sum_{n\geqq 1} 2^{-n} (q_n+1)^{-1} \|g_n\|^{-1} A^n_t.$$

It is evident that A is a predictable AF of X with a bounded one potential and $\{\Delta A > 0\} = \bigcup_n \{(\bar{g}_n - \bar{P}_0 g_n)(X^r_-) > 0\}$. But this latter set is precisely $\{X^r_- \notin E_\Delta\}$ up to evanescence – see [16, 11.16] or [39, §45]. Thus A has the properties claimed in (5.7).

(5.10) *Remark.* The somewhat peculiar looking condition (5.5-iii) will turn out to be technically very useful in section 16. In addition, it turns out that the condition (5.5-iii) is unchanged if we replace $\{X^r_- \notin E_\Delta\}$ by $\{X^r_- \in E\}$, because $\Gamma \equiv \{X^r_- = \Delta\} \cap]\!]0, \zeta[\![$ turns out to be evanescent. To see this, observe that on one hand Γ is obviously in $\mathcal{N}$ and has countable sections, while on the other hand, $X \neq X^r_-$ on Γ implies, by [16, 13.1] applied to X with state space E_Δ, that Γ is a countable union of graphs of totally inaccessible stopping times.

Part II: Processes With an Excessive Initial Measure

6. Exceptional Sets Relative to an Excessive Measure

A measure is *countably finite* provided it is a countable sum of finite measures. A measure m on $(E, \mathcal{E})$ is *excessive* (relative to X or (P_t)) provided it is countably finite and satisfies (i): $mP_t \leqq m$ for each $t \geqq 0$, and (ii) $mP_t \uparrow m$ setwise as $t \downarrow 0$. It is well-known (see the argument at the bottom of page 257 of [4]) that when m is σ-finite, (ii) is a consequence of (i). The advantage of the above definition is if λ is a countably finite measure on E, then λU is excessive, although λU need not be σ-finite even when λ is finite. We now fix an excessive measure m. It is *not* assumed that m is a reference measure. The countable finiteness justifies the use of Fubini's theorem in the arguments involving m which follow. Actually some of these results, particularly in §8, depend only on the inequality (i) and not on (ii). However, we have made no effort to separate off these results, especially since beginning in §9 we suppose that m is, in fact, σ-finite.

Let $\mathscr{E}^m$ denote the completion of $\mathscr{E}$ relative to m. If $f, g \in \mathscr{E}^m$, we write $f = g\,[m]$ to mean $f = g$ a.e. m. If $f, g \in \mathscr{E}_+$ and $f = g\,[m]$, then since $mP_t \leqq m$ and $qmU^q \leqq m$ it follows that $P_t f = P_t g\,[m]$ and $U^q f = U^q g\,[m]$ for $t \geqq 0$ and $q \geqq 0$. (Consider first the case f, g bounded and $q > 0$, and then pass to the general case by taking limits.) Consequently given $f \in \mathscr{E}_+^m$ and choosing $g \in \mathscr{E}_+$ with $f = g\,[m]$, $P_t f$ and $U^q f$ are well defined as the m-equivalence classes containing $P_t g$ and $U^q g$ respectively. Therefore we may regard P_t and U^q as operators on m-equivalence classes. In fact since $(P_t f)^2 \leqq P_t(f^2)$, P_t and U^q send $L^2(m)$ into itself for $t \geqq 0$, $q > 0$. If $f \in b\mathscr{E}^m$ and $g \in b\mathscr{E}$ with $f = g\,[m]$, then $U^q g$, $U^r g$, and $U^r U^q g$ are Borel representatives of the corresponding equivalence classes with g are replaced by f. Hence (U^q) satisfies the resolvent equation on m-equivalence classes. We shall use the notation $(f, g) = \int fg\,dm$ whenever the integral exists. Using the fact that m is countably finite it is a standard argument to show that if $f, g \in \mathscr{E}_+^m$, then $t \to (g, P_t f)$ is Borel measurable and that

$$(g, U^q f) = \int e^{-qt}(g, P_t f)\,dt. \tag{6.1}$$

We shall encounter a number of types of exceptional sets relative to P^m. The definitions below depend on m only through its null sets, which are the same as those of an equivalent finite measure, and this permits us to use some results proved previously for finite measures. However, in general, P^m is only countably finite.

Given a nearly Borel set $F \subset E$ let $T_F \equiv \inf\{t > 0: X_t \in F\}$ denote its hitting time and $S_F = \inf\{0 < t < \zeta: X_{t-}$ exists in E and $X_{t-} \in F\}$ denote its left hitting time. The argument in the first part of the proof of Theorem 13.4 in [16] shows that S_F is a stopping time over $(\mathscr{F}_t)$. Note that $\{S_F < \infty\} = \{S_F < \zeta\}$ and $\{T_F < \infty\} = \{T_F < \zeta\}$ for $F \subset E$. Also because E is a separable metric space and $t \to X_t(\omega)$ is right continuous, the set of t for which $X_{t-}(\omega)$ fails to exist in E or $X_{t-}(\omega)$ exists but is not equal to $X_t(\omega)$ is countable for each ω.

(6.2) *Definition.* Let $F \subset E$ be nearly Borel. Then F is

$$m\text{-polar if } P^m\{T_F < \infty\} = 0; \tag{6.3}$$

$$\text{left } m\text{-polar if } P^m\{S_F < \infty\} = 0; \tag{6.4}$$

(6.5) m-semipolar if there exists a nearly Borel semipolar set G such that $(F - G) \cup (G - F)$ is m-polar.

In (6.5), $F \cap G$ is also nearly Borel semipolar, and therefore every m-semipolar set is the union of a nearly Borel semipolar set and an m-polar set.

(6.6) **Proposition.** *If f is a nearly Borel function on E, then there exist Borel functions g and h with $g \leqq f \leqq h$ and such that $\{g < h\}$ is both m-polar and left m-polar.*

Proof. It is known that the set $\{X_- \neq X\} \equiv \{t: 0 < t < \zeta, X_{t-}$ exists in $E, X_{t-} \neq X_t\}$ $= \cup [\![R_n]\!]$ where each R_n is a stopping time for $n \geqq 1$. See, for example, the proof of (13.4) in [16]. Let μ be a probability equivalent to m and let $\nu_n(dx) = P^\mu[X_{R_n-} \in dx, R_n < \infty\}$ for $n \geqq 1$. Define $\nu = \mu + \sum_{n \geqq 1} 2^{-n} \nu_n$ and choose Borel functions g and h with $g \leqq f \leqq h$ such that $P^\nu[g(X_t) < h(X_t)$ for some $t \geqq 0] = 0$. It is evident that g and h have the properties asserted in (6.6).

One immediate conclusion from (6.6) is the fact that if $G \subset E$ is nearly Borel and semipolar then there exist $G_1, G_2 \in \mathscr{E}$ with $G_1 \subset G \subset G_2$ and $G_2 - G_1$ is m-polar (and left m-polar). In particular, G_1 is semipolar. Consequently, the condition (6.5) is not changed if we require G to be Borel rather than nearly Borel.

Azemá [3] proved, assuming $\zeta < \infty$ a.s., that

(6.7) **Theorem.** *A nearly Borel set F is m-semipolar if and only if $P^m\{X_t \in F$ for uncountably many $t\} = 0$.*

Applying Azéma's result to the q-subprocess of X $(q > 0)$ and then letting $q \to 0$, it is clear that (6.7) is actually valid in complete generality. Observe that the condition in (6.7) is unchanged if X_t is replaced by X_{t-}. That is, left m-semipolar is the same as m-semipolar.

It should be observed that m-polar sets are not nearly as useful as polar sets. For example, if $F \subset E$ is polar, F^c is absorbing (i.e., $P^x\{X_t \in F^c$ for all $t \geqq 0\} = 1$ for all $x \in F^c$) and F may be deleted from the state space without affecting the process on F^c. This is not the case with an m-polar set, and for this reason we define a new type of exceptional set intermediate to polar and m-polar.

(6.8) *Definition.* A nearly Borel set $F \subset E$ is m-inessential if F^c is absorbing and F is m-polar.

It is clear that an m-inessential set F may simply be deleted from E so that X on F^c is a right process and under P^m it is equivalent in law to X under P^m.

It is almost immediate that if $F \subset E$ is nearly Borel and F^c is absorbing (for example, if $F = \{f = \infty\}$ with f q-excessive), then F is m-polar if and only if $m(F) = 0$.

(6.9) **Proposition.** *Let $\{f_n\}$ be a decreasing sequence of q-excessive functions and define $f = \lim f_n$. Then $\{f > 0\}$ is m-polar provided $m\{f > 0\} = 0$.*

Proof. Because $P_T^q f_n \leqq f_n$ for every stopping time T, $P_T^q f \leqq f$ also. In particular $P_T^q f(x) = 0$ if $f(x) = 0$. If $\{f > 0\}$ were not m-polar we could find, by the section theorem, an $\varepsilon > 0$ and a stopping time T with $[\![T]\!] \subset \{f(X) \geqq \varepsilon\}$, $P^m\{T < \infty\} > 0$. Then if $f(x) = 0$

$$0 = P_T^q f(x) \geqq \varepsilon E^x[e^{-qT}].$$

That is, $P^x\{T < \infty\} = 0$ if $f(x) = 0$. Therefore, $m(f > 0) = 0$ implies $P^m\{T < \infty\} = 0$, a contradiction which establishes that $\{f > 0\}$ is m-polar.

(6.10) **Proposition.** (i) *If $F \subset E$ is finely open, nearly Borel, and $m(F) = 0$,* then F is m-polar.

(ii) *If $F \in \mathscr{E}^m$ and $mU^q(F) = 0$, then $m(F) = 0$. In particular if F is m-semipolar, then $m(F) = 0$.*

(iii) *If f and g are q-excessive and $f = g\,[m]$, then $\{f \neq g\}$ is m-polar.*

Proof. For (i), Fubini's theorem and the excessiveness of m give

$$E^m \int_0^\infty 1_F(X_t)\,dt = \int_0^\infty mP_t(F)\,dt = 0\,.$$

Hence F, being finely open, must be m-polar. For (ii) first note that $F \in \mathscr{E}^m$ implies that F is in the completion of $\mathscr{E}$ relative to mP_t and mU^q for each $t \geqq 0$ and $q \geqq 0$. Since $\int_0^\infty e^{-qt}\,mP_t(F)\,dt = mU^q(F) = 0$, it follows that $mP_t(F) = 0$ a.e. Lebesgue in t. But $mP_t(F) \uparrow m(F)$ as $t \downarrow 0$, so $m(F) = 0$. If F is m-semipolar, (6.7) implies that $mU(F) = 0$. Finally setting $F = \{f \neq g\}$, (iii) is immediate from (i).

(6.11) **Proposition.** *If f is q-excessive, then there exist Borel q-excessive g and h with $g \leqq f \leqq h$ such that $\{g < h\}$ is m-polar.*

Proof. Suppose first that $q > 0$ and $f = U^q k$ with $k \in b\mathscr{E}_+^*$. Choose $k_1, k_2 \in b\mathscr{E}_+$ with $k_1 \leqq k \leqq k_2$ and $mU^q(\{k_1 < k_2\}) = 0$. Then $U^q k_1 \leqq f \leqq U^q k_2$ with the outer functions agreeing m a.e. Applying (6.10-iii) the result follows in this case. If f is q-excessive with $q > 0$, choose $f_n \in b\mathscr{E}_+^*$ with $U^q f_n \uparrow f$. By the first case there exist Borel q-excessive g_n, h_n with $g_n \leqq U^q f_n \leqq h_n$ and $\{g_n < h_n\}$ m-polar for each n. Define $\bar{g} = \liminf g_n$, $\bar{h} = \liminf h_n$. Then $\bar{g} \leqq f \leqq \bar{h}$ and $\{\bar{g} < \bar{h}\}$ is m-polar. But $\bar{g}$ and $\bar{h}$ are only q-supermedian (relative to (U^q)). Let $g = \lim_{r\to\infty} rU^{r+q}\bar{g}$, $h = \lim_{r\to\infty} rU^{r+q}\bar{h}$. Then g and h are Borel q-excessive functions with $g \leqq \bar{g}$, $h \leqq \bar{h}$ and $\{g < \bar{g}\}$ has potential zero. Hence $m(\{g < \bar{g}\}) = 0$ by (6.10-ii). Also $g \leqq f \leqq h$. Since $\{g < h\}$ is contained in $\{g < \bar{g}\} \cup \{\bar{g} < \bar{h}\}$, it has m-measure zero, and again the result follows from (6.10-iii). Finally if $q = 0$, f is r-excessive for each $r > 0$. Thus for each n, there exist Borel $1/n$-excessive functions g_n and h_n with $g_n \leqq f \leqq h_n$ and $\{g_n < h_n\}$ m-polar. Let $\bar{g} = \liminf g_n$, $\bar{h} = \liminf h_n$. Then $\bar{g}$ and $\bar{h}$ are Borel and $1/n$-supermedian for each n, and hence (zero) supermedian. Since $\{\bar{g} < \bar{h}\}$ is m-polar one now finishes the argument as in the previous case.

(6.12) **Proposition.** *Let $F \subset E$ be m-polar. Then F is contained in a Borel m-inessential set which may be taken finely open if F is finely open. In particular an m-inessential set is contained in a Borel m-inessential set.*

Proof. By (6.6), F being nearly Borel, we may choose a Borel set H with $F \subset H$ and H m-polar. The following argument shows that if F is finely open, H may be taken finely open. Let $\psi(x) = P^x(T_F < \infty)$. Then ψ is excessive, $\psi = 0\,[m]$ since F is m-polar, and $F \subset \{\psi > 0\}$ because F is finely open. Now using (6.11) choose a Borel excessive function h with $\psi \leqq h$ and $\{\psi < h\}$ m-polar. Then $F \subset \{h > 0\}$ and $h = 0\,[m]$. In view of (6.10-i), $H = \{h > 0\}$ has the desired properties. Now define $\varphi(x) = P^x(T_H < \infty)$. As before $\varphi = 0\,[m]$ and one may choose a Borel excessive function $g \geqq \varphi$ with $g = 0\,[m]$. Let $K = \{g > 0\} \cup H$. Then K is Borel, m-polar, and finely open if H is. To complete the proof we must show that K^c is absorbing. If $g(x) = 0$, then $\varphi(x) = 0$ and so $P^x(T_H < \infty) = 0$. Thus if $x \in K^c = \{g = 0\} \cap H^c$, the process starting from x never enters H. Because $\{g = 0\}$ is absorbing, it is now clear that K^c is also absorbing.

(6.13) **Proposition.** *If F is m-semipolar then F is the union of an m-polar set and a countable number of totally thin Borel sets.*

Proof. As noted following (6.5) F is the union of an m-polar set and a nearly Borel semipolar set. Since the latter is a countable union of thin sets, it suffices to prove (6.13) assuming F thin. Let $\varphi(x) = E^x(e^{-T_F})$. Then $\varphi < 1$. Using (6.6) choose a Borel set $H \subset F$ and a Borel function $h \geqq \varphi$ with $F - H$ and $\{\varphi < h\}$ m-polar. Then

$$H_n \equiv H \cap \{h \leqq 1 - 1/n\} \subset F \cap \{\varphi \leqq 1 - 1/n\} \equiv F_n.$$

Clearly H_n is Borel, $F = \cup F_n$, and $F - \cup H_n \subset \cup (F_n - H_n)$ is m-polar. Hence it suffices to show that each H_n is totally thin. But $T_{H_n} \geqq T_F$ and so if $x \in H_n$

$$E^x(e^{-T_{H_n}}) \leqq E^x(e^{-T_F}) = \varphi(x) \leqq h(x) \leqq 1 - 1/n,$$

completing the proof.

In dealing with X under P^m, one sometimes has to extend the definition of excessiveness.

(6.14) *Definition.* A function $f \in \mathscr{E}_+^m$ is m-q-excessive ($q \geqq 0$) in case both of the following conditions hold:

(6.15) $$e^{-qt} P_t f \leqq f \ m \quad \text{a.e. for each } t \geqq 0;$$

(6.16) $$(g, e^{-qt} P_t f) \to (g, f) \quad \text{as } t \downarrow 0 \text{ for every } g \in \mathscr{E}_+^m.$$

If $f \in \mathscr{E}_+^m$ satisfies (6.15), then since m is excessive, $e^{-qt} P_t f \leqq e^{-qs} P_s f$ a.e. m when $0 \leqq s < t$. Consequently the limit in (6.16) is an increasing limit as $t \downarrow 0$. Moreover if (t_n) is a sequence decreasing to zero $e^{-qt_n} P_{t_n} f$ increases m a.e. to some h and from (6.16), $(g, f) = (g, h)$ for all $g \in \mathscr{E}_+^m$. Hence $f = h\,[m]$ provided m satisfies the following condition:

(6.17) *If $f_1, f_2 \in \mathscr{E}_+^m$ and $\int f_1 g\, dm = \int f_2 g\, dm$ for all $g \in \mathscr{E}_+^m$, then $f_1 = f_2\,[m]$.*

Clearly any σ-finite measure m satisfies (6.17), but there are also interesting non-σ-finite measures for which (6.17) is valid. The following proposition is now immediate.

(6.18) **Proposition.** *Suppose* (6.17) *holds for m. If $f \in \mathscr{E}_+^m$ satisfies* (6.15), *then f is m-q-excessive if and only if for every sequence (t_n) decreasing to zero $e^{-qt_n} P_{t_n} f \to f\, m$ a.e.*

If f is m-q-excessive, then because of (6.1), $(g, rU^{q+r} f)$ increases to (g, f) as $r \to \infty$ for each $g \in \mathscr{E}_+^m$. Thus if m satisfies (6.17), $rU^{q+r} f \leqq sU^{q+s} f \leqq f$, m a.e. for $0 < r < s$, and $rU^{q+r} f$ increases to f m a.e. as $r \to \infty$ through any sequence.

(6.19) **Proposition.** *Suppose m satisfies* (6.17). *Then $f \in \mathscr{E}_+^m$ is m-q-excessive if and only if there exists a Borel q-excessive function $\bar{f}$ with $f = \bar{f}\,[m]$.*

Proof. If there exists such an $\bar{f}$ then f is plainly m-q-excessive. Going the other direction, it is sufficient to suppose f bounded, replacing f by $f \wedge k$ in the general case. Note that it is enough to prove (6.19) for $q > 0$, and we shall so assume from now on. Let $h_n \geqq 0$ be a Borel function with $h_n = n(f - nU^{n+q}f)\,[m]$. It follows from the discussion above (6.1) that

$$U^q h_n = U^q[n(f - nU^{n+q}f)] = nU^{n+q}f\,[m].$$

Let $h = \liminf U^q h_n$. Then h is a Borel q-supermedian function and from remarks above (6.19), $h = f\,[m]$. Let $\bar{f}$ be the q-excessive regularization of h. Clearly $\bar{f}$ is Borel and since $\bar{f} = h$ except on a set of potential zero, $h = \bar{f}[m]$. Hence $f = \bar{f}[m]$, completing the proof.

