Z. Wahrscheinlichkeitstheorie verw. Geb. 4, 232—247 (1965)

Hitting and Martingale Characterizations of One-Dimensional Diffusions * **

By

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Abstract. The main theorem of the paper is that, for a large class of one-dimensional diffusions (i. e. strong Markov processes with continuous sample paths): if $x(t)$ is a continuous stochastic process possessing the hitting probabilities and mean exit times of the given diffusion, then $x(t)$ is Markovian, with the transition probabilities of the diffusion.

For a diffusion $x(t)$ with natural boundaries at $\pm\infty$, there is constructed a sequence $\pi_n(t, x)$ of functions with the property that the $\pi_n(t, x(t))$ are martingales, reducing in the case of the Brownian motion to the familiar martingale polynomials.

It is finally shown that if a stochastic process $x(t)$ is a martingale with continuous paths, with the additional property that

$$\int_0^{x(t)} m(0, y]\, dy - t$$

is a martingale, then $x(t)$ is a diffusion with generator $D_m D^+$ and natural boundaries at $\pm\infty$. This generalizes a martingale characterization given by Lévy for the Brownian motion.

I. Introduction

It is well known that if $x(t)$ is the Brownian motion, then

a) almost all paths of $x(t)$ are continuous,

b) $x(t)$ is a martingale, and

c′) $x(t)^2 - t$ is a martingale.

That the converse is true, i. e., that a stochastic process satisfying a), b), and c′) must be the Brownian motion, has been stated (in slightly different form) by Paul Lévy [*5*]. J. L. Doob [*3*] gives a full proof, which relies heavily on Fourier transforms and the known transition probabilities for the Brownian motion. It is not amenable to generalization.

The Brownian motion is but one of a large class of (one-dimensional) diffusions, i. e., strong Markov processes with continuous sample paths. We were interested in generalizing Lévy's theorem to cover this class. We noted that for a continuous stochastic process, the provision that $x(t)$ and $x^2(t) - t$ be martingales ensures that the process $x(t)$ has the same hitting probabilities and mean exit times as the Brownian motion. We then *conjectured* that: for a large class of one-dimensional diffusions, if $x(t)$ is a continuous stochastic process, then the possession of the

* A revised version of the author's M. I. T. Ph. D. Thesis. I am very grateful to Professor Henry McKean for all that he contributed to the thesis.

** This work has been supported in part by the U.S.Army Signal Corps., the Air Force Office of Scientific Research, and the Office of Naval Research; in part by the National Institutes of Health (Grant NB-01865-05); and in part by the U.S. Air Force (ASD Contract AF 33(616)-7783).

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hitting probabilities and mean exit times of a diffusion implies that $x(t)$ is Markovian, with the transition probabilities of the given diffusion. We found this to be true.

It is well to contrast our result with the theorem, implicit in the paper of BLUMENTHAL, GETOOR, and MCKEAN [2], that if two Markov processes, satisfying HUNT's condition (A), have the same hitting probabilities and mean exit times, then they are identical. Our result is stronger *for the one-dimensional case* in that we need only assume one process Markov. It is tempting to speculate that our result is true without the dimensionality restriction, but the verification of this would require methods quite different from those that we employ here.

Throughout this paper we are concerned with one-dimensional continuous stochastic processes. We may consider such a process as defined on the "path-space" Ω, the set of continuous paths $w:[0, \infty) \to R$. We refer to R as the "state-space", and set $x(t, w) = w(t)$.

We shall denote by $\mathfrak{B}_t$ the usual σ-algebras generated by the sets $\{w | x(u,w) \in \Gamma\}$ $(0 \leqq u \leqq t,\ \Gamma \in \mathfrak{M}$, the σ-algebra generated by the open sets of $R)$. $\mathfrak{M}'$ will be the σ-algebra generated by the intervals on the line $0 \leqq t < \infty$, and $\mathfrak{B}$ will be the σ-algebra of subsets which is generated by the sets $\{x(t, w) \in \Gamma\}$ $(t \geqq 0, \Gamma \in \mathfrak{M})$.