The following properties of excessive measures will be required in later sections. We do not assume (6.17) holds for m.

(6.20) **Proposition.** (i) *Let T be a terminal time. Then $P^m[T = t] = 0$ for every $t > 0$.*

(ii) *Let $\Gamma \subset \mathbb{R}^{++} \times \Omega$ have an expression as $\bigcup_n [\![T^n]\!]$ where the T^n, $n \geqq 1$ are the iterates of a terminal time T. Then $P^m\{t \in \Gamma\} = 0$ for every $t > 0$.*

(iii) *Given $t > 0$, $P^m\{X_{t-}$ doesn't exist in E or X_{t-} exists in E but $X_{t-} \neq X_t\} = 0$.*

Proof. Let $m = \sum m_n$ with $m_n(E) < \infty$. Then $P^{m_n}[T \leqq t] < \infty$ for all $t > 0$, so $P^{m_n}[T = t] > 0$ can occur for only countably many t. It follows that $P^m[T = t] > 0$ for only countably many t. However, $P^m[T = t + s] = P^m[T > t, T \circ \theta_s = t] = P^m[P^{X_s}(T = t); T > t] \leqq P^{mP_s}[T = t] \leqq P^m[T = t]$. From this inequality, (i) follows at once. Part (ii) is a simple consequence of (i) because $\bigcup_n [\![T^n]\!] \subset \bigcup_r [\![r + T \circ \theta_r]\!]$ where the second union is over all rationals $r \geqq 0$. Standard arguments (see, for example, the proof (13.4) in [16]) show that the set of (s, ω) for which either $X_{s-}(\omega)$ does not exist in E or $X_{s-}(\omega)$ exists in E but $X_{s-}(\omega) \neq X_s(\omega)$ is a countable union of sets of the type in (ii). Therefore (iii) follows from (ii).

7. Homogeneous Random Measures

When working with Markov processes with a distinguished excessive initial measure m one is led to construct functionals whose shift properties hold only up to P^m-null sets. For this reason it is interesting to consider the following extension of the notion of a HRM.

(7.1) *Definition.* A random measure κ carried by $]\!]0, \zeta[\![$ is a weak HRM of X (relative to m) if for every $t \geqq 0$,

$$\kappa(\theta_t\omega, B) = \kappa(\omega, B + t) \quad \text{for all } B \in \mathscr{R}^+ \tag{7.2}$$

holds except on a P^m null set, possibly depending on t.

We shall prove later (7.4) that in many cases, a weak HRM may be perfected to give a genuine HRM as defined in (4.5).

Suppose now that κ is a weak HRM of X (relative to m). The q-potential of κ is defined by $u_\kappa^q(x) \equiv E^x \int_0^\infty e^{-qt}\,\kappa(dt)$.

(7.3) **Proposition.** *The q-potential of a weak HRM is m-q-excessive (as defined in (6.14)).*

Proof. Given $t \geqq 0$ the weak additivity (7.2) gives

$$E^x \int_{]t,\infty[} e^{-qs}\,\kappa(ds) = E^x\left[e^{-qt}\left(\int_0^\infty e^{-qs}\,\kappa(ds)\right)\circ\theta_t\right]$$

for m a.a. x. The right side is equal to $e^{-qt}P_t u_\kappa^q(x)$. Therefore, if $g \in \mathscr{E}_+^*$, $(g, e^{-qt}P_t u_\kappa^q) = \left(g, E^\bullet \int_{]t,\infty[} e^{-qs}\,\kappa(ds)\right)$. From this, (6.15) and (6.16) are evident.

(7.4) **Theorem.** *Let κ be a natural weak HRM whose q-potential u_κ^q is finite m a.e. Then there exists a natural perfect HRM γ which is P^m indistinguishable from κ. That is, κ may be perfected relative to P^m.*

Proof. The argument that follows is in the same spirit as those in (3.3) and (4.11). Let u denote the m-q-excessive function u_κ^q, and recalling (6.19), let f denote a Borel q-excessive function with $m\{u \neq f\} = 0$. The set $F \equiv \{f < \infty\}$ is absorbing and $m(F^c) = 0$ so F^c is m-inessential. Denote by X' the process X restricted to F, defined on $\Omega' \equiv \{\omega \in \Omega\colon X_t(\omega) \in F \cup \{\Delta\}$ for all $t \geqq 0\}$. The restriction κ' of κ to Ω' is P^m a.s. finite valued on compact subsets of $[0, \zeta[$. One checks that κ' is natural over X'.

In what follows we assume $m(E) = 1$, replacing m by an equivalent probability law if necessary. Denote by Z' the (P^m, Ω')-potential of $\int_{]0,t]} e^{-qs}\,\kappa'(ds)$, so that for every $t \geqq 0$

$$Z_t' = E^m\left\{\int_t^\infty e^{-qs}\,\kappa'(ds) \mid \mathscr{F}_t'\right\}.$$

Then by (7.2), $Z_t' = e^{-qt}u(X_t')$ P^m a.s. for every $t \geqq 0$, so $Z_t' = e^{-qt}f(X_t')$ P^m a.s. By right continuity of both sides in t, the last equality holds up to P^m evanescence. As f is finite on F it follows as in the proof of (3.3) that there exists a unique predictable AF B' of X', possibly charging ζ', such that $e^{-qt}f(X_t') + \int_0^t e^{-qs}\,dB_s'$ is a P^μ local martingale for every probability law μ carried by F. As κ' is natural, then as pointed out in §4, $(\kappa')^p$ exists. Since $\int_0^t e^{-qs}(\kappa')^p(ds)$ also has Z' as its P^m potential, B' and $(\kappa')^p$ are P^m-indistinguishable on Ω' in view of the uniqueness of the Doob-Meyer decomposition. Observe that as κ' is carried by $]\!]0,\zeta'[\![$, $(\kappa')^p$ is carried by $\{l' > 0\} \equiv \Lambda'$ (in the notation of §2) so that $(\kappa')^p$ and $1_{\Lambda'} * B'$ are P^m-indistinguishable on Ω'. According now to (2.15-ii) and (2.16), $\kappa' = ((\kappa')^p)^n$ is P^m-indistinguishable (on Ω') from $(1_{\Lambda'} * B')^n$. However, in view of (2.30), $(1_{\Lambda'} * B')^n$ may be taken to be a (perfect) NAF of X'. We now invoke (4.8) to get a natural perfect HRM γ of X extending $(1_{\Lambda'} * B')^n$. Clearly γ is P^m-indistinguishable from κ, so the proof is complete.

We turn now to characterizations of HRM's by their potentials.

(7.5) **Lemma.** *Let κ_1, κ_2 be HRM's of X. Suppose κ_1 and κ_2 are P^m indistinguishable and suppose that $\Omega_0 \equiv \{\omega: \kappa_1(\omega, B) \neq \kappa_2(\omega, B)$ for some $B \in \mathscr{R}^{++}\}$ belongs to $\mathscr{F}$. Then $\{x: P^x(\Omega_0) > 0\}$ is m-inessential.*

Proof. Additivity of κ_1 and κ_2 gives $\theta_t^{-1}(\Omega_0) = \{\omega: \kappa_1(\omega, B) \neq \kappa_2(\omega, B)$ for some $B \in \mathscr{R}^{++}$ with $B \subset]t, \infty[\}$, so $\theta_t^{-1}(\Omega_0)$ increases to Ω_0 as $t \downarrow 0$. If we set $\varphi(x) = P^x(\Omega_0)$, this leads to $P_t\varphi(x) \uparrow \varphi(x)$ as $t \downarrow 0$, so φ is excessive. But $m(\varphi) = 0$ since $P^m(\Omega_0) = 0$ by hypothesis. Therefore $\{\varphi > 0\}$ is m-inessential, by (6.10).

(7.6) *Remark.* The condition $\Omega_0 \in \mathscr{F}$ in (7.5) is satisfied if, for example, there exist a strictly positive measurable process M such that $\int_r^t M_s \kappa_j(ds) < \infty$ a.s. for $j = 1,2$ for each $0 < r < t < \infty$, for then κ_1 and κ_2 are countable sums of finite kernels.

(7.7) **Theorem.** *Let κ_1, κ_2 be natural HRM's whose respective q-potentials u_1^q, u_2^q are both finite m a.e. Then κ_1 and κ_2 are P^m indistinguishable if and only if $u_1^q = u_2^q$ a.e. $[m]$.*

Proof. Assume $u_1^q = u_2^q$ a.e. Both $\{u_1^q = \infty\}$ and $\{u_2^q = \infty\}$ are obviously m-inessential and (6.10) shows that $\{u_1^q \neq u_2^q\}$ is m-polar. We may, by (6.12), choose a Borel m-inessential set $F \supset \{u_1^q = \infty\} \cup \{u_2^q = \infty\} \cup \{u_1^q \neq u_2^q\}$. Let X' denote X restricted to F^c, $\Omega' = \{\omega \in \Omega: X_t(\omega) \in F^c \cup \{\Delta\}$ for all $t \geqq 0\}$, and let κ_1', κ_2' be the restrictions of κ_1, κ_2 to Ω'. Then κ_1', κ_2' have finite q-potentials $u_1^q|_{F^c}$, $u_2^q|_{F^c}$. Because NAF'S with finite q-potentials are uniquely determined by their q-potentials, it follows that κ_1' and κ_2' are indistinguishable on Ω'. Because $P^m(\Omega - \Omega') = 0$, κ_1 and κ_2 are therefore P^m indistinguishable. The reverse implication is obvious.

8. The Revuz Measure of a HRM

For this section we require the additional hypothesis that X_{t-} exists in E for all $t \in]0, \zeta[$. Again, m is a fixed excessive measure.

Given a HRM κ, as defined in (4.5), and given $f \in \mathscr{E}_+^*$, let $f * \kappa$ (resp., $f_- * \kappa$) denote the HRM $(f * \kappa)(\omega, dt) = f(X_t(\omega))\, \kappa(\omega, dt)$ (resp., $(f_- * \kappa)(\omega, dt) = f(X_{t-}(\omega))\, \kappa(\omega, dt)$.)

Excessiveness of m gives one

$$\begin{aligned} E^m(\kappa\,]s, s+t]) &= E^m(\kappa(]0, t]) \circ \theta_s) = E^{mP_s}(\kappa(]0, t]) \\ &\leqq E^m \kappa(]0, t]). \end{aligned}$$

Therefore, if one defines $\varphi(t) \equiv E^m \kappa(]0, t])$, it follows that φ is subadditive: $\varphi(t+s) \leqq \varphi(t) + \varphi(s)$. Clearly φ is increasing with values in $[0, \infty]$. If $\varphi(t_0) < \infty$ for some $t_0 > 0$ then $\varphi(t) < \infty$ for all $t > 0$, φ is right continuous on $\mathbb{R}^+$ and $\varphi(0) = 0$. It is an elementary and standard fact that if $S = \sup_{t>0} \varphi(t)/t$, then $\lim_{t \downarrow 0} \varphi(t)/t = S$ and that $2^n \varphi(2^{-n})$ increases to S as $n \to \infty$. See, for example, the proof of II.1 in [35]. The *Revuz measure* ν_κ^m of κ (relative to m) is defined by

(8.1)
$$\nu_\kappa^m(f) = \sup_{t>0} t^{-1} E^m[(f_- * \kappa)(]0,t])] = \lim_{t\to 0} t^{-1} E^m \int_{]0,t]} f(X_{s-})\,\kappa(ds),$$

for $f \in \mathscr{E}_+^*$. It is immediate from the facts listed above that ν_κ^m is a measure on $(E, \mathscr{E})$. For a fixed m we often write ν_κ in place of ν_κ^m when there is no danger of misinterpretation.

A HRM κ is called *integrable* if $\nu_\kappa(E) < \infty$, *σ-integrable* if ν_κ is σ-finite. (This is quite different from being σ-integrable over $\mathscr{O}$, $\mathscr{P}$ or $\mathscr{N}$, as defined in §4.) In case $\kappa(dt) = dA_t$ with A a NAF of X, Revuz [36] showed that if A has uniformly bounded jumps then κ is σ-integrable.

(8.2) **Lemma.** *If κ is integrable and $q > 0$, then $\int m(dx)\, u_\kappa^q(x) \leqq \nu_\kappa(E)/q$. In particular, $u_\kappa^q < \infty$ a.e. (m).*

Proof. Integrability of κ gives $E^m \kappa\,]0,t] \leqq t\,\nu_\kappa(E)$ for all $t > 0$, and together with excessiveness of m this yields $E^m \kappa(]s, s+t]) \leqq t\,\nu_\kappa(E)$. Consequently,

$$\int m(dx)\, u_\kappa^q(x) = E^m \int_0^\infty e^{-qt}\,\kappa(dt) \leqq \nu_\kappa(E) \int_0^\infty e^{-qt}\,dt = q^{-1}\,\nu_\kappa(E).$$

The following lemma contains a preliminary calculation which will be used several times in the sequel.

(8.3) **Lemma.** *Let $\varphi \in \mathscr{R}^+$ and $f \in \mathscr{E}$ be positive, and let η be a countably finite measure on E. If κ is a HRM, then for each $t > 0$*

(8.4)
$$\int_0^\infty \varphi(s)\,ds\, E^{\eta P_s}(f(X_0)\,\kappa(]0,t])) = E^\eta \int_0^\infty \kappa(dr) \int_0^\infty ds\,\varphi(s)\, f(X_s)\, 1_{[r-t,\,r[}(s).$$

Proof. We may assume that $\eta(E) < \infty$. The left side of (8.4) is equal to

$$\begin{aligned}
&\int_0^\infty \varphi(s)\,ds \int \eta(dx) \int P_s(x,dy)\, E^y[f(X_0)\,\kappa(]0,t])] \\
&= \int \eta(dx) \int_0^\infty \varphi(s)\,ds\, E^x[f(X_s)\,\kappa(]0,t])\circ\theta_s] \\
&= E^\eta \int_0^\infty ds\,\varphi(s)\, f(X_s)\,\kappa(]s, t+s]) \\
&= E^\eta \int_0^\infty \kappa(dr) \int_0^\infty ds\,\varphi(s)\, f(X_s)\, 1_{[r-t,\,r[}(s),
\end{aligned}$$

the countable finiteness condition on κ permitting Fubini's theorem to be applied in the last step.

(8.5) **Corollary.** *With η and κ as in* (8.3) *and $q \geqq 0$, one has*

(8.6) $$\lim_{t\to 0} t^{-1} E^{\eta U^q}\kappa(]0,t]) = E^\eta \int_0^\infty e^{-qs}\kappa(ds).$$

Proof. Applying (8.3)

$$E^{\eta U^q}\kappa(]0,t]) = \int_0^\infty e^{-qs}\,ds\,E^{\eta P_s}\kappa(]0,t]) = E^\eta \int_0^\infty \kappa(dr) \int_{(r-t)^+}^r e^{-qs}\,ds.$$

But $t^{-1}\int_{(r-t)^+}^r e^{-qs}\,ds \to e^{-qr}$ as $t \to 0$ when $r > 0$, and it is dominated by

$$t^{-1} e^{-q(r-t)^+}[r-(r-t)^+] \leqq e^{-q(r-t)^+}.$$

If $t \leqq 1$, this is in turn dominated by $e^q e^{-qr}$. Hence, by the dominated convergence theorem, (8.6) holds whenever $E^\eta \int_0^\infty e^{-qr}\kappa(dr) < \infty$. On the other hand, if this integral is infinite, observe that

$$t^{-1}\int_{(r-t)^+}^r e^{-qs}\,ds \geqq t^{-1} e^{-qr}[r-(r-t)^+].$$

But $t^{-1}[r-(r-t)^+] \uparrow 1$ as $t \downarrow 0$ for each $r > 0$, and so

$$\liminf_{t\to 0} t^{-1} E^{\eta U^q}\kappa(]0,t]) \geqq E^\eta \int_0^\infty e^{-qr}\kappa(dr) = \infty,$$

completing the proof.

(8.7) **Theorem.** *Let κ be an integrable HRM of X and let $f \in \mathscr{E}^*$ be bounded and positive. Suppose that $t \to f(X_{t-})$ is P^m a.s. left continuous on $]0,\zeta[$. Then*

(8.8) $$\lim_{t\downarrow 0} t^{-1} E^m \int_0^t f(X_{s-})\,\kappa(ds) = \lim_{t\downarrow 0} t^{-1} E^m[f(X_0)\,\kappa(]0,t])].$$

Proof. For the first part of the argument we do *not* assume that κ is integrable nor that f is bounded. (This is of importance in (8.11).) Note that the discussion at the beginning of this section shows that the limit on the left side of (8.8) exists (possibly $+\infty$) for any HRM and any positive $f \in \mathscr{E}^*$. Since m is excessive, $u^{-1}\int_0^u mP_s\,ds \leqq m$. Therefore if $u > 0$

(8.9) $$E^m[f(X_0)\,\kappa(]0,t])] \geqq u^{-1}\int_0^u ds\,E^{mP_s}[f(X_0)\,\kappa(]0,t])].$$

To evaluate the right side of (8.9) we use (8.3) with $\varphi(s) = u^{-1} 1_{]0,u]}(s)$, and obtain $t^{-1}E^m[f(X_0)\,\kappa(]0,t])] \geqq u^{-1} E^m \int \kappa(dr)\,J_t(r)$, where $J_t(r) \equiv t^{-1}\int_0^u 1_{[r-t,r[}(s)$ $f(X_s)\,ds$ vanishes if $r > u$ for $t < r-u$. If $0 < r \leqq u$ then for sufficiently small t,

$$J_t(r) = t^{-1}\int_{r-t}^r f(X_s)\,ds = t^{-1}\int_{r-t}^r f(X_{s-})\,ds.$$

The last expression converges P^m a.s. to $f(X_{r-})$ as $t \to 0$ provided $0 < r < \zeta$, because of the hypothesis on f. Hence, by Fatou's lemma,

$$\liminf_{t\to 0} t^{-1} E^m [f(X_0)\, \kappa(]0,t])] \geqq u^{-1} E^m \int_0^u f(X_{r-})\, \kappa(dr).$$

As previously remarked, the right side has a limit as $u \downarrow 0$ and so

$$\text{(8.10)} \qquad \liminf_{t\to 0} t^{-1} E^m [f(X_0)\, \kappa(]0,t])] \geqq \lim_{u\to 0} u^{-1} E^m \int_0^u f(X_{r-})\, \kappa(dr).$$

We suppose now that κ is integrable and that $0 \leqq f \leqq 1$. Applying (8.10) to $1-f$ (which also satisfies the hypothesis of (8.7)) and using $\nu_\kappa(E) < \infty$, we see that

$$\limsup_{t\to 0} t^{-1} E^m [f(X_0)\, \kappa(]0,t])] \leqq \lim_{u\to 0} u^{-1} E^m \int_0^u f(X_{r-})\, \kappa(dr).$$

From this and (8.10), (8.8) follows at once.

(8.11) **Corollary.** *Let κ be a σ-integrable HRM (relative to m) and suppose $f \in \mathscr{E}^*$ is positive and $t \to f(X_{t-})$ is P^m a.s. left continuous on $]0,\zeta[$. If, in addition, fm is an excessive measure then* (8.8) *obtains.*

Proof. Choose $E_n \uparrow E$ with $\kappa_n = 1_{E_n-} * \kappa$ integrable for each n. Clearly $\kappa_n \uparrow \kappa$. Since fm is excessive

$$L \equiv \nu_\kappa^{fm}(1) = \lim_{t\to 0} t^{-1} E^m [f(X_0)\, \kappa(]0,t])]$$

exists and, in fact, $2^k \mathrm{E}^{fm}\kappa(]0,2^{-k}]) \uparrow L$ as $k \to \infty$. Let $f_p \equiv f \wedge p$. Then f_p satisfies the hypotheses of (8.7) and

$$L = \lim_k \lim_n \lim_p 2^k E^{f_p m} \kappa_n(]0,2^{-k}]).$$

The latter limit is actually increasing in k, n and p so we may interchange the order of limit operation at will. Using (8.7) on f_p and κ_n gives

$$L = \lim_n \lim_p \lim_{t\to 0} t^{-1} E^m \int_0^t f_p(X_{s-})\, 1_{E_n}(X_{s-})\, \kappa(ds)$$
$$L \leqq \lim_{t\to 0} t^{-1} E^m \int_0^t f(X_{s-})\, \kappa(ds).$$

But the opposite inequality also obtains because of (8.10) which, it should be recalled, was proved without assuming f bounded or κ integrable. These two inequalities establish (8.11).

(8.12) *Remark.* Corollary (8.11) is especially useful under the weak duality hypotheses of Part III, since it will turn out that any co-excessive function f such that fm is a countably finite measure satisfies the hypotheses of (8.11) (see §9).

(8.13) *Remark.* If we drop the hypothesis that X_{t-} exists in E for $0 < t < \zeta$, the proof of (8.7) repeated almost word for word gives the following: if κ is integrable and $f \in b\mathscr{E}^*$ is positive and $t \to f(X_t)$ has left limits on $]0, \zeta[$ a.s. (P^m) then

$$\lim_{t \downarrow 0} t^{-1} E^m [f(X_0)\, \kappa(]0, t])] = \lim_{t \downarrow 0} t^{-1} E^m \int_0^t f(X_s)_- \, \kappa(ds). \tag{8.14}$$

The corresponding statement in (8.11) is likewise valid.

(8.15) *Remark.* There is a version of this result which includes both (8.7) and (8.13). Left limits are not assumed to exist in E. Suppose $Y = (Y_t)_{t>0}$ is a bounded positive measurable process which is homogeneous on $]0, \infty[$: $(\Theta_t Y)_s = Y_s$ on $]t, \infty[$, or equivalently, $Y_s \circ \theta_t = Y_{s+t}$ for all $s > 0$, $t \geqq 0$. Suppose further that $t \to Y_t$ is left continuous on $]0, \zeta[$ a.s. P^m, and that there exists $Y_0 \in b\mathscr{F}_+$ such that $E^m \int |Y_0 \circ \theta_t - Y_t|\, dt = 0$. Then if κ is an integrable HRM

$$\lim_{t \downarrow 0} t^{-1} E^m [Y_0\, \kappa(]0, t])] = \lim_{t \downarrow 0} t^{-1} E^m \int_0^t Y_s\, \kappa(ds). \tag{8.16}$$

The proof of (8.16) is almost identical to that of (8.7) if one uses the obvious analogue of (8.3).

Part III: Weak Duality

For the remainder of this paper we shall work under the weak duality hypotheses (9.1), (9.2) below, unless explicitly stated otherwise.

9. The Revuz Formula

From now on we assume given two Borel right processes X, $\hat{X}$ on a common state space E with transition semigroups (P_t), $(\hat{P}_t)$, together with a *σ-finite measure* m on $(E, \mathscr{E})$. Writing $(f, g) \equiv \int fg\, dm$ for f, g positive functions in $\mathscr{E}^*$, weak duality of (P_t), $(\hat{P}_t)$ relative to m means

$$(P_t f, g) = (f, \hat{P}_t g) \tag{9.1}$$

for all $t \geqq 0, f, g \geqq 0$ in $\mathscr{E}^*$. Weak duality of X, $\hat{X}$ will mean (9.1) together with

$$X_{t-} \text{ (resp. } \hat{X}_{t-}) \textit{ exists in } E \textit{ for all } t \in]0, \zeta[\text{ (resp., }]0, \hat{\zeta}[). \tag{9.2}$$

Actually, (9.1) implies that (9.2) holds except on sets in Ω, $\hat{\Omega}$ which are respectively P^m, $\hat{P}^m$ null. This was shown by Walsh [47]. Assuming (9.2) identically on Ω (resp., $\hat{\Omega}$) relieves us from carrying one more exceptional set everywhere.

The condition (9.1) implies that m is excessive, for if $f \in \mathscr{E}^*_+$

$$mP_t(f) = (1, P_t f) = (\hat{P}_t 1, f) \leqq (1, f) = m(f).$$

Similarly, m is co-excessive. As is customary, we use the prefix "co" to describe quantities associated with $\hat{X}$. Thus U^q is the resolvent (for X) and $\hat{U}^q$ denotes the co-resolvent (the resolvent for $\hat{X}$). Note also that for each $t > 0$, $P^m(X_t \neq X_{t-}; t < \zeta) = \hat{P}^m(\hat{X}_t \neq \hat{X}_{t-}; t < \hat{\zeta}) = 0$. This follows from (6.20), but the proof is simpler under the present hypotheses.

We aim to get the version appropriate to weak duality hypotheses of a formula obtained by Revuz [35] under strong duality hypotheses. His result stated that if A is a σ-integrable NAF with Revuz measure ν_A then its q-potential is given by $u_A^q(x) = \int u^q(x,y)\,\nu_A(dy)$, u^q being the potential kernel density. As processes in weak duality need not have such densities, we have to settle for a weaker result. The above formula shows that in the case of strong duality at least,

$$\begin{aligned} u_A^q(x)\, m(dx) &= \int \nu_A(dy)\,(m(dx)\, u^q(x,y)) \\ &= \nu_A \hat{U}^q(dx). \end{aligned}$$

That is, $\nu_A \hat{U}^q \ll m$ and u_A^q is a version of $d(\nu_A \hat{U}^q)/dm$. We shall prove that this same result obtains just under weak duality, $\nu_A \hat{U}^q$ being defined by $\nu_A \hat{U}^q(\cdot) = \int \nu_A(dx)\, \hat{U}^q(x, \cdot)$. To be more precise, we prove

(9.3) **Theorem.** *Let κ be a σ-integrable HRM with Revuz measure ν_κ. If $F(t,x)$ is a positive Borel function on $\mathbb{R}^+ \times E$, then*

$$\int m(dx)\, E^x \int_0^\infty F(t,x)\,\kappa(dt) = \int_0^\infty dt \int \nu_\kappa \hat{P}_t(dx)\, F(t,x). \tag{9.4}$$

In particular for each $q \geqq 0$

$$\nu_\kappa \hat{U}^q(dx) = u_\kappa^q(x)\, m(dx). \tag{9.5}$$

Proof. First note that applying (9.4) with $F(t,x) = e^{-qt} f(x)$ gives $\int f(x)\, u_\kappa^q(x)\, m(dx) = \int \nu_\kappa \hat{U}^q(dx) f(x)$ which is (9.5). Thus it suffices to prove (9.4). In the course of the argument for (9.4) we shall require the following fact:

(9.6) If h is q-co-excessive then $t \to h(X_{t-})$ is P^m a.s. left continuous on $]0, \zeta[$.

This is a version of a theorem of Weil [48], proved originally under strong duality hypotheses, which is valid in weak duality. The first part of Mitro's proof [29, §6] of Weil's theorem applies directly to give (9.6). (Once we discuss the associated stationary process in §10, (9.6) will be practically obvious.)

For notational convenience let $\nu \equiv \nu_\kappa$. It suffices to establish (9.4) when $\nu(E) < \infty$. Let $c_t(x) = E^x(\kappa\,]0,t])$. Then $\int c_t\, dm \leqq t\,\nu(E) < \infty$ and $c_{t+s} = c_t + P_t c_s$. Let f be a bounded q-coexcessive function. Then $e^{-qt}\hat{P}_t f$ is again such a function and so by (9.6) and (8.7) we see that

$$\nu(\hat{P}_t f) = \lim_{s \downarrow 0} \frac{1}{s} (\hat{P}_t f, c_s).$$

Consequently

$$\begin{aligned} \frac{1}{s}(f, c_{t+s} - c_t) &= \frac{1}{s}(f, P_t c_s) \\ &= \frac{1}{s}(\hat{P}_t f, c_s) \to \nu(\hat{P}_t f) \end{aligned}$$

as $s \downarrow 0$. Therefore $v\hat{P}_t(f)$ is the right derivative of (f, c_t). If M is a bound for f,

$$(f, c_{t+s} - c_t) = (\hat{P}_t f, c_s) \leqq M(1, c_s) \leqq M s v(E)$$

shows that $t \to (f, c_t)$ is absolutely continuous. Combining these observations with $(f, c_0) = 0$ gives

$$(9.7) \qquad (f, c_t) = \int_0^t v\hat{P}_s(f)\, ds.$$

Given a positive bounded continuous h, $f = q\hat{U}^q h$ with $q > 0$ is a bounded q-coexcessive function and so (9.7) holds for such f. But $q\hat{U}^q h \to h$ boundedly as $q \to \infty$ and consequently (9.7) holds for all bounded, positive, continuous f, and hence for $f \in b\mathscr{E}$ since both sides are finite measures in f. Now (9.7) is just (9.4) when $F(s, x) = 1_{]0,t]}(s) f(x)$, and since both sides of (9.7) and hence (9.4) are finite for such F with $t < \infty$ and $f \in b\mathscr{E}$, it follows that (9.4) holds for $F \geqq 0$, $F \in \mathscr{R}^+ \times \mathscr{E}$. This establishes (9.3).

(9.8) *Remark.* Let κ be a σ-integrable HRM and $f \in \mathscr{E}^*$ bounded and positive. Then the HRM $f_- * \kappa$ has Revuz measure $f v_\kappa$. Applying (9.5) we get the formula

$$(9.9) \qquad (f v_\kappa)\hat{U}^q(dx) = \left(E^x \int_0^\infty e^{-qt} f(X_{t-})\kappa(dt)\right) m(dx).$$

In case $\kappa(dt) = dA_t$ with A a RAF of X which lives on $\{X_- = X\}$, $E^x \int_0^\infty e^{-qt} f(X_{t-})\kappa(dt) = E^x \int_0^\infty e^{-qt} f(X_t)\, dA_t$ is what is usually denoted $U_A^q f(x)$, the q-potential operator for A, so (9.9) determines this operator up to m-null sets.

(9.10) **Theorem.** *Let κ, γ be σ-integrable HRM's. Then $v_\kappa = v_\gamma$ if and only if $\kappa^n = \gamma^n$ up to P^m evanescence.*

Proof. It is clear using (2.14) extended to random measures that κ and κ^n have the same Revuz measure, so we may as well assume κ and γ natural, and that $v_\kappa = v_\gamma$. Writing $E = \cup E_n$ (disjoint) with $v_\kappa(E_n) < \infty$ for each n, it is enough to prove that for each n, $1_{E_n-} * \kappa = 1_{E_n-} * \gamma$ up to P^m evanescence. In other words, we may as well take $E_n = E$ and assume $v_\kappa(E) < \infty$. Then (9.5) gives us $u_\kappa^1 = u_\gamma^1$ a.e. (m), both functions being finite a.e. (m) by (8.2). The theorem is then an immediate consequence of (7.7).

10. The Stationary Process

Part of the theory of dual processes is best understood by means of a certain auxiliary stationary process, which we now discuss. In what follows, it is to use different burial points Δ, $\hat{\Delta}$ for X, $\hat{X}$ respectively, and we shall do this systematically in the remaining sections. In addition we shall suppose that X is realized as the coordinate process on the space Ω of maps ω: $\mathbb{R}^+ \to E \cup \{\Delta\}$ admitting Δ as a trap, and with ω being right continuous on $[0, \zeta[$ and having left limits in E during $]0, \zeta[$. Similarly, $\hat{X}$ is the coordinate process on the corresponding path space $\hat{\Omega}$.

The *stationary process* constructed from the data $(P_t, \hat{P}_t, m)$ may be described as follows. Adjoin Δ, $\hat{\Delta}$ to E as isolated points, and let W denote the space of all maps $w\colon \mathbb{R} \to E \cup \{\Delta, \hat{\Delta}\}$ such that (setting $\sup \phi = -\infty$ and $\inf \phi = \infty$

(10.1) (i) $w(t) = \hat{\Delta}$ *implies* $w(s) = \hat{\Delta}$ *for all* $s < t$;
(ii) $w(t) = \Delta$ *implies* $w(s) = \Delta$ *for all* $s > t$;
(iii) *if* $\alpha(w) \equiv \sup\{t\colon w(t) = \hat{\Delta}\}$ *and* $\beta(w) \equiv \inf\{t\colon w(t) = \Delta\}$ *then the restriction of* w *to* $]\alpha(w), \beta(w)[$ *is rcll in* E, *and* $w(\alpha(w)) = \hat{\Delta}$ *provided* $\alpha(w) \in \mathbb{R}$, $w(\beta(w)) = \Delta$ *provided* $\beta(w) \in \mathbb{R}$.

Note that $\alpha(w) \leqq \beta(w)$, and $\alpha(w) < \beta(w)$ if either is finite. Let $Z_t(w) = w(t)$ denote the coordinate maps on W. Then α, β are thought of as the birth and death times respectively of the process Z. Let $\theta_t\colon W \to W$ denote the shift operators, so that $Z_s \circ \theta_t = Z_{s+t}$.

By $Z_{t-}(w)$ we mean the left limit of $u \to Z_u(w)$ at t if $\alpha(w) < t < \beta(w)$, with $Z_{t-}(w)$ defined to be Δ if $t \geqq \beta(w)$, $\hat{\Delta}$ if $t \leqq \alpha(w)$. Thus $Z_{t-}(w) \in E$ if and only if $\alpha(w) < t < \beta(w)$. Next define $\hat{Z}_t\colon \mathbb{R} \to E \cup \{\Delta, \hat{\Delta}\}$ by

(10.2) $$\hat{Z}_t(w) = Z_{(-t)-}(w), \; t \in \mathbb{R}.$$

The trajectories $t \to \hat{Z}_t(w)$ are precisely the reverses of $t \to Z_t(w)$ made *rcll* on the lifetime interval $]\hat{\alpha}(w), \hat{\beta}(w)[\equiv]-\beta(w), -\alpha(w)[$. Of course, $\hat{Z}_t$ has birth point Δ and death point $\hat{\Delta}$.

Let $\mathscr{A}^0$ denote the σ-algebra on W generated by the maps Z_t $(-\infty < t < \infty)$ or alternatively by the $\hat{Z}_t$. One may then, following Mitro [29] or Kuznetzov [23] construct a σ-finite measure P on $(W, \mathscr{A}^0)$ which is invariant under θ_u and, in a manner to be made precise in (10.5) below, makes $Z_t (0 \leqq t < \beta)$ on $\{Z_0 \in E\}$ a copy of $X_t (0 \leqq t < \zeta)$ under P^m, and $\hat{Z}_t (0 \leqq t < \hat{\beta})$ on $\{\hat{Z}_0 \in E\}$ a copy of $\hat{X}_t (0 \leqq t < \hat{\zeta})$ under $\hat{P}^m$, the forward and backward processes being conditionally independent on $\{Z_0 \in E\} = \{\hat{Z}_0 \in E\} = \{\alpha < 0 < \beta\}$.

In order to set this description in precise form which is useful for computations we consider the maps $\Pi_t\colon W \to \Omega$ and $\hat{\Pi}_t\colon W \to \hat{\Omega}$ defined for $t \in \mathbb{R}$ by

(10.3) $$\begin{aligned} \Pi_t w(s) &= w(t+s), \; s \geqq 0 \text{ if } \alpha(w) < t < \beta(w) \\ &= [\Delta] \; (\textit{the path constantly} = \Delta) \; \textit{otherwise}. \end{aligned}$$

(10.4) $$\begin{aligned} \hat{\Pi}_t w(s) &= w[(t-s)-], \; s \geqq 0 \text{ if } \alpha(w) < t < \beta(w); \\ &= [\hat{\Delta}] \; \textit{otherwise}. \end{aligned}$$

That is, provided w is alive at time t, $\Pi_t w$ and $\hat{\Pi}_t w$ represent its forward and backward halves as viewed from t. If $\Pi \equiv \Pi_0$, $\hat{\Pi} \equiv \hat{\Pi}_0$, then it is evident that $\Pi_t = \Pi \circ \theta_t$, $\hat{\Pi}_t = \hat{\Pi} \circ \theta_t$ for $t \in \mathbb{R}$. Recalling that $\mathscr{F}^0$, $\hat{\mathscr{F}}^0$ are the uncompleted σ-algebras on Ω, $\hat{\Omega}$ generated by X, $\hat{X}$ respectively, P may be characterized by the formula

(10.5) $$P[F \circ \Pi_u \hat{F} \circ \hat{\Pi}_u; \alpha < u < \beta] = \int m(dx) P^x(F) \hat{P}^x(\hat{F})$$

valid for all $u \in \mathbb{R}$, $F \in \mathscr{F}^0$, $\hat{F} \in \hat{\mathscr{F}}^0$ both positive. It is clear from (10.5) that P is stationary:

(10.6) $$\theta_u P = P \textit{ for every } u \in \mathbb{R}.$$

There is another useful formulation of (10.5) in terms of the joining operation $(\hat{\omega}, u, \omega) \to \hat{\omega}: u: \omega$ defined by

$$(\hat{\omega}: u: \omega)(t) = \hat{\omega}((u-t)-) \text{ if } t < u, = \omega(t-u) \text{ if } t \geqq u.$$

In other words, if $\omega \neq [\Delta]$ and $\hat{\omega} \neq [\hat{\Delta}]$,

(10.7) $$\Pi_u(\hat{\omega}: u: \omega) = \omega, \; \hat{\Pi}_u(\hat{\omega}: u: \omega) = \hat{\omega}.$$

It is clear that $(\hat{\omega}, u, \omega) \to \hat{\omega}: u: \omega$ is a measurable isomorphism of $(\hat{\Omega} - [\hat{\Delta}]) \times \mathbb{R} \times (\Omega - [\Delta])$ onto $\{\alpha < u < \beta\} \subset W$. Note that $w = \hat{\Pi}_t w: t: \Pi_t w$ if $\alpha(w) < t < \beta(w)$. By a monotone class argument (10.5) then leads to

(10.8) $$P[H(\hat{\omega}: u: \omega); \alpha < u < \beta] = \int m(dx) \int P^x(d\omega) \int \hat{P}^x(d\hat{\omega}) \, H(\hat{\omega}: u: \omega)$$

for every positive $H \in \mathscr{A}^0$.