A (continuous, one-dimensional) stochastic process $X = (\Omega, \mathfrak{B}_t, P_x)$ is then given by a family of probability measures P_x, one for each $x \in R$, such that $P_x\{x(0, w) = x\} = 1$, and that for any $B \in \mathfrak{B}$, the function $P_x(B)$ is an $\mathfrak{M}'$-measurable function of x. P_x may be interpreted as the probability measure induced on Ω by the process if started at point x at time 0. Thus a process in our sense is really a 'family' of processes, one for each $x \in R$.

Let $\xi(w)$ be a $\mathfrak{B}$-measurable function on Ω. We shall denote by $E_x\{\xi\}$ the integral

$$\int_\Omega \xi(w)\, P_x(dw).$$

We say that the non-negative random quantity $\tau = \tau(w)$ is a *stopping time* if, for any t, the set $\{\tau(w) \leqq t\}$ belongs to $\mathfrak{B}_t$. The sets $A \in \mathfrak{B}$ which for arbitrary t satisfy the condition $\{\tau(w) \leqq t\} \cap A \in \mathfrak{B}_t$ form a σ-algebra that we shall call $\mathfrak{B}_\tau$. The conditional expectations with respect to the σ-algebra $\mathfrak{B}_\tau$ will be denoted by the symbols $E_x[- \mid \mathfrak{B}_\tau]$. (The definition and properties of conditional probabilities and expectations with respect to a σ-algebra may be found, e. g., in Doob's "Stochastic Processes".)

A stochastic process is called a *Markov process* (homogeneous in time) if, for any positive *constant* τ, and for any $\Gamma_1, \Gamma_2, \ldots, \Gamma_n \in \mathfrak{M}$, $0 \leqq t_1 < t_2 \cdots < t_n$, we have for each x,

$$P_x[x(\tau + t_1) \in \Gamma_1, \ldots, x(\tau + t_n) \in \Gamma_n \mid \mathfrak{B}_\tau] = P_{x(\tau)}\{x(t_1) \in \Gamma_1, \ldots, x(t_n) \in \Gamma_n\}.$$

If this condition is satisfied not only for every constant τ but also for any stopping time $\tau = \tau(w)$, then we shall call X a *strong Markov process*. In the case developed here where X is a strong Markov process with continuous paths, we shall call X a *diffusion*.

In Section III, we verify our conjecture. The idea of the proof is very simple. Essentially, we show that for a large family of functions f, and any starting point x_0,

$$E_{x_0}[f(x(t + \sigma)) \mid \mathfrak{B}_\sigma] = \hat{E}_{x(\sigma)}[f(x(t))] \text{ a. e. } P_{x_0} \tag{1}$$

where $\hat{X}$ is a given diffusion, and X is a process with the same hitting characteristics. The left-hand side of (1) is an expectation taken with respect to the probability measure P_{x_0} of X, while the right-hand side is with respect to the probability measure $\hat{P}_{x(\sigma)}$ of $\hat{X}$. We choose the family of f' s so large that the left-hand sides determine the distribution of $x(t+\sigma)$, given $\mathfrak{B}_\sigma$. Since the right-hand sides are functions of $x(\sigma)$, our process X is Markov with the same transition function as X.

The only problems of any technical difficulty are the choice of the class of functions f, and the proof of (1).

The Brownian motion has generator $1/2\, d^2/dx^2$, and is but one of the class of diffusions $x(t)$ with natural boundaries at $\pm\infty$, and generators $D_m D^+$. In Section IV, which is concerned with martingales and diffusions with natural boundaries at $\pm\infty$, we shall see that, for such an $x(t)$, there is a sequence $\pi_n(t, x)$ of functions with the property that the $\pi_n(t, x(t))$ are martingales, reducing in the case of the Brownian motion to the familiar martingale polynomials. In particular,

$$\pi_1(t,x) = x$$

and

$$\pi_2(t,x) = \int_0^x m(0,y]\,dy - t\,.$$

We then see that a stochastic process $x(t)$ satisfies the conditions

a) almost all paths of $x(t)$ are continuous,

b) $x(t)$ is a martingale, and

c) $\int_0^{x(t)} m(0,y]\,dy - t$ is a martingale

if and only if it is a diffusion with generator $D_m D^+$, and has natural boundaries at $\pm\infty$. This generalizes LÉVY's martingale characterisation of the Brownian motion [c) clearly yields c') when setting $m(0,y] = 2y$].