Let $\mathscr{A}$ denote the P-completion of $\mathscr{A}^0$ and let $\mathscr{J}$ denote the P-null sets in $\mathscr{A}$. For arbitrary $t \in \mathbb{R}$ let $\mathscr{A}_t^0 \equiv \sigma\{Z_s: s \leqq t\}$ and $\mathscr{A}_t \equiv \mathscr{A}_t^0 \vee \mathscr{J}$. By a *stopping time for* Z is meant a random time $T: W \to \mathbb{R} \cup \{-\infty, \infty\}$ with $\{T \leqq t\} \in \mathscr{A}_t$ for all $t \in \mathbb{R}$. Approximating such a T from above by countable valued stopping times, it is easy to see that

(10.9) **Lemma.** *If T is a stopping time for Z then there exists a P-equivalent random time T^0 with $\{T^0 \leqq t\} \in \mathscr{A}_{t+}^0$ for every $t \in \mathbb{R}$.*

Dually, let $\hat{\mathscr{A}}_t^0 \equiv \sigma\{Z_s: s \geqq t\}$ and $\hat{\mathscr{A}}_t \equiv \hat{\mathscr{A}}_t^0 \vee \mathscr{J}$. By a *co-stopping time for* Z is meant a random time $R: W \to \mathbb{R} \cup \{-\infty, \infty\}$ such that $\{R \geqq t\} \in \hat{\mathscr{A}}_t$ for every $t \in \mathbb{R}$. Co-stopping times may be approximated from below by countable valued co-stopping times.

In later sections we shall need to use the following strong Markov properties of Z (see Mitro [29, 31]). Of course, $(\Pi_T)(w) \equiv \Pi_{T(w)}(w)$ if $T(w)$ is finite with $\hat{\Pi}_R$ defined similarly.

(10.10) **Proposition.** *Let T be a stopping time for Z. Then P is σ-finite on the trace of $\mathscr{A}_T$ on $\{\alpha < T < \beta\}$, and for all positive $F \in \mathscr{F}^0$, $\hat{F} \in \hat{\mathscr{F}}^0$ one has*

(10.11) $$P\{F \circ \Pi_T \hat{F} \circ \hat{\Pi}_T; \alpha < T < \beta\} = P\{P^{Z(T)}(F) \hat{F} \circ \hat{\Pi}_T; \alpha < T < \beta\}.$$

The formula (10.11) may be extended by monotone classes (σ-finiteness of P on $\mathscr{A}_{T|\{\alpha < T < \beta\}}$ being critical) to give the following formula, valid for all positive functions $H(\hat{\omega}, t, \omega)$ in $\hat{\mathscr{F}}^0 \times \mathscr{R} \times \mathscr{F}^0$,

(10.12) $$P\{H(\hat{\Pi}_T, T, \Pi_T); \alpha < T < \beta\} = \int P(dw) \int P^{Z(T(w))}(d\omega) \, H(\hat{\Pi}_T(w), T(w), \omega) \, 1_{\{\alpha < T < \beta\}}(w).$$

Dually, if R is a co-stopping time, one obtains the corresponding formula

(10.13) $$P\{F \circ \Pi_R \hat{F} \circ \hat{\Pi}_R; \alpha < R < \beta\} = P\{\hat{P}^{Z(R-)}(\hat{F}) F \circ \Pi_R; \alpha < R < \beta\},$$

and more generally, for H as described above,

(10.14) $$P\{H(\hat{\Pi}_R, R, \Pi_R); \alpha < R < \beta\} = \int P(dw) \int \hat{P}^{Z(R(w)-)}(d\hat{\omega}) \, H(\hat{\omega}, R(w), \Pi_R(w)) \, 1_{\{\alpha < R < \beta\}}(w).$$

The remaining results will require standardness of X. We show first how Z inherits quasi left continuity from X.

(10.15) **Proposition.** *Suppose X is standard, and let T_n, T be stopping times for Z with $T_n \uparrow T$. Then $Z(T_n) \to Z(T)$ a.s. on $\{\alpha < T < \beta\}$.*

Proof. According to (10.9) it may be assumed that T_n, T are all $(\mathscr{A}_{t+}^0)$ stopping times. It suffices to prove that for every fixed $u \in \mathbb{R}$. $P\{Z(T_n) \not\to Z(T);\ \alpha < u < T < \beta,\ Z_u \in B\} = 0$ for every $B \in \mathscr{E}$ with $m(B) < \infty$. Now, for any stopping time S over $(\mathscr{A}_{t+}^0)$ it is easy to see that

$$\text{(10.16)} \qquad \omega \to [S(\hat{\omega}: u: \omega) - u]^+ \text{ is an } (\mathscr{F}_{t+}^0) \text{ stopping time;}$$

$$\text{(10.17)} \qquad \hat{\omega} \to S(\hat{\omega}: u: \omega) \text{ is } \hat{\mathscr{F}}^0 \text{ measurable.}$$

In addition, $Z_S(\hat{\omega}: u: \omega) = X_{S(\hat{\omega}: u: \omega) - u}(\omega)$ if $S(\hat{\omega}: u: \omega) > u$ so we have, using (10.8)

$$\begin{aligned} &P\{Z(T_n) \not\to Z(T);\ \alpha < u < T < \beta,\ Z_u \in B\} \\ &= \int_B m(dx) \int \hat{P}^x(d\hat{\omega}) P^x\{X_{T_n(\hat{\omega}: u: \cdot) - u}(\cdot) \not\to X_{T(\hat{\omega}: u: \cdot) - u}(\cdot); \\ &\qquad 0 < T(\hat{\omega}: u: \cdot) - u < \zeta(\cdot)\} \\ &= 0 \end{aligned}$$

by quasi left continuity of X and (10.16).

(10.18) *Remark.* Dually if $\hat{X}$ is standard and (R_n) is a sequence of co-stopping times decreasing to R, then $Z(R_n-) \to Z(R-)$ a.s. on $\{\alpha < R < \beta\}$.

11. Switching Identities and a Characterization of Revuz Measures

Recall from §6 that if $B \subset E$ is nearly Borel, T_B and S_B denote respectively the hitting times of B by X, X_- respectively. We shall, as usual, denote by P_B^q the q-order hitting operator: $P_B^q f(x) = E^x\{e^{-qT_B} f(X_{T_B})\}$. Similarly, let P_{B-}^q denote the q-order *left* hitting operator: $P_{B-}^q f(x) = E^x\{e^{-qS_B} f(X_{S_B})\}$. It is clear from the correspondence between X, $\hat{X}$ and the stationary process Z that P_B^q is related to $\hat{P}_{B-}^q$ and our first aim in this section is to establish the precise connection between them. First of all, recalling from §6 the meanings of m-polar and left m-polar, we obtain the following preliminary comparison. (The restriction here to Borel subsets of E rather than nearly Borel is no real loss because of (6.12) and (6.13), in view of the fact that we shall be working with P^m, and not an arbitrary initial measure.)

(11.1) **Proposition.** *Let $B \in \mathscr{E}$. Then B is left m-polar if and only if B is m-copolar.*

Proof. Suppose B is left m-polar. The event $\{Z_{t-} \in B$ for some $t \in]\alpha, \beta[\}$ may be covered by a countable union of events of the form $\{r \in]\alpha, \beta[,\ Z_{t-} \in B$ for some $t > r\}$. However, the latter event is P-null, by (10.5). Reversing t, this proves that $P\{\hat{Z}_t \in B$ for some $t \in]\hat{\alpha}, \hat{\beta}[\} = 0$, which implies that B is m-copolar. The converse is evident by reversing the steps of the argument.

(11.2) **Proposition.** *Let $B \in \mathscr{E}$. Then B is m-semipolar if and only if B is m-cosemipolar.*

Proof. The same type of covering argument used above, in conjunction with (6.7), shows that each condition on B is in fact equivalent to

$$P\{Z_{t-} \in B \text{ for uncountably many } t \in]\alpha, \beta[\} = 0.$$

(See the remarks following (6.7)).

The next result gives us the weak duality version of Hunt's switching identity [4, VI(1.16)].

(11.3) **Theorem.** *If $f, g \in \mathscr{E}^*$ are positive and if $B \in \mathscr{E}$ then for all $q \geqq 0$*

$$(11.4) \qquad (P_B^q U^q g, f) = (g, \hat{P}_{B-}^q \hat{U}^q f).$$

Proof. Using the relationship between X, Z, and $\hat{X}$, and $\hat{P}^m(\hat{X}_t \neq \hat{X}_{t-}) = 0$ for each fixed t, one has

$$\begin{aligned} E^m[f(X_0)g(X_t); T_B < t] &= E^m[f(X_0)g(X_t); \exists s,\ 0 < s < t \text{ with } X_s \in B] \\ &= P[f(Z_0)g(Z_t); \exists s,\ 0 < s < t \text{ with } Z_s \in B] \\ &= P[f(Z_{-t})g(Z_0); \exists s,\ -t < s < 0 \text{ with } Z_s \in B] \\ &= \hat{E}^m[f(\hat{X}_{t-})g(\hat{X}_{0-}); \exists s,\ 0 < s < t \text{ with } \hat{X}_{s-} \in B] \\ &= \hat{E}^m[f(\hat{X}_t)g(\hat{X}_0); S_B < t]. \end{aligned}$$

Multiply this by e^{-qt} and integrate in t over $]0, \infty[$ to obtain (11.4).

Remark. Of course one also has the dual of (11.4), namely,

$$(11.5) \qquad (P_{B-}^q U^q f, g) = (f, \hat{P}_B^q \hat{U}^q g).$$

The next result generalizes (9.3).

(11.6) **Theorem.** *Let κ be a σ-integrable HRM with Revuz measure $\nu = \nu_\kappa$, and let B be a Borel set. Then*

$$(11.7) \qquad \nu \hat{P}_{B-}^q \hat{U}^q(dx) = P_B^q u_\kappa^q(x)\, m(dx);$$

$$(11.8) \qquad \nu \hat{P}_B^q \hat{U}^q(dx) = P_{B-}^q u_\kappa^q(x)\, m(dx).$$

Proof. It suffices to prove this when $q > 0$ and $\nu(E) < \infty$. Let $u = u_\kappa^q$. Then $\int u\, dm < \infty$ and so u is finite a.e. m. Let $u_k = u \wedge k$ and $g_{n,k} = n(u_k - P_{1/n}^q u_k)$. Clearly

$$U^q g_{n,k} = n \int_0^{1/n} P_s^q u_k\, ds$$

increases with both n and k, and so $u = \lim_k \lim_n U^q g_{n,k} = \lim_n \lim_k U^q g_{n,k}$. Suppose h is a bounded q-coexcessive function. Using (9.3),

$$\begin{aligned} \int g_{n,k} h\, dm &= n(u_k - P_{1/n}^q u_k, h) \\ &= n(u_k, h - \hat{P}_{1/n}^q h) \\ &\to n(u, h - \hat{P}_{1/n}^q h) \text{ as } k \to \infty \\ &= n\nu(\hat{U}^q h - \hat{U}^q \hat{P}_{1/n}^q h) \\ &= \nu\Big(n \int_0^{1/n} \hat{P}_s^q h\, ds\Big) \nearrow \nu(h) \text{ as } n \to \infty. \end{aligned}$$

Therefore

(11.9) $$\nu(h) = \lim_n \lim_k (g_{n,k}, h)$$

whenever h is a bounded q-coexcessive function. If $\varphi \in b\mathscr{E}^*$ is positive, $\hat{P}^q_{B-} \hat{U}^q \varphi$ is a bounded q-coexcessive function and from (11.9) and (11.3)

$$\begin{aligned}\nu(\hat{P}^q_{B-} \hat{U}^q \varphi) &= \lim_n \lim_k (g_{n,k}, \hat{P}^q_{B-} \hat{U}^q \varphi)\\ &= \lim_n \lim_k (P^q_B U^q g_{n,k}, \varphi)\\ &= (P^q_B u, \varphi).\end{aligned}$$

Consequently $\nu \hat{P}^q_{B-} \hat{U}^q(dx) = P^q_B u(x)\, m(dx)$, proving (11.7). Taking $h = \hat{P}^q_B \hat{U}^q \varphi$ in (11.9) the same argument yields (11.8).

(11.10) *Remark.* The proof of (11.6) in case $\nu(E) < \infty$ and $q > 0$ used only the fact that $\nu \hat{U}^q(dx) = u(x)\, m(dx)$ with u q-excessive. The fact that $\nu = \nu_\kappa$ and $u = u^q_\kappa$ did not enter explicitly. This remark is used in the next theorem characterizing Revuz measures.

(11.11) **Theorem.** *Let ν be a σ-finite measure on E not charging m-copolar sets. Then there exists a HRM, κ, of X with $\nu_\kappa = \nu$.*

Proof. It suffices to prove the result when $\nu(\mathrm{E}) < \infty$. Fix $q > 0$. Then $\nu \hat{U}^q(1) \leqq q^{-1} \nu(E)$. If $B \in \mathscr{E}$ with $m(B) = 0$, then $\hat{U}^q 1_B = 0$ a.e. m. But $\{\hat{U}^q 1_B = 0\}$ is absorbing for $\hat{X}$, and hence cofinely open. Since it has m measure zero, it is m-copolar by (6.10). Since ν does not charge m-copolars $\nu \hat{U}^q(B) = 0$. In other words $\nu \hat{U}^q \ll m$. Let f be a positive density. Given $g \in b\mathscr{E}^*$ with $g \geqq 0$,

$$(g, P^q_t f) = (\hat{P}^q_t g, f) = \nu \hat{U}^q \hat{P}^q_t g \uparrow \nu \hat{U}^q g = (g, f) \text{ as } t \downarrow 0.$$

Therefore f is m-q-excessive and so by (6.19) we may suppose $\nu \hat{U}^q(dx) = u(x)\, m(dx)$ where u is q-excessive and Borel. Clearly $\int u\, dm < \infty$.

We shall show that u satisfies the hypotheses of (4.11). If $g \in b\mathscr{E}^*$ with $g \geqq 0$, then $(g, P^q_t u) = \nu \hat{U}^q \hat{P}^q_t g \to 0$ as $t \to \infty$ and so $P^q_t u \to 0$ a.e. m as $t \to \infty$. Next let (D_n) be a decreasing sequence of nearly Borel sets with $\lim T_{D_n} \geqq \zeta$ a.s. P^m. Because of (6.6) one may suppose that each D_n is, in fact, Borel. Then by (11.10), for a bounded positive $g \in \mathscr{E}$

(11.12) $$(g, P^q_{D_n} u) = \nu \hat{P}^q_{D_n-} \hat{U}^q g.$$

We claim that $\lim \hat{S}_{D_n} \geqq \hat{\zeta}$ a.s. $\hat{P}^m$. We shall use this to complete the proof and then establish this claim. Since

$$\hat{P}^q_{D_n-} \hat{U}^q g(x) = \hat{E}^x \int_{\hat{S}_{D_n}}^{\hat{\zeta}} e^{-qt} g(\hat{X}_t)\, dt,$$

it follows that $\hat{P}^q_{D_n-} \hat{U}^q g \downarrow 0$ a.e. m as $n \to \infty$ if $\lim \hat{S}_{D_n} \geqq \hat{\zeta}$ a.s. $\hat{P}^m$. Using (6.9) we find that $\hat{P}^q_{D_n-} \hat{U}^q g \downarrow 0$ except on an m-copolar set. Since ν does not charge such sets it is immediate from (11.12) that $P^q_{D_n} u \to 0$ a.e. m as $n \to \infty$. Hence (4.11) asserts the existence of a NHRM, κ with $u_\kappa^q = u$ a.e. m. Using (9.3)

$$\nu_\kappa \hat{U}^q = u_\kappa^q dm = u dm = \nu \hat{U}^q,$$

and consequently by the uniqueness theorem for potentials of measures $\nu_\kappa = \nu$.

Thus to complete the proof we must show that $\lim \hat{S}_{D_n} \geqq \hat{\zeta}$ a.s. $\hat{P}^m$. Given $0 < u < v$ let

$$W_{u,v} = \left\{ w \in W\colon \bigcap_n \{\exists t \in]u, v[\colon Z_t(w) \in D_n\} \neq \phi;\ \alpha(w) < u < v < \beta(w) \right\}.$$

Then if $m(B) < \infty$

$$\begin{aligned} &P[Z_u \in B, W_{u,v}] \\ &\quad = P^m\left[X_0 \in B,\ \bigcap_n \{X_t \in D_n \text{ for some } t \in]0, v-u[\} \neq \phi;\ v-u < \zeta \right] \\ &\quad \leqq \lim_n P^m(X_0 \in B, T_{D_n} < v-u < \zeta) = 0. \end{aligned}$$

Since m is σ-finite $P(W_{u,v}) = 0$ for all pairs u, v. But

$$\begin{aligned} 0 &= P(W_{u,v}) \\ &= \hat{P}^m\left[\bigcap_n \{\hat{X}_{t-} \in D_n \quad \text{for some} \quad t \in]0, v-u[\} \neq \phi;\ v-u < \hat{\zeta} \right] \\ &\geqq \hat{P}^m[\lim \hat{S}_{D_n} < v-u < \hat{\zeta}], \end{aligned}$$

and since u and v are arbitrary this completes the proof of (11.11).

12. HRM's over X, $\hat{X}$ and Z

We investigate now the connections between HRM's over X, $\hat{X}$ and Z. These methods are needed for some later sections. The correspondence we discuss below was first considered by Mitro [30], and our discussion will be similar to hers. To begin with, a random measure K over Z is defined to be a kernel $K(w, dt)$ from $(W, \mathscr{A})$ to $(\mathbb{R}, \mathscr{R})$ such that for every $w \in W$, $K(w, \cdot)$ is a countably finite measure carried by $]\alpha(w), \beta(w)[$. Such a K is *homogeneous* in case

(12.1) *except for a null set of w's, $K(w, B) = K(\theta_t w, B - t)$ for all $t \in \mathbb{R}$, $B \in \mathscr{R}$.*

Note that homogeneity here means *perfect* homogeneity: the exceptional set may be chosen independent of t. If one starts with a perfect HRM (§4) κ for X one may embed κ in a HRM K for Z by the following procedure. With notation as in §10 so that $X_t(\Pi_s w) = Z_{t+s}(w)$ for $t \geqq 0$ and $\alpha(w) < s < \beta(w)$, $X_t(\Pi_s w) = \Delta$ otherwise, set

(12.2) $$K(w, B) = \sup_{s \in \mathbb{R}} \kappa(\Pi_s w, B - s); \quad B \in \mathscr{R}.$$

Observe first that because $\kappa([\Delta], \cdot) = 0$, $\kappa(\Pi_s w, B - s) = 0$ if $s \leqq \alpha(w)$ or $s \geqq \beta(w)$. Then note that if $\alpha(w) < u < v < \beta(w)$ and if $B \subset]v, \beta(w)[$ then additivity of κ gives

$$\kappa(\Pi_u w, B-u) = \kappa(\theta_{v-u}\Pi_u w, B-u-(v-u))$$
$$= \kappa(\Pi_v w, B-v).$$

That is, the random measures $\kappa(\Pi_s w, B-s)$ have enough consistency to show that $K(w, \cdot)$ is indeed a countably finite measure carried by $]\alpha(w), \beta(w)[$. The homogeneity of K follows easily from these remarks. The following examples are easily checked.

(12.3) If $\kappa(dt) = f(X_t)\,dt$ $[f \in \mathscr{E}_+]$ then $K(dt) = f(Z_t)\,dt$.

(12.4) If $\kappa(dt) = \sum_{0<s<\zeta} f(X_s)\,\varepsilon_s\,dt)$ with $f \in \mathscr{E}_+$ vanishing off a semipolar set in E then $K(dt) = \sum_{\alpha<s<\beta} f(Z_s)\,\varepsilon_s(dt)$.

In the next examples and in later sections we use the notation, for $B \in \mathscr{E}$,

(12.5)
$$L_B = \sup\{t: X_t \in B\}$$
$$M_B = \sup\{t < \zeta: X_{t-} \in B\}$$

where, as usual, $\sup\phi \equiv 0$. Note that $L_B \leqq \zeta$ and $M_B \leqq \zeta$.

For the objects over Z corresponding to T_B, S_B, L_B and M_B we use $\tau_B, \sigma_B, \lambda_B$ and μ_B respectively. Thus, for example, $\sigma_B = \inf\{t \in]\alpha, \beta[: Z_{t-} \in B\}$, $\lambda_B = \sup\{t \in]\alpha, \beta[: Z_t \in B\}$, with $\sup\phi = -\infty$, $\inf\phi = +\infty$ in this case. With this notation we obtain the following examples which will be used in the next section.

(12.6) If $L \equiv L_B$ and $\lambda \equiv \lambda_B (B \in \mathscr{E})$ and if $\kappa(dt)$
$\equiv 1_{\{0<L<\zeta\}}\varepsilon_L(dt)$ then $K(dt) = 1_{\{\alpha<\lambda<\beta\}}\varepsilon_\lambda(dt)$.

(12.7) If $M \equiv M_B$ and $\mu \equiv \mu_B (B \in \mathscr{E})$ and if $\kappa(dt)$
$\equiv 1_{0<M<\zeta\}}\varepsilon_M(dt)$ then $K(dt) = 1_{\{\alpha<\mu<\beta\}}\varepsilon_\mu(dt)$.

If one starts instead with a HRM $\hat{\kappa}$ for $\hat{X}$, there is an analogous procedure for generating a HRM K of Z, simply replacing Π_s by $\hat{\Pi}_s$ in (12.2). Instead of using the notation $\hat{\tau}_B$, $\hat{\sigma}_B$ etc. for the objects over $\hat{Z}$ corresponding to τ_B, σ_B we shall use systematically $\tau_B, \sigma_B, \lambda_B, \mu_B$ with the obvious relations

(12.8)
$$\hat{\tau}_B = -\mu_B; \ \hat{\sigma}_B = -\lambda_B$$

etc.

The duals of (12.6), (12.7) may then be stated in the form, with $B \in \mathscr{E}$,

(12.9) if $\hat{L} \equiv \hat{L}_B$ and $\sigma \equiv \sigma_B$ then $\hat{\kappa}(dt) \equiv 1_{\{0<\hat{L}<\zeta\}}\varepsilon_{\hat{L}}(dt)$
extends to $K(dt) \equiv 1_{\{\alpha<\sigma<\beta\}}\varepsilon_\sigma(dt)$:

(12.10) if $\hat{M} \equiv \hat{M}_B$ and $\tau \equiv \tau_B$ then $\hat{\kappa}(dt) \equiv 1_{\{0<\hat{M}<\zeta\}}\varepsilon_{\hat{M}}(dt)$
extends to $K(dt) \equiv 1_{\{\alpha<\tau<\beta\}}\varepsilon_\tau(dt)$.

The following computation gives a simple interpretation of the Revuz measure of κ in terms of K.

(12.11) **Theorem.** *Let K be a HRM over Z and let κ be a HRM over X such that for every $u > 0$, $K(]0,u]) = \kappa(]0,u]) \circ \Pi_0$ P a.s. on $\{\alpha < 0 < \infty\}$, as in* (12.2). *Then $\nu_\kappa(E) = P\{K(]0,1])\}$.*

Proof. As $n \to \infty$ one finds

$$\sum_{k=0}^{2^n-1} 1_{]k2^{-n},(k+1)2^{-n}]} 1_{\{\alpha < k2^{-n}\}} \uparrow 1_{]0,1] \cap]\alpha, \infty[}.$$

Therefore

$$PK(]0,1]) = \lim_{n\to\infty} P \sum_{k=0}^{2^n-1} K\{]k2^{-n}, (k+1)2^{-n}]\} 1_{\{\alpha < k2^{-n}\}}.$$

As K vanishes on $[\beta, \infty[$, the indicator on the right may be replaced by $1_{\{\alpha < k2^{-n} < \beta\}}$. By hypothesis, P a.s., $K(]0, 2^{-n}]) = \kappa(]0, 2^{-n}]) \circ \Pi_0$ if $\alpha < 0 < \beta$. Using the homogeneity of K and stationarity of P,

$$\begin{aligned} &P\{K(]k2^{-n}, (k+1)2^{-n}]); \alpha < k2^{-n} < \beta) \\ &\quad = P\{K(]0, 2^{-n}]); \alpha < 0 < \beta\} = P^m\{\kappa(]0, 2^{-n}])\}, \end{aligned}$$

where we used (10.5) to get the last equality. Thus

$$PK(]0,1]) = \lim 2^n E^m \kappa(]0, 2^{-n}]),$$

the right side being $\nu_\kappa(E)$ by the very definition (8.1) of ν_κ.

The preceding theorem has also been obtained independently by Atkinson and Mitro [2].

(12.12) **Theorem.** *Let $f \in \mathscr{R} \times \mathscr{E}$ be positive and let κ be a σ-integrable HRM for X. Then*

$$P\int_\alpha^\beta f(t, Z_{t-}) K(dt) = \int_E \int_{-\infty}^{\infty} f(t,x)\, dt\, \nu_\kappa(dx). \tag{12.13}$$

Proof. The case $f(t,x) = 1_{]0,1]}(t) 1_E(x)$ is established in (12.11). On the other hand, stationarity of P and homogeneity of K imply that the measure $B \to PK(B)$ on $\mathscr{R}$ is translation invariant, and hence is a multiple of Lebesgue measure. It follows that (12.13) holds in case $f(t,x) = 1_B(t) 1_E(x)$. It is then easy to see that if $g \in b\mathscr{E}_+$ then $g_- * \kappa$ has Revuz measure $g \cdot \nu_\kappa$, $g(Z_{t-}) K(dt)$ being the corresponding HRM over Z, and this gives (12.13) in case $f(t,x) = 1_B(t) g(x)$. A routine monotone class argument completes the proof, taking into account the σ-integrability of κ.

It follows from (12.12) that, in essence, what Dynkin [11] calls the characteristic measure of a homogeneous random measure may be identified with the Revuz measure. See also [2] in this regard.

13. Capacities

We propose to give an interpretation of capacities in terms of Z. Recall [6, 17] the interpretation of capacity of a set F in terms of last hitting time L_F of F by X. We begin with a lemma, one form of which has been obtained independently in [2].

(13.1) **Lemma.** *Let κ, γ be σ-integrable HRM's of X, $\hat{X}$ respectively. Then for every $q \geq 0$*

$$\nu_\kappa(\hat{u}_\gamma^q) = \hat{\nu}_\gamma(u_\kappa^q). \tag{13.2}$$

Proof. We remark first that in the case of classical duality, each side of (13.2) is equal to $\int\int \nu_\kappa(dx)\, u^q(x,y)\, \hat{\nu}_\gamma(dy)$ because of the discussion preceding (9.3). That is, (13.2) amounts just to Fubini's theorem in this case. In the weak duality setting, we rely instead on (9.5). Using $n\,\hat{U}^{q+n}\hat{u}^q_\gamma \uparrow \hat{u}^q_\gamma$ as $n \to \infty$ in the first equality and (9.5) in the second and third, we find

$$\begin{aligned}\nu_\kappa(\hat{u}^q_\gamma) &= \lim_n \int \nu_\kappa(dx)\, n\hat{U}^{q+n}\hat{u}^q_\gamma(x)\\ &= \lim_n \int n u^{q+n}_\kappa(x)\, m(dx)\, \hat{u}^q_\gamma(x)\\ &= \lim_n \int n u^{q+n}_\kappa(x)\, \hat{\nu}_\gamma U^q(dx) = \lim_n \int \hat{\nu}_\gamma(dx)\, n\, U^q u^{q+n}_\kappa(x)\,.\end{aligned}$$

However, a routine calculation requiring only σ-integrability of κ shows that

$$\begin{aligned}nU^q u^{q+n}_\kappa(x) &= E^x \int_0^\infty n e^{nt}\, dt \int_t^\infty e^{-(q+n)s}\kappa(ds)\\ &= E^x \int_0^\infty \int_0^s n e^{nt}\, dt\, e^{-(q+n)s}\kappa(ds)\\ &= E^x \int_0^\infty e^{-qs}(1-e^{-ns})\,\kappa(ds)\,,\end{aligned}$$

which increases to $u^q_\kappa(x)$ as $n \to \infty$. This establishes (13.1).

Recall now the definitions ((12.5) and the subsequent paragraph) of the random times L_B, M_B, τ_B, σ_B, λ_B, μ_B etc., for $B \in \mathscr{E}$.

(13.3) *Definition.* A set $B \in \mathscr{E}$ is strongly m-transient (resp., strongly left m-transient) for X if $P^m\{L_B = \zeta\} = 0$ (resp. $P^m\{M_B = \zeta\} = 0$).

For any $B \in \mathscr{E}$ let π_B, π^-_B denote the Revuz measures for the respective HRM's $1_{\{0<L_B<\zeta\}}\varepsilon_{L_B}(dt)$, $1_{\{0<M_B<\zeta\}}\varepsilon_{M_B}(dt)$. We shall call π_B, π^-_B the right and left *capacitary measures* for B, and the numbers $c(B)$, $c^-(B)$ defined by $c(B) = \pi_B(E)$, $c^-(B) = \pi^-_B(E)$, will be called the right and left *capacities* of B respectively. The following formulas are immediate consequences of (9.9).

(13.4) $$\begin{aligned}&\int g(x)\, m(dx)\, E^x\{f(X(L^-_B))\, e^{-qL_B};\, 0 < L_B < \zeta\}\\ &\quad = \int g(x)\,(f\pi_B)\,\hat{U}^q(dx)\,,\end{aligned}$$

(13.5) $$\begin{aligned}&\int g(x)\, m(dx)\, E^x\{f(X(M^-_B))\, e^{-qM_B};\, 0 < M_B < \zeta\}\\ &\quad = \int g(x)\,(f\pi^-_B)\,\hat{U}^q(dx)\,,\end{aligned}$$

where $f, g \in \mathscr{E}^*$ are positive.

Observe too that if $B \in \mathscr{E}$ is strongly m-transient, then B is m-polar if and only if $c(B) = 0$, and that if B is strongly left m-transient then B is left m-polar ($= m$-copolar) if and only if $c^-(B) = 0$.

There is a rather obvious formulation of the transience conditions in (13.3) using Z and the times τ_B, σ_B etc. defined after (12.5).

(13.6) **Proposition.** *A set $B \in \mathscr{E}$ is strongly m-transient for X if and only if $P\{\lambda_B = \beta\} = 0$, strongly left m-transient for X if and only if $P\{\mu_B = \beta\} = 0$.*

Using the examples (12.6) and (12.7), Theorem 12.12 specializes to give us the following formulas, assuming for (13.7) (resp. (13.8)) that B is strongly m-transient (resp. strongly left m-transient), and that $f \in (\mathscr{R} \times \mathscr{E})_+$.

(13.7) $$P\{f(\lambda_B, Z(\lambda_B^-)); \lambda_B > \alpha\} = \int_E \int_{-\infty}^{\infty} f(t,x)\, dt\, \pi_B(dx);$$

(13.8) $$P\{f(\mu_B, Z(\mu_B^-)); \mu_B > \alpha\} = \int_E \int_{-\infty}^{\infty} f(t,x)\, dt\, \pi_B^-(dx).$$

These formulas are more attractive in their respective differential forms

(13.9) $$P\{\lambda_B \in dt, Z(\lambda_{B^-}) \in dx\} = dt\, \pi_B(dx);$$

(13.10) $$P\{\mu_B \in dt, Z(\mu_{B^-}) \in dx\} = dt\, \pi_B^-(dx).$$

In particular, under the respective transience hypotheses described before (13.7) we have

(13.11) $$P\{\lambda_B \in dt\} = c(B)\, dt;\ P\{\mu_B \in dt\} = c^-(B)\, dt.$$

The formulas dual to (13.7)–(13.11) are likewise valid. If $B \in \mathscr{E}$ is strongly m-transient relative to $\hat{X}$, or equivalently, $P\{\sigma_B \leqq \alpha\} = 0$, then if $f \in (\mathscr{R} \times \mathscr{E})_+$

(13.12) $$P\{f(\sigma_B, Z(\sigma_B)); \sigma_B < \beta\} = \int_E \int_{-\infty}^{\infty} f(t,x)\, dt\, \hat{\pi}_B(dx).$$

That is

(13.13) $$P\{\sigma_B \in dt, Z(\sigma_B) \in dx\} = dt\, \hat{\pi}_B(dx),$$

(13.14) $$P\{\sigma_B \in dt\} = \hat{c}(B)\, dt,$$

where, of course, $\hat{\pi}_B$ is the Revuz measure for the HRM of $\hat{X}$ putting unit mass at $\hat{L}_B$ provided $0 < \hat{L}_B < \hat{\zeta}$, and $\hat{c}(B) = \hat{\pi}_B(E)$. We are using here (12.9). There are corresponding formulas, which we shall not record, in case $B \in \mathscr{E}$ is strongly left m-transient relative to $\hat{X}$. This is obtained using (12.10), substituting τ_B for σ_B and $\hat{\pi}_B^-$ for $\hat{\pi}_B$ in (13,12).

The connection between capacities and cocapacities will be given in (13.20) below. We begin with an important computation.

(13.15) **Theorem.** *Let $B \in \mathscr{E}$ be strongly m-transient for $\hat{X}$. That is, we suppose $P\{\sigma_B = \alpha\} = 0$. Then, for every $t > 0$*

(13.16) $$\begin{aligned} t\,\hat{c}(B) &= t \int \hat{\pi}_B(dx) P^x\{L_B > 0\} + P\{\sigma_B = \lambda_B \in]0,t[\} \\ &= t \int \hat{\pi}_B(dx) P^x\{M_B > 0\} + P\{\sigma_B = \mu_B \in]0,t[\}. \end{aligned}$$

Proof. Because σ_B is a stopping time for Z and $\alpha < \sigma_B < \beta$ a.s. on $\{\sigma_B < \infty\}$, the strong Markov property (10.11) applies to compute

$$P\{\sigma_B = \lambda_B \in]0,t[\} = P\{P^{Z(\sigma_B)}\{L_B = 0\}; \sigma_B \in]0,t[\}.$$

On the other hand, (13.13) gives

$$P\{P^{Z(\sigma_B)}\{L_B>0\};\ \sigma_B\in]0,t[\} = \int\int P^x\{L_B>0\}\,1_{]0,t[}(s)\,ds\,\hat{\pi}_B(dx) = t\int \hat{\pi}_B(dx)P^x\{L_B>0\}.$$

Adding the last two equations and using (13.14) we obtain (13.16). Exactly the same proof with μ_B replacing λ_B gives the second equality.

There are three other pairs of formulas analogous to (13.16), the proofs being modifications of the proof above, and we simply record the results. For (13.17), (13.18), (13.19) below it is assumed respectively that $B\in\mathscr{E}$ is strongly left m-transient for $\hat{X}$, strongly m-transient for X, strongly left m-transient for X:

(13.17) $$t\hat{c}^-(B) = t\int \hat{\pi}_B^-(dx)P^x\{L_B>0\} + P\{\tau_B=\lambda_B\in]0,t[\} = t\int \hat{\pi}_B^-(dx)P^x\{M_B>0\} + P\{\tau_B=\mu_B\in]0,t[\}.$$

(13.18) $$tc(B) = t\int \pi_B(dx)\hat{P}^x\{\hat{L}_B>0\} + P\{\sigma_B=\lambda_B\in]0,t[\} = t\int \pi_B(dx)\hat{P}^x\{\hat{M}_B>0\} + P\{\tau_B=\lambda_B\in]0,t[\}.$$

(13.19) $$tc^-(B) = t\int \pi_B^-(dx)\hat{P}^x\{\hat{L}_B>0\} + P\{\sigma_B=\mu_B\in]0,t[\} = t\int \pi_B^-(dx)\hat{P}^x\{\hat{M}_B>0\} + P\{\tau_B=\mu_B\in]0,t[\}.$$

(13.20) **Theorem.** *Let $B\in\mathscr{E}$ be strongly m-transient for $\hat{X}$ and strongly left m-transient for X – that is, $P\{\lambda_B=\beta$ or $\sigma_B=\alpha\}=0$ – then $\hat{c}(B)=c^-(B)$.*

Proof. We use (13.16) and (13.19). The formula (13.2) gives us $\int\hat{\pi}_B(dx)$ $P^x\{M_B>0\} = \int \pi_B^-(dx)\hat{P}^x\{\hat{L}_B>0\}$. Using the first equation in (13.19) and the second in (13.16), the equality $\hat{c}(B)=c^-(B)$ is now evident.

Dually if $B\in\mathscr{E}$ is strongly m-transient for X and strongly left m-transient for $\hat{X}$, then $c(B)=\hat{c}^-(B)$.

The nicest form of the results above occurs when both X and $\hat{X}$ are standard (see (15.14) and (15.16).

14. Nagasawa's Theorem

As an application of some of the results developed in the preceding sections we present a version of Nagasawa's theorem that is valid for Borel right processes under weak duality. The proof is similar in outline to the proof given in [41], and we shall refer to [41] for some of the details.

We suppose only that X and $\hat{X}$ are Borel right processes in weak duality with respect to m. We fix an initial probability measure μ on E not charging m-polars. Then by (the dual of) (11.11), there exists a NHRM, $\hat{\lambda}$ of $\hat{X}$ having μ as its Revuz measure. Let $\hat{u}(x)=\hat{E}^x\{\hat{\lambda}(]0,\infty[)\}$. Then according to (9.3)

(14.1) $$\mu U(dx) = \hat{u}(x)\,m(dx).$$

Finally recall that $L\colon\Omega\to[0,\infty]$ is co-optional for X provided L is $\mathscr{F}^*$ measurable, $L\leq\zeta$, and $L\circ\theta_t=(L-t)^+$.