II. Preliminaries

This section recapitulates, in a form convenient for our use, relevant material from the paper by DYNKIN [4].

Let us denote by B the space of $\mathfrak{M}$-measurable bounded functions on R, whose norm is

$$\|f\| = \sup_{x\in R} |f(x)|.$$

Given a Markov process X, the formula

$$T_t f(x) = E_x[f(x(t))]$$

defines uniquely a one parameter semigroup of linear operators in B, such that $\|T_t f\| \leqq \|f\|$, and for all $s \geqq 0$, $t \geqq 0$, $T_{s+t} = T_s T_t$.

Let f_n, $f \in B$. We write $f = \lim f_n$ if $\|f_n - f\| \to 0$. The set of all $f \in B$ for which $\lim\limits_{t\to 0} T_t f = f$ shall be denoted B_0. If

$$h(x) = \lim_{t\to 0} \frac{T_t f(x) - f(x)}{t} \quad (f, h \in B_0)$$

we write $f \in D_{\mathfrak{G}}$, and $\mathfrak{G} f = h$.

$\mathfrak{G}$ is called the (strong (infinitesimal)) generator of the Markov process.

We next define hitting probabilities and mean exit times for a random process X. Given an interval (x_1, x_2) and a stopping time T, we define

$$T + \tau(T, x_1, x_2) \text{ to be } \min(t \geqq T \mid x(t) \notin (x_1, x_2))$$

which is itself a stopping time.

Let x_0 be an arbitrary starting point that we can hold fixed. $E_{x_0}[\tau(T) \mid \mathfrak{B}_T]$ is referred to as a mean exit time. (If T is a time for which $x(T) \equiv x$, and $x(t)$ is a *diffusion*, then this mean exit time reduces to $E_x[\tau(0)]$.) The hitting probabilities are

$$p(T, x_1, x_2) = P_{x_0}[x(\tau(T) + T) = x_1 \mid \mathfrak{B}_T]$$
$$p(T, x_2, x_1) = P_{x_0}[x(\tau(T) + T) = x_2 \mid \mathfrak{B}_T].$$

We write $p(x_0, x_1, x_2)$, etc., if $T = 0$.

We shall consider below triples (T, x_1, x_2) in which the path is almost sure to leave (x_1, x_2) so that we have

$$p(T, x_1, x_2) + p(T, x_2, x_1) = 1.$$

A process X and a diffusion $\hat{X}$ are said to have the same *hitting characteristics* if for each starting point x_0, stopping time T, and interval (x_1, x_2)

$$E_{x_0}[\tau(T) \mid \mathfrak{B}_T] = \hat{E}_{x(T)}(\tau(0))$$
$$p(T, x_1, x_2) \quad = \hat{P}_{x(T)}[x(\tau(0)) = x_1]$$
$$p(T, x_2, x_1) \quad = \hat{P}_{x(T)}[x(\tau(0)) = x_2]$$

these statements all holding a. e. P_{x_0}.

Note that, in our general definition of the hitting probabilities and mean exit times, we have conditioned on $\mathfrak{B}_T$, that is, on the past up to some stopping time T, not merely on hitting x as in the case of a diffusion. The necessity for this stronger conditioning was shown to me by D. B. Ray with an example of a stochastic process that is continuous, but *not* a diffusion, while having the same $E_x[\tau(0, x_1, x_2)]$ as a diffusion but not the same $E_{x_0}[\tau(T, x_1, x_2) \mid \mathfrak{B}_T]$, etc. In fact, take a standard Brownian motion $b(t, w_1)$, $w_1 \in \Omega$. Then such a process is $x(t, w)$, $w = (w_1, w_2) \in \Omega \times [0,2)$, with

$$x(2n + t, w) = b(w_2 + 3n + t, w_1)$$
$$x(2n + 1 + t, w) = b(w_2 + 3n + 1 + 2t, w_1) \quad 0 \leqq t < 1$$

where we have the measure $1/2\, dx$ on $[0,2)$, Brownian measure on Ω, and the product measure on $\Omega x[0,2)$.