(14.2) **Theorem.** *Let μ and $\hat{u}$ be as above. Let L be a co-optional time for X. Define $\tilde{X}$ by*

$$\begin{aligned}\tilde{X}_t &= X_{(L-t)-} \quad \text{if } 0 < t < L < \infty \\ &= \Delta \quad \text{otherwise}.\end{aligned}$$

Then under P^μ, the process $(\tilde{X}_t)_{t>0}$ is a homogeneous Markov process on E with semigroup $(\tilde{P}_t)$ given by

(14.3)
$$\begin{aligned}\tilde{P}_t f(y) &= \hat{P}_t(\hat{u}f)(y)/\hat{u}(y) \quad \text{if } 0 < \hat{u}(y) < \infty \\ &= 0 \qquad\qquad \text{if } \hat{u}(y) = 0 \text{ or } \hat{u}(y) = \infty.\end{aligned}$$

Proof. Arguing exactly as in §3 of [41], it suffices to show that

(14.4)
$$\begin{aligned}&E^\mu\{Y_L \int_0^L e^{-qt} g(\tilde{X}_t)\,dt; 0 < L < \infty\} \\ &= E^\mu\{Y_L \tilde{U}^q g(X_{L-}); 0 < L < \infty\}\end{aligned}$$

for g a bounded positive continuous function, Y a bounded positive process homogeneous on $]0, \infty[$, and for all co-optional times L of X satisfying $L < \zeta$ almost surely on $\{L < \infty\}$. In (14.4), $\tilde{U}^q$ is the resolvent of $(\tilde{P}_t)$ and is given by

(14.5)
$$\begin{aligned}\tilde{U}^q f(y) &= \hat{U}^q(\hat{u}f)(y)/\hat{u}(y) \quad \text{if } 0 < \hat{u}(y) < \infty \\ &= 0 \qquad\qquad \text{if } \hat{u}(y) = 0 \text{ or } \hat{u}(y) = \infty.\end{aligned}$$

Again exactly as in [41], the left side of (14.4) is equal to $E^\mu \int_0^\infty h(X_t)\,g(X_t)\,dt = \mu U(hg)$ with

(14.6)
$$\begin{aligned}h(x) &= E^x\{e^{-qL} Y_L; 0 < L < \infty\} \\ &= E^x \int e^{-qt}\,dB_t,\end{aligned}$$

where

(14.7)
$$B_t = Y_L 1_{[L,\infty[}(t)\, 1_{\{0 < L < \infty\}}$$

is a raw AF of X. Let ν denote the Revuz measure of B. Using (14.1) and the corresponding fact that $\nu \hat{U}^q(dx) = h(x)\,m(dx)$ we see that the left side of (14.4) is given by

(14.8)
$$\begin{aligned}\mu U(hg) &= \int \hat{u}(x)\,h(x)\,g(x)\,m(dx) \\ &= \int \hat{u}(x)\,g(x)\,\nu\hat{U}^q(dx) = \nu\hat{U}^q(\hat{u}g).\end{aligned}$$

We next compute the right side of (14.4). Let $dC_t = \tilde{U}^q g(X_{t-})\,dB_t$ so that the Revuz measure ν_C of C is given by $\nu_C(dx) = \tilde{U}^q g(x)\,\nu(dx)$. The right side of (14.4) may be written (with $u_C(x) \equiv E^x(C_\infty)$)

$$\begin{aligned}E^\mu \int \tilde{U}^q g(X_{t-})\,dB_t &= \mu(u_C) \\ &= \int \hat{u}(x)\,\tilde{U}^q g(x)\,\nu(dx) \\ &= \nu\hat{U}^q(\hat{u}g),\end{aligned}$$

where the second equality follows from (13.1). This establishes (14.4) and completes the proof of Theorem (14.2).

15. Specialization to m-Standard Processes

In the last part of this section we shall set down some improvements in some of the results of the last few sections due to standardness assumptions on X and $\hat{X}$. We begin without duality hypotheses, obtaining a comparison between m-standardness and standardness. *Excessiveness of m is required throughout this section except in* (15.8)–(15.11).

(15.1) **Theorem.** *Let X be m-standard, with m excessive. Then there exists a Borel m-inessential set $F \subset E$ such that the restriction X' of X to $E' \equiv E - F$ is P^x-standard for all $x \in E'$. (It follows then from the discussion in* §5 *that X' becomes standard once we delete from Ω' the null set $\{X_{t-}$ fails to exist in E for some $t < \zeta\}$.)*

Proof. Define $g(x) = P^x\{X_{t-}$ fails to exist in E for some $t \in]0, \zeta[\}$. Evidently g is excessive and, $m\{g > 0\} = 0$ follows from (5.5) and the hypothesis. Therefore, $\{g > 0\}$ is m-inessential. Following (6.12) choose a Borel m-inessential set $G \supset \{g > 0\}$. *Let* R denote the hitting time of the absorbing set G^c. Set $\psi(x) = P^x$ $\{$there exists $t > R$ with $t < \zeta$, $X_{t-} \neq X_t$ but either $X^r_{t-} \notin E_\Delta$ or $X^r_{t-} = X_t\}$. Because $R \circ \theta_s = (R - s)^+$ it is easy to see that ψ is excessive. In addition, the m-standardness of X and (5.5) yield $m\{\psi > 0\} = 0$ so that $\{\psi > 0\}$ is m-inessential. Choose a Borel m-inessential set $H \supset \{\psi > 0\}$. If we take x in the absorbing set $E' \equiv G^c \cap H^c$, then using the fact that (5.5iii) implies (5.5ii) we see that X is P^x-standard. It is then immediate that the restriction X' of X to E' is P^x-standard for all $x \in E'$.

In view of the preceding theorem and the remarks at the beginning of §5, we may apply results known for genuine standard processes to m-standard processes as long as we make allowance for an m-inessential set.

We discuss first the occupation time properties for X, X_- assuming X to be m-standard. In what follows, if $B \subset E \cup \Delta$, $\{X \in B\}$ denotes $\{(t, \omega)\colon t > 0, X_t(\omega) \in B\}$ and $\{X_- \in B\} \equiv \{(t, \omega)\colon t > 0, X_{t-}(\omega)$ exists and belongs to $B\}$. For any subset Γ of $\mathbb{R}^+ \times \Omega$, Γ^- will denote the random set whose ω-section is the closure of the ω-section of Γ.

We emphasize that the next result makes no use of duality hypotheses.

(15.2) **Lemma.** *Let X be standard (resp., m-standard with m excessive). If $B \subset E$ is nearly Borel, then*

$$\{X_- \in B\}^- \cap]\!]0, \zeta[\![\subset \{X \in B\}^- \cap]\!]0, \zeta[\![$$

*up to evanescence (*resp., *P^m-evanescence).*

Proof. In view of the remarks above (15.2) it is enough to prove the standard case. It is proved in [4, I-10.20] that

$$\{X \in B \cup \Delta\} = \{X \in B \cup \Delta\} \cup \{X_- \in B \cup \Delta\}$$

up to evanescence. It follows that, up to evanescence

$$\{X \in B \cup \Delta\}^- = [\{X \in B \cup \Delta\} \cup \{X_- \in B \cup \Delta\}]^-$$

and consequently $\{X_- \in B \cup \Delta\}^- \subset \{X \in B \cup \Delta\}^-$ also up to evanescence. Because $\{X_- = \Delta\} \cap]\!]0, \zeta[\![$ is evanescent, the desired inclusion now obtains.

(15.3) **Proposition.** *Let X be m-standard with m excessive. If $B \in \mathscr{E}$ is m-semipolar then $\{X_- \in B\} \cap]\!]0, \zeta[\![\subset \{X_- = X\}$ up to P^m evanescence.*

Proof. Because of (15.2) an m-polar set is also left m-polar. It is therefore enough to prove (15.3) in case $B \in \mathscr{E}$ is totally thin. As $\{X \in B\}$ is then a.s. discrete, (15.2) gives $\{X_- \in B\} \cap]\!]0, \zeta[\![\subset \{X \in B\}$ up to P^m evanesecence. Let d be a metric on E compatible with the topology of E. Given $\varepsilon > 0$ write $B = \cup B_k$ where the B_k are disjoint Borel sets with diam $(B_k) < \varepsilon$ for all k. Up to P^m evanescence, $\{X_- \in B_k\} \subset \{X \in B_k\}$ so if $X_{t-} \in B$, $X_{t-} \in B_k$ for some k implies $X_t \in B_k$ and therefore $d(X_{t-}, X_t) < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, this completes the proof of (15.3).

It will be proved later (16.15) that, under weak duality hypotheses, the condition in (15.3) is sufficient for m-standardness of X. That is, if $\{X_- \in B\} \cap]\!]0, \zeta[\![\subset \{X = X_-\}$ up to P^m-evanescence for every m-semipolar $B \in \mathscr{E}$ then X is m-standard.

We assume for the rest of this section that X, $\hat{X}$ are in weak duality relative to m.

(15.4) **Proposition.** *Let $B \in \mathscr{E}$. If X is m-standard, then up to P-evanescence*

(15.5) $$\{Z \in B\}^- \cap]\!]\alpha, \beta[\![\supset \{Z_- \in B\}^- \cap]\!]\alpha, \beta[\![.$$

On the other hand, if $\hat{X}$ is m-standard then

(15.6) $$\{Z \in B\}^- \cap]\!]\alpha, \beta[\![\subset \{Z_- \in B\}^- \cap]\!]\alpha, \beta[\![.$$

Proof. The left side of (15.5) may be expressed as

$$\cup \{[\{Z \in B\} \cap]\!]r, \beta[\![]^- \cap]\!]\alpha, \beta[\![: r \in \mathbf{Q}, r > \alpha\} .$$

For a fixed $r \in \mathbf{Q}$, (15.2) applied to $t \to Z_{t+r}$ on $\{r > \alpha\}$ gives that

$$\{Z \in B\}^- \cap]\!]r, \beta[\![\supset \{Z_- \in B\}^- \cap]\!]r, \beta[\![$$

up to P-evanescence on $\{r > \alpha\}$. Therefore, up to P-evanescence, (15.5) holds if X is m-standard. Interchanging the roles of X, $\hat{X}$ yields instead (15.6), after a simple time reversal.

Recall that T_B denotes the hitting time of B and S_B denotes $\inf\{0 < t < \zeta: X_{t-}(\omega) \in B\}$. Using the relationship between X and Z the following result is evident from (15.2) and (15.4).

(15.7) **Proposition.** *Let $B \in \mathscr{E}$. If $\hat{X}$ is m-standard then $\{X \in B\}^- \cap]\!]0, \zeta[\![\subset \{X_- \in B\}^- \cap]\!]0, \zeta[\![$ up to P^m evanescence, so in particular $P^m\{S_B > T_B\} = 0$. If both X and $\hat{X}$ are m-standard then $\{X \in B\}^- \cap]\!]0, \zeta[\![= \{X_- \in B\}^- \cap]\!]0, \zeta[\![$ up to P^m-evanescence, so in particular $P^m\{S_B \neq T_B\} = 0$.*

Note the connections with Hunt's hypothesis (B) for a right process X. (No duality or excessive measure is assumed in the discussion of (15.8)–(15.11).) Hypothesis (B) states:

(15.8) *If $A \subset G \subset E$ with A nearly Borel, G open, and if $q \geq 0$ then $P^q_G P^q_A = P^q_A$.*

It is proved in Meyer [26, III, T17–19] that for a standard process, each of the following is equivalent to hypothesis (B):

(15.9) *If A is semipolar and nearly Borel and if $A \subset G$ with G open in E then* $P^x\{X(T_G) \in A\} = 0$ *for every* $x \notin A$.

(15.10) *If $A \in \mathscr{E}$ is semipolar then $\{X \in A\} \subset \{X = X_-\}$ up to evanescence.*

It is also known [38, 6.2] for example, that under standardness plus the hypothesis of absolute continuity (i.e., Meyer's hypothesis (L)), hypothesis (B) implies

$$S_B = T_B \text{ a.s.} \tag{15.11}$$

It is easy to see that (15.11) implies hypothesis (B), and it seems that (15.11) is in general a stronger condition. Our result (15.7) gives a weaker form of (15.11) appropriate to the weak duality setting. See Azéma [3, 5.1] for a related result. The following result gives the appropriate weakening of (15.10) in the present situation.

(15.12) **Proposition.** *Let $B \in \mathscr{E}$ be m-semipolar. If $\hat{X}$ is m-standard, then*

$$\{X \in B\} \subset \{X = X_-\} \text{ up to } P^m \text{ evanescence}; \tag{15.13}$$

$$\{Z \in B\} \cap \rrbracket \alpha, \beta \llbracket \subset \{Z = Z_-\} \cap \rrbracket \alpha, \beta \llbracket \text{ up to } P\text{-evanescence}. \tag{15.14}$$

Proof. The dual of (15.3) gives us $\{\hat{X}_- \in B\} \cap \rrbracket 0, \zeta \llbracket \subset \{\hat{X} = \hat{X}_-\}$ up to $\hat{P}^m$ evanescence. Arguing then as in the first part of the proof of (15.4) one sees that this implies that, up to P-evanescence,

$$\{\hat{Z}_- \in B\} \cap \rrbracket \hat{\alpha}, \hat{\beta} \llbracket \subset \{\hat{Z} = \hat{Z}_-\} \cap \rrbracket \hat{\alpha}, \hat{\beta} \llbracket .$$

The latter inclusion is identical to (15.14), by time reversal. The inclusion (15.13) is an obvious consequence of (15.14).

From now to the end of §15 *we shall be assuming that both X and $\hat{X}$ are m-standard.* [In view of (16.15) this condition is equivalent to requiring that for every m-semipolar set $B \in \mathscr{E}$, X is P^m a.s. continuous on $(\{X_- \in B\} \cup \{X \in B\}) \cap \rrbracket 0, \zeta \llbracket$.]

(15.15) **Theorem.** *Let X and $\hat{X}$ be m-standard and let $B \in \mathscr{E}$. Then the following are equivalent*:

(i) *B is m-polar*;
(ii) *B is left m-polar*;
(iii) *$\{Z \in B\}$ is P evanescent*;
(iv) *$\{Z_- \in B\}$ is P evanescent*;
(v) *B is m-copolar.*

Proof. The equivalence of (i) and (ii) is an immediate consequence of (15.7). If B is m-polar, then clearly $P(Z_t \in B$ for some $t) = 0$ and so $\hat{P}^m(\hat{X}_{t-} \in B$ for some $t) = 0$. Thus using the equivalence of (i) and (ii) for $\hat{X}$, we see that B is m-copolar. Finally, as in the second sentence of this proof, we see that $\{Z_- \in B\}$ is P evanescent. Consequently the assertions (i) through (v) are equivalent.

It is evident from (15.7) that the hitting operators $P_B^q(x, \cdot)$, $P_{B-}^q(x, \cdot)$ are identical for (m) a.a. x. Therefore the switching identity (11.4) now takes the form

(15.16) $$(P_B^q U^q f, g) = (f, \hat{P}_B^q \hat{U}^q g)$$

with $q \geqq 0$, f and g in $\mathscr{E}_+^*$.

The results of §12 and §13 are considerably simpler under m-standardness of X and $\hat{X}$. It is clear from (15.7) that one has the identifications

(15.17) $$P^m\{T_B \neq S_B\} = P^m\{L_B \neq M_B\} = P\{\tau_B \neq \sigma_B\} = P\{\lambda_B \neq \mu_B\} = 0.$$

It follows that, using the terminology of (13.3), strong m-transience is identical to strong left m-transience. In addition, the capacitary measures π_B, π_B^- are equal for every $B \in \mathscr{E}$, so $c(B) = c^-(B)$, and the corresponding dual equalities likewise obtain. In particular, (13.15) now has the following form.

(15.18) **Theorem.** *Let X and $\hat{X}$ be m-standard, and let $B \in \mathscr{E}$ be strongly m-transient for $\hat{X}$. Then*

(15.19) $$t\hat{c}(B) = t \int \hat{\pi}_B(dx) P^x\{L_B > 0\} + P\{\sigma_B = \mu_B \in]0, t[\}.$$

The last term in (15.19) is also equal to $P\{\tau_B = \lambda_B \in]0, t[\}$, and this is just the P measure of those paths w which enter B exactly once, the encounter happening during $]0, t[$.

Finally, (13.20) now states, assuming X, $\hat{X}$ both m-standard,

(15.20) **Theorem.** *If $B \in \mathscr{E}$ is strongly m-transient for both X and $\hat{X}$, then $\hat{c}(B) = c(B)$.*

This result is well known [4, VI–4.4] in the case of classical duality, this being one of Hunt's original results in the subject.

16. Structure Theorems

Unless stated otherwise, X and $\hat{X}$ will be supposed in weak duality relative to m. In this situation one may describe rather precisely the structure of certain homogeneous functionals of X, up to P^m evanescence. Recall [39, §24] that $\mathscr{H}^g$, $\mathscr{H}^d$ denote the σ-algebras of processes generated by $\mathscr{I}$ and by processes Y which are *lcrl* (resp. *rcll*) and perfectly homogeneous on $\mathbb{R}^{++}$ (resp. $\mathbb{R}^+$). The latter conditions mean of course $Y_{t+s} = Y_t \circ \theta_s$ for all $t > 0$, $s \geqq 0$ (resp. all $t \geqq 0$, $s \geqq 0$). It is known [39, 24.28] for example, that assuming only that X is a right process, $\mathscr{O} \cap \mathscr{H}^d = \mathscr{X}^e \vee \mathscr{I}$, $\mathscr{X}^e$ denoting $\sigma\{f(X): f \in \mathscr{E}^e\}$. and that the trace of $\mathscr{H}^d$ on $]\!]0, \infty[\![$ is contained in $\mathscr{H}^g$. Similarly, $\mathscr{X}_-^e$ denoting $\sigma\{f(X)_-: f\, q\text{-excessive for some } q \geqq 0\}$, $\mathscr{P} \cap \mathscr{H}^g = \mathscr{X}_-^e \vee \mathscr{I}$ [39, §24]. In addition, $\mathscr{O} \cap \mathscr{H}^g = (\mathscr{P} \cap \mathscr{H}^g) \vee \mathscr{X}^e$ [39, §45]. Let $\mathscr{I}^m$ denote the P^m evanescent sets.

The central result of this section is a characterization of the trace of $\mathscr{X}_-^e$ on $]\!]0, \zeta[\![$, modulo $\mathscr{I}^m$, as $\sigma\{f(X_-): f \in \mathscr{E}\}$. See (16.4).

(16.1) **Lemma.** *Let $Y: \mathbb{R}^+ \times \Omega \to \mathbb{R}$ be a bounded measurable process satisfying*

(i) *Y is adapted to $(\mathscr{F}_t^m)$;*
(ii) *P^m a.s., $Y_{t+s} = Y_t \circ \theta_s$ for all $t, s \geqq 0$;*
(iii) *$t \to Y_t$ is P^m a.s. rcll.*

Then there exists $f \in b\,\mathscr{E}$ such that Y and $f(X)$ are P^m-indistinguishable.