In a way, then, our Theorem 1 below may be interpreted as saying that, to make a continuous stochastic process Markov, it suffices to make its exit times and hitting probabilities possess the Markovian property of being independent of the past, once the present is given.

We say that a point x_0 is a right transition point (or left) for a diffusion $x(t)$ if there is some t for which $P_{x_0}\{x(t) > x_0\} > 0$ (or $P_{x_0}\{x(t) < x_0\} > 0$). We say that $x(t)$ is regular on the segment $[a, b]$ if all points of (a, b) are both left and right transition points.

Theorems 2.1 and 13.1 of DYNKIN [4] link mean exit times, hitting probabilities, and the generator of a *diffusion* as follows: Let X be a diffusion regular on the segment $[a, b]$. Then

1. The function $p(x) = p(x, b, a)$ is continuous, strictly increasing, and satisfies the conditions

$$\lim_{x \to a+0} p(x) = 0, \quad \lim_{x \to b-0} p(x) = 1.$$

For all $a \leqq x_1 < x < x_2 \leqq b$, we have

$$p(x, x_1, x_2) = \frac{p(x_2) - p(x)}{p(x_2) - p(x_1)}.$$

2. $n(x) = E_x(\tau(0, a, b))$ is continuous on $[a, b]$, and the derivative

$$D_p^+ n(x) = \lim_{y \to x+0} \frac{n(y) - n(x)}{p(y) - p(x)}$$

exists for all $x \in (a, b)$ and is continuous on the right, and monotonically decreasing. We set

$$m(x) = -D_p^+ n(x).$$

Thus $m(x)$ is continuous on the right and monotonically increasing. Since it is continuous on the right, we set $m(x, y] = m(y) - m(x)$.

3. If $f \in D_{\mathfrak{G}}$, then for any $x \in (a, b)$, the derivative $D_p^+ f(x)$ exists and is continuous on the right. In the interval (a, b) we have

$$\mathfrak{G} f(x) = D_m D_p^+ f(x).$$

The meaning of this is perhaps most clearly expressed by the equation

$$f^+(y) - f^+(x) = \int_{(x,y]} (\mathfrak{G} f)(\xi)\, m(d\xi); \; f^+ \equiv D_p^+ f.$$

Since $p(x)$ is continuous and strictly increasing, it is permissible to make the change of coordinates on our state space: $p(x) \to x$. Of course, this change affects m too, but we shall still denote the new function by m. Our diffusion is then so standardized that, on the regular interval $[a, b]$, if $a \leqq x_1 < x < x_2 \leqq b$, we have:

$$p(x, x_1, x_2) = \frac{x_2 - x}{x_2 - x_1}, \quad p(x, x_2, x_1) = \frac{x - x_1}{x_2 - x_1} \tag{2}$$

$$E_x(\tau(0, x_1, x_2)) = \frac{x_2 - x}{x_2 - x_1} \int_x^{x_1} m(x, y]\, dy + \frac{x - x_1}{x_2 - x_1} \int_x^{x_2} m(x, y]\, dy. \tag{3}$$

Furthermore, after this standard change of scale, the generator of a diffusion on a regular interval always has the form

$$\mathfrak{G} f = D_m D^+ f.$$

III. The hitting characterization

This section is devoted to the proof of the following theorem.