Proof. Let $g(x) = E^x Y_0 \in \mathscr{E}^*$. Then

$$P^m\{Y_t \neq g(X_t)\} = P^m\{\{Y_0 \neq g(X_0)\} \circ \theta_t\} \leqq P^m\{Y_0 \neq g(X_0)\} = 0.$$

Using Fubini's theorem we conclude that for m a.a. x,

$$(16.2)\qquad E^x \int_0^\infty q e^{-qt} Y_t\, dt = E^x \int_0^\infty q e^{-qt} g(X_t)\, dt = q U^q g(x).$$

From (ii) we see that, up to P^m evanescence in the variable s,

$$(16.3)\qquad \int_0^\infty q e^{-qt} Y_{t+s}\, dt = \left(\int_0^\infty q e^{-qt} Y_t\, dt\right) \circ \theta_s.$$

Now set $f(x) \equiv \limsup_{n\to\infty} n U^n g(x) \in \mathscr{E}^e$. As $q \to \infty$ through integral values, the left side of (16.3) tends boundedly to Y_s. On the other hand, (16.2) shows that the optional projection of the right side of (16.3) is $qU^q g(X_s)$. It follows that $f(X)$ is a P^m-optional projection of Y, and $f \in \mathscr{E}^e$. Because Y is P^m optional, the result follows after taking (6.6) into account.

(16.4) **Theorem.** *For every $Y \in \mathscr{P} \cap \mathscr{H}^g$ there exists a Borel function g such that Y and $g(X_-)$ are P^m indistinguishable on $]\!]0, \zeta[\![$.*

Proof. It is well known, [39, 8.7] for example, that a product of two bounded q-excessive functions may be expressed as a difference of two q-excessive functions. Therefore the generators $f(X)_-$ of $\mathscr{X}^e_-$ described above (16.1) have differences forming an algebra, and consequently the monotone class theorem shows that it is enough to prove the above representation for Y of the form $f(X)_-$ with f bounded and q-excessive. In view of (6.11) we may assume $f \in \mathscr{E}$. Seeing that $t \to f(X_t)_-$ is a.s. *lcrl* on $]0, \infty[$, it follows that $t \to f(Z_t)_-$ is P-a.s. *lcrl* on $]\alpha, \beta[$. Fix w with $t \to f(Z_t(w))_-$ *lcrl* and set $\varphi(t) = f(Z_t(w))_-$. It is clear that $t \to \varphi(-t)$ is then *rcll* on $]\hat\alpha(w), \hat\beta(w)[$. However,

$$\varphi(-t) = \lim_{s\uparrow\uparrow -t} f(Z_s(w)) = \lim_{s\uparrow\uparrow -t} f(\hat Z_{(-s)-}(w)) = \lim_{u\downarrow\downarrow t} f(\hat Z_{u-}(w)).$$

This proves that $t \to \lim_{u\downarrow\downarrow t} f(\hat X_{u-})$ is $\hat P^m$ a.s. *rcll* on $]0, \hat\zeta[$. The process $V_t \equiv \limsup_{u\downarrow\downarrow t} f(\hat X_{u-})$ clearly satisfies the hypotheses of (16.1) relative to $\hat X$ and therefore V_t is $\hat P^m$ indistinguishable from $g(\hat X_t)$ for some $g \in \mathscr{E}$. This in turn leads to $g(\hat Z_t)$ being P-indistinguishable from $\lim_{u\downarrow\downarrow t} f(\hat Z_{u-})$ and so, reversing t again, $g(Z_{t-})$ is P indistinguishable from $f(Z_t)_-$. This shows that $[f(X_t)_- - g(X_{t-})]\, 1_{]\!]0,\zeta[\![} \in \mathscr{I}^m$, completing the proof.

Theorem 16.4 is a version, appropriate to the present hypotheses, of a result of Azéma [3, §6]. Note the following corollary of (16.4), the proof of which is immediate from the remarks preceding (16.1).

(16.5) **Theorem.** *Given $Y \in \mathscr{O} \cap \mathscr{H}^g$ there exists $f \in \mathscr{E} \times \mathscr{E}$ such that Y and $f(X_-, X)$ are P^m-indistinguishable on $]\!]0, \zeta[\![$.*

A more satisfying way to state (16.4) uses the natural σ-algebra.

(16.6) **Theorem.** *$\mathcal{N} \cap \mathcal{H}^g$ is generated by X_- (on $]\!]0, \zeta[\![$), up to P^m evanescent sets.*

Proof. We need only show that if $Y \in \mathcal{N} \cap \mathcal{H}^g$ then there exists $V \in \mathcal{P} \cap \mathcal{H}^g$ with $Y = V 1_{]\!]0,\zeta[\![}$. Recall that $Y \in \mathcal{H}^g$ implies ${}^pY \in \mathcal{H}^g$ [39, §24] so, using the remarks on the process l_t at the end of §2, it follows that $V_t \equiv {}^pY_t / l_t$ belongs to $\mathcal{P} \cap \mathcal{H}^g$. However, $Y \in \mathcal{N}$ gives us $Y = {}^nY = V 1_{]\!]0,\zeta[\![}$, proving (16.6).

Recall that if κ is a random measure then $\kappa\{t\}$ is the mass of κ at the singleton t.

(16.7) **Proposition.** *Let κ be a HRM with q-potential function u and let T denote the hitting time of the absorbing set $\{u < \infty\}$. Then if κ is natural* (resp., *optional*) *the process $\kappa\{t\} 1_{]T,\infty[}(t)$ is in $\mathcal{N} \cap \mathcal{H}^g$* (resp., $\mathcal{O} \cap \mathcal{H}^g$).

Proof. We shall suppose κ natural, the optional case being quite analogous. By the definition of κ being natural in §4, it is clear that $\kappa\{t\} \in \mathcal{N}$. It is also clear that $1_{]T,\infty[} \in \mathcal{P}$ and, since $\{u < \infty\}$ is absorbing, $1_{]T,\infty[} \in \mathcal{H}^g$. Define now $H = \int_0^\infty e^{-qs} \kappa(ds)$. By additivity of κ,

$$H \circ \theta_t = e^{qt} \int_{]t,\infty[} e^{-qs} \kappa(ds).$$

By Lebesgue's dominated convergence theorem, $t \to V_t \equiv H(\theta_t \omega)$ is *rcll* on $[r, \infty[$ whenever $V_r(\omega) < \infty$. For every $\varepsilon > 0$, $u(X_{T+\varepsilon}) < \infty$ a.s. on $\{T < \infty\}$ and consequently $t \to V_t$ is a.s. *rcll* on $[T + \varepsilon, \infty[$. As $\varepsilon > 0$ is arbitrary this proves that $t \to V_{t-}$ is a.s. *lcrl* on $]T, \infty[$. It follows, using homogeneity of V_{t-}, that $1_{]T,\infty[}(t) V_{t-} \in \mathcal{H}^g$. However,

$$1_{]T,\infty[}(t) V_t = \lim_{\varepsilon \downarrow\downarrow 0} 1_{]T,\infty[}(t) V_{(t+\varepsilon)-}$$

is then also in $\mathcal{H}^g$ and consequently $1_{]T,\infty[}(t)(V_t - V_{t-}) \in \mathcal{H}^g$. Because $V_{t-} = e^{qt} \int_{[t,\infty[} e^{-qs} \kappa(ds)$ if $t > T$, this proves that $1_{]T,\infty[}(t) e^{qt} e^{-qt} \kappa\{t\} \in \mathcal{H}^g$. Combining this with the first observations in the proof yields $1_{]T,\infty[}(t) \kappa\{t\} \in \mathcal{N} \cap \mathcal{H}^g$, as claimed.

(16.8) **Theorem.** *Let κ be a σ-integrable NHRM. Then*

(i) *there exists $g \in \mathcal{E}_+$ vanishing off an m-semipolar set such that $\kappa\{t\} = g(X_{t-})$ up to P^m evanescence on $]\!]0, \zeta[\![$.*

(ii) *κ is P^m* a.s. *diffuse if and only if ν_κ doesn't charge m-semipolar sets.*

(iii) *κ is P^m* a.s. *purely discontinuous if and only if ν_κ is carried by an m-semipolar set.*

Proof. We may assume, without loss of generality, that κ is integrable. Then $u_\kappa^1 < \infty$ a.e. (m) and so, if T denotes the hitting time of $\{u_\kappa^1 < \infty\}$, $P^m(T > 0) = 0$. By (16.7), $\kappa(t\} 1_{]T,\infty[}(t) \in \mathcal{N} \cap \mathcal{H}^g$ and so (16.6) gives us $g \in \mathcal{E}^+$ with $g(X_{t-})$ and $\kappa(t) 1_{]T,\infty[}(t)$ being P^m indistinguishable on $]\!]0, \zeta[\![$. Since $P^m(T > 0) = 0$, this establishes (i). It is evident that $\{t > 0 : g(X_{t-}) > 0\}$ is P^m a.s. countable, so by (6.7) and the subsequent remarks, $\{g > 0\}$ is m-semipolar. Define κ^d to be the discrete part of κ, so that $\kappa^d(B) = \sum_{t>0} \kappa\{t\} 1_B(t)$. Clearly ν_{κ^d} is carried by $\{g > 0\}$. From this and the fact (9.10) that ν_κ determines κ up to P^m evanescence, (ii) and (iii) are clear.

(16.9) **Theorem.** *Let κ be a σ-integrable optional HRM over X. Then there exists $f \in (\mathscr{E} \times \mathscr{E})_+$ with $\{x\colon f(x,x) > 0\}$ m-semipolar such that $\kappa\{t\}$ is P^m-indistinguishable from $f(X_{t-}, X_t) 1_{]0,\zeta[}(t)$.*

Proof. It may be assumed that κ is integrable and so $u^1_\kappa < \infty$ a.e. (m). Then (16.7) gives $\kappa\{t\} 1_{]T,\infty[}(t) \in \mathcal{O} \cap \mathscr{H}^g$, T being the hitting time of $\{u^1_\kappa < \infty\}$. As $P^m\{T > 0\} = 0$, the assertion comes directly from (16.5).

Theorem (16.8) has a partial converse.

(16.10) **Proposition.** *Let $F \in \mathscr{E}$ be m-semipolar. Then there exists a perfect integrable NHRM, κ, such that $\{X_- \in F\} \cap]\!]0, \zeta[\![$ and $\{t\colon \kappa\{t\} > 0\}$ are P^m-indistinguishable.*

Proof. We know (11.2) that F is also m-cosemipolar and so, by the dual of (6.13), there exist disjoint totally cothin Borel sets F_n with $F - (\cup F_n)$ m-copolar. In view of (11.1) it suffices to produce, for a fixed n, an integrable perfect NHRM, κ, such that $\{X_- \in F_n\} \cap]\!]0, \zeta[\![= \{t\colon \kappa\{t\} > 0\}$ up to P^m evanescence. Fix n and set $G = F_n$. As G is totally cothin, $\hat{A}_t \equiv \sum_{0<s\leq t} 1_G(\hat{X}_s)$ defines an AF of $\hat{X}$ having bounded q-potential for some $q > 0$ and having uniformly bounded jumps. Therefore $\hat{A}$ is σ integrable over $\hat{X}$ by [36, 1.3]. Hence, we may find a strictly positive $f \in \mathscr{E}$ such that $\hat{B} \equiv f_- * \hat{A}$ is integrable over $\hat{X}$. Let K be the HRM of Z generated by $\hat{B}$, so that for P a.e. w,

$$K(w, dt) = \sum_{\alpha(w) < s < \beta(w)} f(Z_s(w)) 1_G(Z_{s-}(w)) \varepsilon_s(dt).$$

Because $\hat{B}$ is integrable, $PK]0,1] < \infty$ by (12.11). Let $\Lambda = \{\omega \in \Omega\colon X_{s-}(\omega) \in G$ only countably often$\}$. Then $\Lambda \in \mathscr{F}$ because of the measurability of penetration times [9, VI–D22]. By (11.2), $P^m(\Lambda^c) = 0$. Let $\psi(x) = P^x(\Lambda^c)$ so that $m\{\psi > 0\} = 0$. Clearly ψ is excessive and hence $\{\psi = 0\}$ is absorbing. Let $R = \inf\{t\colon \psi(X_t) = 0\}$ and set

$$\gamma(\omega, dt) = \sum_{0<s<\infty} 1_{]R,\zeta[}(s) f(X_s(\omega)) 1_G(X_{s-}) \varepsilon_s(dt).$$

By the definition of Λ, $X_{s-} \in G$ only countably often for $s > R$, and so γ is a random measure which is σ-integrable on $\mathscr{N}$, as discussed in §4. In particular, γ has a dual predictable projection κ. Seeing that $\{\psi = 0\}$ is absorbing, γ is in fact a perfect optional HRM over X and therefore κ is a NHRM over X. Because γ generates K over Z, (12.11) gives us $\nu_\gamma(E) = PK]0,1] < \infty$ and therefore $\nu_\kappa(E) = \nu_\gamma(E) < \infty$. By (8.2), $u^1_\kappa < \infty$ a.e. $[m]$ so (7.4) shows that κ may be assumed perfect. By construction, $\{t\colon \gamma\{t\} > 0\} = \{X_- \in G\} \cap]\!]0, \zeta[\![$, up to P^m evanescence. It follows that $\{t\colon \kappa\{t\} > 0\}$ is also P^m indistinguishable from $\{X_- \in G\} \cap]\!]0, \zeta[\![$, since this set is in $\mathscr{N}$.

We examine next the connection between optional homogeneous closed random subsets of $]0, \infty[$ and exact terminal times. By a homogeneous random set in $]0, \infty[$ is meant a measurable subset M of $]0, \infty[\times \Omega$ such that for a.a. ω, $s + t \in M(\omega)$ if and only if $s \in M(\theta_t \omega)$ for every pair $s > 0$, $t \geqq 0$. Given an optional homogeneous closed random set M in $]0, \infty[$, its debut $T \equiv T_M \equiv \inf\{t > 0\colon t \in M\}$ is an exact terminal time for X. See [18] for a fairly complete discussion of the connections between M and T. In particular, M is the closure in $]0, \infty[$ of $\{t + T \circ \theta_t\colon t > 0\}$. If we start with an exact terminal time T then the closure M of

$\{t+T\circ\theta_t\colon\ t>0\}$ is an optional homogeneous random set, and if we set $D_t \equiv \lim_{s\uparrow\uparrow t}(s+T\circ\theta_s)$ then $t\to D_t - t$ is the left limit of $t\to T\circ\theta_t$, which is a.s. *rcll*, hence in $\mathscr{H}^d$. Therefore $D_t - t\in\mathscr{H}^g$. As is shown in [18], $M=\{t\colon D_t=t\}$, and consequently $M\in\mathcal{O}\cap\mathscr{H}^g$.

It is easy to see that M is the closure in $\mathbb{R}^{++}$ of $\{t+T\circ\theta_t\colon t>0, \text{rational}\}$. It follows from this remark that if $T^1=T^2$ a.s. (resp., a.s. P^m) then the associated optional homogeneous closed random sets M^1, M^2 are indistinguishable (resp., P^m indistinguishable).

The following result is valid for an arbitrary right process.

(16.12) **Lemma.** *Let T be a thin natural terminal time with M the associated optional, closed, homogeneous random set. Let $u(x)=E^x e^{-T\wedge\zeta}$. Then $M\cap\,]\!]0,\zeta[\![$ is indistinguishable from $\{u(X)_-=1\}\cap\,]\!]0,\zeta[\![$, which a.s. contains no strictly decreasing sequence.*

Proof. It is evident that u is 1-excessive and $u(x)<1$ for all $x\in E$. The set $\{u(X)_-=1\}$ is obviously left closed. In addition, if $t_n\downarrow\downarrow t$ with $u(X_{t_n})_-=1$ we would obtain the absurdity $u(X_t)=1$. That is, $\{u(X)_-=1\}$ a.s. does not contain any strictly decreasing sequence. These observations prove that $\{u(X)_-=1\}$ is a closed, optional, homogeneous random set. Let S denote its debut. It follows from (3.1-ii) that $S\wedge\zeta=T\wedge\zeta$ a.s., and this implies the stated result.

Going back now to weak duality hypotheses we obtain the following consequence of (16.12) and (16.6).

(16.13) **Theorem.** *Let T be a thin natural terminal time. Then there exists $B\in\mathscr{E}$ with B m-semipolar such that $M\cap\,]\!]0,\zeta[\![$ and $\{X_-\in B\}\cap\,]\!]0,\zeta[\![$ are P^m indistinguishable. In particular $T\wedge\zeta=S_B\wedge\zeta$ a.s. P^m.*

In the same way, the observations prior to (16.12) lead to the following result, which uses (16.5) rather than (16.6).

(16.14) **Theorem.** *Let T be an exact terminal time for X. Then there exists $B\in\mathscr{E}\times\mathscr{E}$ such that, P^m a.s., $T\wedge\zeta=\inf\{t\colon 0<t<\zeta,\,(X_{t-},X_t)\in B\}\wedge\zeta$.*

Another important consequence of (16.6) is a converse to (15.2), mentioned already following (15.3).

(16.15) **Theorem.** *Suppose that for every m-semipolar set $B\in\mathscr{E}$ the inclusion* (16.16) (resp. (16.17)) *holds up to P^m-evanescence*:

(16.16) $$\{X_-\in B\}\cap\,]\!]0,\zeta[\![\,\subset\{X=X_-\};$$

(16.17) $$\{X\in B\}\subset\{X=X_-\}.$$

Then X (resp., $\hat{X}$) is m-standard.

Proof. Assume first that (16.16) holds for every m-semipolar set $B\in\mathscr{E}$. Let X^r_- denote the left limit of X in a Ray compactification of E_Δ. Because $X^r_-\in\mathscr{P}\cap\mathscr{H}^g$, one has $\{X^r_-\notin E_\Delta\}\cap\,]\!]0,\zeta[\![\,\in\mathscr{N}\cap\mathscr{H}^g$ and so, by (16.6) there exists an m-semipolar set $B\in\mathscr{E}$ with $\{X^r_-\notin E_\Delta\}\cap\,]\!]0,\zeta[\![\,=\{X_-\in B\}\cap\,]\!]0,\zeta[\![$ up to P^m evanescence. By hypothesis, this gives $\{X^r_-\notin E_\Delta\}\cap\,]\!]0,\zeta[\![\,\subset\{X=X_-\}$ up to P^m evanescence. The set $\{X=X^r_-, X\neq X_-\}\cap\,]\!]0,\zeta[\![$ may be expressed in an obvious way as a countable union of graphs of stopping times, and as every stopping time with graph in $\{X=X^r_-\}$ is predictable, the set $\{X=X^r_-, X \neq X_-\} \cap]\!]0,\zeta[\![$ is in $\mathcal{N} \cap \mathcal{H}^g$. It follows, again from (16.6), that there exists an m-semipolar set $B \in \mathscr{E}$ with $\{X=X^r_-, X \neq X_-\} \cap]\!]0,\zeta[\![= \{X_- \in B\}$ up to P^m-evanescence. The hypothesis $\{X_- \in B\} \subset \{X = X_-\}$ can be satisfied only if $\{X = X^r_-, X \neq X_-\} \cap]\!]0,\zeta[\![$ is P^m-evanescent. This proves $\{X = X^r_-\} \cap]\!]0,\zeta[\![\subset \{X_- = X\} \cap]\!]0,\zeta[\![$ up to P^m evanescence. We have now proved that X satisfies the criterion (5.5-iii), and consequently X is m-standard. Assume next that (16.17) holds. Then $\{Z \in B\} \subset \{Z = Z_-\}$ up to P-evanescence. Time reversal makes the last inclusion equivalent to $\{\hat{Z}_- \in B\} \subset \{\hat{Z} = \hat{Z}_-\}$ up to P-evanescence, and this in turn implies $\{\hat{X}_- \in B\} \cap]\!]0,\hat{\zeta}[\![\subset \{\hat{X} = \hat{X}_-\}$ up to $\hat{P}^m$-evanescence. The first part of the proof above shows shows now that $\hat{X}$ is m-standard.