Theorem 1. *Let us be given a diffusion $\hat{Y}$ and a stochastic process Y (with continuous paths) having as common state space some interval of the real line. Let*

(a, b) be an interval of regularity for the diffusion. Let $\hat{X}$ be our diffusion stopped at the end points of the interval (a, b), that is,

$$\hat{x}(t, w) = \begin{cases} \hat{y}(t, w) & t < \tau(0, a, b) \\ \hat{y}(\tau(0, a, b), w) & t \geqq \tau(0, a, b). \end{cases}$$

Similarly, let X be the given stochastic process stopped at the end points of (a, b).

If the given stochastic process has the same hitting characteristics as the diffusion, then X is a strong Markov process, and thus a diffusion, and has the same transition probabilities as $\hat{X}$.

Let X have generator $D_m D^+$ in the domain under consideration. (The standard change of scale of Section II does not vitiate the truth of our theorem.) We know, from McKEAN [6], that if $p(t, s, x)$ $(a < s < b)$ is the fundamental solution of the "generalized heat equation",

$$D_m D^+ u(t, x) = \frac{\partial}{\partial t} u(t, x), \tag{4}$$

subject to the boundary conditions that $u(x, t)$ is 0 for $x \notin (a, b)$, then

$$\int_\Gamma p(t, y, x)\, m(dx) = \hat{P}_y(x(t) \in \Gamma).$$

We also know that if $f(x)$ is any continuous function of support in (a, b), then

$$\psi(t, x) = \int_a^b p(t, x, s)\, f(s)\, m(ds) \qquad (t > 0) \tag{5}$$

is a solution of (3) with $\psi(0, x) = f(x)$; and $\psi(t, x) = 0$ when $t \leqq a$ or $t \geqq b$.

We consider the class $C(m, a, b)$ of functions $\psi(t, x)$ of the form (5) with the properties that $\psi_1 = \partial\psi/\partial t$ is uniformly continuous and that both ψ_1 and $D^+\psi$ are bounded, less than M say, in any bounded interval of x and t. (It suffices that f be in the domain of the generator.)

For such a ψ, we have

$$\begin{aligned} \hat{E}_{x(\sigma)}[\psi(t - s, x(s))] &= \int \psi(t - s, \xi)\, p(s, x(\sigma), \xi)\, m(d\xi) \\ &= \iint f(\eta)\, p(t - s, \xi, \eta)\, p(s, x(\sigma), \xi)\, m(d\eta)\, m(d\xi) \\ &= \int f(\eta)\, p(t, x(\sigma), \eta)\, m(d\eta), \\ &= \psi(t, x(\sigma)). \end{aligned}$$

We shall soon prove

Lemma 1. *For all ψ of the class $C(m, a, b)$ defined above, $0 \leqq s < t$,*

$$E[\psi(t - s, x(s + \sigma)) - \psi(t, x(\sigma)) \mid \mathfrak{B}_\sigma] = 0.$$

[Throughout the rest of this section we shall write E and P as abbreviations for E_{x_0} and P_{x_0}, where x_0 is an arbitrary starting point that we can hold fixed.]

Let us first observe, however, that from Lemma 1 we may deduce Theorem 1. Lemma 1 tells us that

$$E[\psi(t - s, x(s + \sigma)) \mid \mathfrak{B}_\sigma] = \hat{E}_{x(\sigma)}[\psi(t - s, x(s))] \tag{6}$$

for all functions ψ of the class $C(m, a, b)$. We wish to see that these equalities determine the distribution of $x(s + \sigma)$, given $\mathfrak{B}_\sigma$. But

$$\psi(t - s, x) = \int f(y)\, p(t - s, x, y)\, m(dy),$$

where f may clearly approximate any continuous function, vanishing outside (a, b), arbitrarily closely. Now, $t - s$ may be made arbitrarily small without affecting the freedom of choice of s and σ. Since $\psi(0, x) = f(x)$, it follows, by continuity of ψ in $t - s$, that the $\psi' s$ form a determining set. We thus obtain (1), whence the theorem follows as in Section I.

To prove Lemma 1 we must estimate our quantity in terms of exit times, and so forth. To this end, we sub-divide the interval (a, b) into 2^n equal disjoint intervals.