Before stating the final results of this section we remind the reader that a Borel right process X is defined to be *special* if $T_n \uparrow T$ (stopping times) implies $\mathscr{F}^\mu_T = \bigvee_n \mathscr{F}^\mu_{T_n}$ for every initial law μ. We shall call X *μ-special* if $\mathscr{F}^\mu_T = \bigvee_n \mathscr{F}^\mu_{T_n}$ whenever $T_n \uparrow T$. It is shown in [45, 13] and in [16, 13.2] that X is μ-special if and only if $P^\mu\{X^r_{t-} \notin E \cup \Delta$ for some $t > 0\} = 0$, X^r_{t-} denoting the Ray left limit of X in some Ray compactification of $E \cup \Delta$. If we define $\varphi(x) = P^x\{X^r_{t-} \notin E \cup \Delta$ for some $t > 0\}$, then φ is obviously excessive, so if X is m-special, $\{\varphi > 0\}$ is m-inessential. That is, we may delete a Borel m-inessential set from E so that the restricted process X' is in fact special, and not just m-special. Recall too that a special process can also be described as a right process which is a Hunt process in its Ray topology (see [45] or [16, 13.3]).

Recalling from §5 the meaning of μ-standardness, we shall say that X is μ-special standard provided it is both μ-special and μ-standard.

The next lemma requires no duality, and is valid for an arbitrary right process X and an arbitrary initial measure μ. As usual, X^r_- will denote the Ray left limit of X in a Ray compactification of E_Δ.

(16.18) **Lemma.** *X is μ-special standard if and only if X_- exists in E a.s. P^μ on $]\!]0,\zeta[\![$, and X^r_-, X_- are P^μ indistinguishable on $]\!]0,\zeta[\![$. The latter condition is equivalent to the P^μ indistinguishability of X^r_-, X_- on $]\!]0,\zeta[\![\cap (\{X^r_- = X\} \cup \{X^r_- \notin E_\Delta\})$.*

Proof. The last sentence follows from the fact (16, 13.4) that $X^r_- = X_-$ on $\{X^r_- \in E_\Delta, X^r_- \neq X\}$. If X is μ-special then, as remarked above, $X^r_- \in E_\Delta$ a.s. on $]0,\zeta[$. Therefore, if one knows that X is μ-special, the necessity and sufficiency of the conditions in (16.18) is a direct consequence of (5.5), using the criterion (5.5-iii). Thus it suffices to prove that the conditions of the lemma imply that X is μ-special. Let $S = \inf\{t > 0\colon X^r_{t-} \notin E \cup \Delta\}$. Because $X^r_{t-} = \Delta$ for all $t > \zeta$, the hypothesis on X^r_- and X_- implies that $\{t > 0\colon X^r_{t-} \notin E \cup \Delta\} \subset [\![\zeta]\!]$ up to P^μ-evanescence. In particular, $\{X^r_- \notin E \cup \Delta\}$ P^μ a.s. contains its debut and it follows from [9, IV–T16] that S is P^μ-predictible. As $1_\Delta(X_S) = 1$ a.s. P^μ on $\{S < \infty\}$, taking predictable projections leads to $\bar{P}_0(X^r_{S-}, \Delta) = 1$ a.s. P^μ on $\{S < \infty\}$. It is known [45, §2] that there are no degenerate branch points, that is, $\bar{P}_0(x, \cdot) = \varepsilon_y$ implies $x = y$. Therefore $X^r_{S-} = \Delta$ a.s. P^μ on $\{S < \infty\}$. This proves that $X^r_{t-} \in E_\Delta$ for all $t > 0$, a.s. P^μ, completing the proof.

The following result requires no duality, but is of greatest interest under duality hypotheses because of the subsequent result (16.21).

(16.19) **Theorem.** *Suppose X_- exists in E a.s. P^μ on $]\!]0, \zeta[\![$. Suppose also that every NAF of X is P^μ a.s. continuous. Then X is μ-special standard.*

Proof. According to (5.7) there exists a NAF with bounded 1-potential such that $\{\Delta A > 0\} = \{X^r_- \notin E_\Delta\} \cap]\!]0, \zeta[\![$. In view of our hypothesis, this implies that $\{X^r_- \notin E_\Delta\} \cap]\!]0, \zeta[\![$ is P^μ evanescent. Using now (16.18) we see that X will be μ-special standard once we prove that $\{X^r_- = X\} \cap]\!]0, \zeta[\![\subset \{X_- = X\} \cap]\!]0, \zeta[\![$ up to P^μ-evanescence. Fix a metric d for E, and for $\varepsilon > 0$, let

$$A_t = \sum_{0 < s \leqq t} 1_{\{X^r_{s-} = X_s\}} 1_{\{d(X_{s-}, X_s) \geqq \varepsilon\}} 1_{\{s < \zeta\}}.$$

As every stopping time with graph in $\{X^r_- = X\}$ is predictable [16, 13.1] it is clear that A is a NAF of X. By hypothesis, A vanishes a.s. P^μ. It follows that $\{X^r_- = X, d(X_-, X) > 0\} \cap]\!]0, \zeta[\![$ is P^μ-evanescent, and this establishes that X is μ-special standard..

We return now to weak duality hypotheses. We seek to establish a number of equivalents to the condition that every NAF of X is P^m a.s. continuous.

In what follows, if $F \in \mathscr{E}$, let reg$(F) \equiv \{x \in E: P^x\{T_F = 0\} = 1\}$ denote the set of regular points for F, with coreg (F) the coregular points for F. The following is the weak duality version of one of the basic facts [4, VI (1.25)] concerning classical duality.

(16.20) **Proposition.** *Let $F \in \mathscr{E}$. Then the fine and cofine closures of F differ by an m-semipolar set.*

Proof. The fine closure $F \cup \text{reg}(F)$ of F has the property that $\{X \in F \cup \text{reg}(F)\}$ is a.s. the right closure in $[0, \zeta[$ of $\{X \in F\}$. As reg$(F) \in \mathscr{E}^e$ we may choose by (6.6) $G \in \mathscr{E}$ such that $(F \cup \text{reg}(F)) \,\Delta\, G$ is both m-polar and left m-polar. Then $\{Z \in G\}$ is P a.s. the right closure in $]\alpha, \beta[$ of $\{Z \in F\}$. In view of (11.1), the dual of this fact is that we may choose $H \in \mathscr{E}$ such that $(F \cup \text{coreg}(F)) \,\Delta\, H$ is both m-polar and left m-polar, so that $\{Z_- \in H\}$ is P a.s. the left closure in $]\alpha, \beta[$ of $\{Z_- \in F\}$. It is now evident that $\{Z \in G\} \,\Delta\, \{Z \in H\}$ is P a.s. countable. The conclusion of (16.20) then comes from (6.7).

(16.21) **Theorem.** *The following conditions on X are equivalent.*

(i) *Every m-semipolar set is left m-polar (= m-copolar).*
(ii) *If $F \in \mathscr{E}$ then the fine and cofine closures of F differ by a left m-polar set.*
(iii) *If T is a thin natural terminal time for X then $P^m\{T < \zeta\} = 0$.*
(iv) *Every NAF of X is P^m a.s. continuous.*
(v) *Every σ-integrable NHRM of X is P^m a.s. diffuse.*

Proof. We shall prove that (1) ⇒ (ii) ⇒ (iii) ⇒ (iv) ⇒ (i) and (i) ⇒ (v) ⇒ (iv). That (i) ⇒ (ii) is obvious from (16.20). Assume now that (ii) holds, and let T be a thin natural terminal time. As in (3.1), set $D_n \equiv \{x: E^x e^{-T \wedge \zeta} \geqq 1 - 1/n\}$ so that each D_n is finely closed. By (ii), D_n differs from a cofinely closed set by an m-copolar set, and consequently, as noted in the proof of (16.20), $\{X_- \in D_n\}$ is P^m a.s. left closed in $]\!]0, \zeta[\![$. However, we observed in (3.1) that $\{X \in D_n\}$ includes an interval $]T-\varepsilon, T[$ provided $T < \zeta$, and so $T \in \cap \{X_- \in D_n\}$ P^m a.s. on $\{0 < T < \zeta\}$. That is $E^{X(T-)} e^{-T\wedge\zeta} = 1$ a.s. P^m on $\{0 < T < \zeta\}$. Because $E^x e^{-T\wedge\zeta} < 1$ on E this implies $P^m\{T < \zeta\} = 0$. That is, (ii)$\Leftrightarrow$(iii). Now suppose (iii) holds, and let A be a NAF for X. Then for any $\varepsilon > 0$, $T \equiv \inf\{t: \Delta A_t \geqq \varepsilon\}$ is a thin natural terminal time, so by (iii), $P^m\{T < \zeta\} = 0$. Thus (iii) $\Rightarrow$ (iv). Assume now that (iv) holds, and let $F \in \mathscr{E}$ be m-semipolar. If F is in fact m-polar then because (iv) implies by (16.19) that X is m-special standard, (15.2) shows that F is left m-polar. Suppose next that F is totally thin. Once again using m-standardness of X, (15.3) gives us $\{X_- \in F\} \subset \{X_- = X\}$ up to P^m evanescene. However, since X is m-special standard, (16.18) shows that X_- and X^r_- are P^m indistinguishable on $]\!]0, \zeta[\![$. It follows that, up to P^m evanescence,

$$\{X_- \in F\} \subset \{X_- \in F\} \cap \{X^r_- = X\} \cap]\!]0, \zeta[\![.$$

As every stopping time with graph in $\{X^r_- = X\}$ is predictable,

$$A_t \equiv \sum_{0<s\leqq t} 1_F(X_{s-})\, 1_{\{X^r_{s-} = X_s\}}$$

is a NAF over X. According to (iv), A is P^m evanescent, so $\{X_- \in F\}$ is likewise P^m-evanescent. This proves that F is left m-polar whenever F is m-semipolar, assuming (iv). It is obvious from (16.8) that (i) $\Rightarrow$ (v), and (v) $\Rightarrow$ (iv) is evident, completing the proof of (16.21).

(16.22) *Remark.* In light of (16.19), each of the conditions in (16.21) implies that X is m-special standard.

References

1. Atkinson, B.: Generalized strong Markov properties and applications. Z. Wahrscheinlichkeitstheorie Verw. Geb. **60**, 71–78 (1982)
2. Atkinson, B., Mitro, J.: Applications of Revuz and Palm type measures for additive functionals in weak duality. Seminar on Stochastic Processes 1982, 23–50. Boston: Birkhäuser 1983
3. Azéma, J.: Théorie générale des processus et retournement du temps. Ann. Sci. de l'Ecole Norm. Sup. **6**, 459–519 (1973)
4. Blumenthal, R.M., Getoor, R.K.: Markov Processes and Potential Theory. New York: Academic Press 1968
5. Blumenthal, R.M., Getoor, R.K.: Dual processes and potential theory. Proc. 12th Bienial Seminar of the Canadian Math. Congress. Canadian Math. Soc. (1970)
6. Chung, K.L.: Probability approach to the equilibrium problem in potential theory. Ann. Inst. Fourier **23**, 313–322 (1973)
7. Chung, K.L., Walsh, J.B.: Meyer's theorem on predictability. Z. Wahrscheinlichkeitstheorie Verw. Geb. **29**, 253–256 (1974)
8. Cinlar, E., Jacod, J., Protter, P., Sharpe, M.: Semimartingales and Markov processes. Z. Wahrscheinlichkeitstheorie verw. Geb. **54**, 161–219 (1980)
9. Dellacherie, C.: Capacités et Processes Stochastiques. Berlin-Heidelberg-New York: Springer 1972
10. Dellacherie, C., Meyer, P.-A.: Probabilités et Potentiel. Paris: Herman 1975
11. Dynkin, E.B.: Green's and Dirichlet spaces for a symmetric Markov transition function. Lecture Notes Lond. Math. Soc. 1982
12. Dynkin, E.B.: Green's and Dirichlet spaces associated with fine Markov processes. J. Funct. Anal. **47**, 381–418 (1982).

13. Fukushima, M.: Dirichlet Forms and Markov Processes. Amsterdam: North Holland 1980
14. Getoor, R.K.: Multiplicative functionals of dual processes. Ann. Inst. Fourier **21**, 43–83 (1971)
15. Getoor, R.K.: Duality of Lévy Systems, Z. Wahrscheinlichkeitstheorie verw. Geb. **19**, 257–270 (1971)
16. Getoor, R.K.: Markov Processes: Ray Processes and Right Processes. Springer Lecture Notes 440. Berlin- Heidelberg-New York: Springer-Verlag 1975
17. Getoor, R.K., Sharpe, M.J.: Last exit times and additive functionals. Ann. Probability **1**, 550–569 (1973)
18. Getoor, R.K., Sharpe, M.J.: Balayage and multiplicative functionals. Z. Wahrscheinlichkeitstheorie verw. Geb. **28**, 139–164 (1974)
19. Getoor, R.K., Sharpe, M.J.: Excursions of dual processes. Advances in Math. **45**, 259–309 (1982)
20. Getoor, R.K., Sharpe, M.J.: Two results on dual excursions. In Seminar on Stochastic Processes. 31–52. Boston-Basel-Stuttgart: Birkhäuser 1982
21. Glover, J.: Compactifications for dual processes. Ann. Probability **8**, 1119–1135 (1980)
22. Hunt, G.A.: Markov processes and potentials I, II, III. Ill. J. Math. **1**, 44–93 and 316–369 (1957), **2**, 151–213 (1958)
23. Kuznetsov, S.E.: Construction of a regular split process. Theor. Probability Appl. **22**, 773–793 (1977)
24. Le Jan, Y.: Dual Markovian semigroups and processes. In Functional Analysis in Markov Processes. Springer Lecture Notes 923, pp. 47–75. Berlin-Heidelberg-New York: Springer 1982
25. Meyer, P.A.: Fonctionelles multiplicatives et additives de Markov. Ann. Inst. Fourier **12**, 125–230 (1962)
26. Meyer, P.A.: Processus de Markov. Lecture Notes in Mathematics 26. Berlin-Heidelberg-New York: Springer 1967
27. Meyer, P.A.: Martingales locales fonctionnelles additives I. Lecture Notes in Math. **649**, 775–785. Berlin-Heidelberg-New York: Springer-Verlag 1978
28. Meyer, P.A.: Note sur l'interpretation des mesures d'équilibre. Lecture Notes in Math. **321**, Heidelberg: Springer-Verlag 1973
29. Mitro, J.B.: Dual Markov processes: construction of a useful auxiliary process, Z. Wahrscheinlichkeitstheorie verw. Geb. **47**, 139–156 (1979)
30. Mitro, J.B.: Dual Markov processes: applications of a useful auxiliary process. Z. Wahrscheinlichkeitstheorie verw. Geb. **48**, 97–114 (1979)
31. Mitro, J.: Exit systems for dual Markov processes. Z. Wahrscheinlichkeitstheorie verw. Gebiete **66**, 259–267 (1984)
32. Nagasawa, M.: Time reversions of Markov processes. Nagoya Math. J. **24**, 177–204 (1964)
33. Pitman, J.W.: Lévy systems and path decompositions. In Seminar on Stochastic Processes. 79–110. Boston-Basel-Stuttgart: Birkhäuser 1982
34. Port, S., Stone, C.: Infinitely divisible processes and their potential theory. Ann. Inst. Fourier **21**, (2), 157–275 and (4), 179–265 (1971)
35. Revuz, D.: Mesures associées aux fonctionelles additives de Markov I. Trans. Amer. Math. Soc. **148**, 501–531 (1970)
36. Revuz, D.: Mesures associées aux fonctionelles additive de Markov II. Z. Wahrscheinlichkeitstheorie verw. Geb. **16**, 336–344 (1970)
37. Sharpe, M.J.: Discontinuous additive functionals of dual processes. Z. Wahrscheinlichkeitstheorie verw. Geb. **21**, 81–95 (1972)
38. Sharpe, M.J.: Exact multiplicative functionals in duality. Indiana Univ. Math. J. **21**, 27–60 (1971)
39. Sharpe, M.J.: General Theory of Markov Processes. (Forthcoming book)
40. Sharpe, M.J.: Homogeneous extensions of random measures. Lecture Notes **465**. 496–514. Berlin-Heidelberg-New York: Springer 1975
41. Sharpe, M.J.: Some transformations of diffusions by time reversal. Ann. Probability **8**, 1157–1162 (1980)
42. Silverstein, M.L.: Boundary Theory for Symmetric Markov Processes. Lecture Notes in Mathematics 516. Berlin-Heidelberg-New York: Springer 1976
43. Smythe, R.T.: Remarks on the hypotheses of duality. Lecture Notes in Mathematics 381. Berlin-Heidelberg-New York: Springer 1974
44. Smythe, R.T., Walsh, J.B.: The existence of dual processes. Invent. Math. **19**, 113–148 (1973)

45. Walsh, J.B., Meyer, P.A.: Quelques applications des résolventes de Ray. Invent. Math. **14**, 143–166 (1971)
46. Walsh, J.B.: The perfection of multiplicative functionals. Lecture Notes in Math. 258, 233–242. Berlin-Heidelberg-New York: Springer 1972
47. Walsh, J.B.: Markov processes and their functionals in duality. Z. Wahrscheinlichkeitstheorie verw. Geb. **24**, 229–246 (1972)
48. Weil, M.: Propriétés de continuité fine des fonctions coexcessives. Z. Wahrscheinlichkeitstheorie verw. Geb. **12**, 75–86 (1969)

Received February 1983

Note Added in Proof

The remark in the sentence just above (8.2) is only true if X is a standard process. For a general right process the argument in [36] only shows that ν_κ is a countable sum of finite measures. Further discussion and examples illustrating this point may be found in "Riesz decompositions in Markov process theory" by R.K. Getoor and J. Glover which is to appear in Trans. Amer. Math. Soc. As a result the proof of (16.10) is valid as written only when $\hat{X}$ is standard. We have another proof of (16.10) which is correct in complete generality.

In each of (13.16) through (13.19), only one of the claimed equalities is valid: namely, the second one in (13.16), the first in (13.17), the second in (13.18), and the first in (13.19). However, these are precisely the relations that are required for the proof of Theorem 13.20 and the remark following it, so the proof of (13.20) remains correct. The proof of (13.15) may be made valid by replacing λ_B and L_B by μ_B and M_B respectively.