We define $\sigma + e_0^n = \min \{t \geqq \sigma \,|\, x(t)$ is an end point of an interval of our sub-division$\}$. Then, provided $x(\sigma + e_k^n) \neq a$ or b, and $e_k^n < s$, we define

$$\sigma + e_{k+1}^n = \begin{cases} \min\left\{t \geqq \sigma + e_k^n \,|\, x(t) = x(\sigma + e_k^n) \pm \frac{1}{2^n}\right\} \\ \infty \text{ if no such } t \text{ exists.} \end{cases}$$

We thus obtain a sequence of stopping times

$$\sigma + e_0^n \leqq \sigma + e_1^n \leqq \cdots \leqq \sigma + e_{m(n)}^n \equiv \sigma + e_k^n \quad \text{for all } k > m\,,$$

where

$$e_m^n = \min\{\tau(\sigma, a, b), e_{\#}^n\}$$
$$\# = \min\{k \,|\, e_{k-1}^n < s, e_k^n \geqq s\}.$$

$\mathfrak{B}_k^n$ will now be used as an abbreviation for $\mathfrak{B}_\sigma + e_k^n$, whilst x_k^n will be short notation for $x(\sigma + e_k^n)$. Furthermore, we shall usually suppress the superscript n.

We shall need to use the fact that $E[m(n) \,|\, \mathfrak{B}_\sigma] < \infty$ for each n. To see this, first note that

$$E[e_k^n - e_{k-1}^n \,|\, x_{k-1}^n \neq a \quad \text{or} \quad b;\, \mathfrak{B}_\sigma]$$
$$\geqq \min_{0 \leqq q \leqq 2^n - 1} \int_{a+q(b-a)2^{-n}}^{a+(q+1)(b-a)2^{-n}} m(a + q(b-a)2^{-n}, y]\, dy$$
$$= i(n) > 0\,,$$

since m is strictly increasing on (a, b). Thus

$$E[\tau(\sigma, a, b) \,|\, \mathfrak{B}_\sigma] \geqq E[\sum (e_k^n - e_{k-1}^n) \,|\, \mathfrak{B}_\sigma]$$
$$\geqq i(n)\, E[m(n) \,|\, \mathfrak{B}_\sigma].$$

Hence

$$E[m(n) \,|\, \mathfrak{B}_\sigma] \leqq \frac{1}{i(n)} E[\tau(\sigma, a, b) \,|\, \mathfrak{B}_\sigma] < \infty.$$

We also have, from (2) and (3)

$$E[x_k - x_{k-1} \,|\, \mathfrak{B}_{k-1}] = 0 \tag{7}$$

and

$$E[\int_{x_{k-1}}^{x_k} m(x_{k-1}, y]\, dy - (e_k - e_{k-1}) \,|\, \mathfrak{B}_{k-1}] = 0 \tag{8}$$

We now compare

$$E[\psi(t - s, x(\sigma + s)) - \psi(t, x(\sigma)) \,|\, \mathfrak{B}_\sigma]$$

with

$$E[\psi(t - e_m, x_m) - \psi(t - e_0, x_0) \,|\, \mathfrak{B}_\sigma]$$
$$= E\left[\sum_{k=1}^{\infty} [\psi(t - e_k, x_k) - \psi(t - e_{k-1}, x_{k-1})] \,|\, \mathfrak{B}_\sigma\right] \tag{9}$$

The difference between the two expressions is at most

$$E[|\psi(t-e_0,x_0)-\psi(t,x(\sigma))||\mathfrak{B}_\sigma]+E[|\psi(t-e_m,x_m)-\psi(t-s,x(\sigma+s))||\mathfrak{B}_\sigma],$$

We shall prove that the second term tends to zero as $n\to\infty$. The proof for the first term is simpler, and is omitted. The second term is less than or equal to

$$E[|\psi(t-e_m,x_m)-\psi(t-e_m,x(\sigma+s))||\mathfrak{B}_\sigma] + E[|\psi(t-e_m,x(\sigma+s))-\psi(t-s,x(\sigma+s))||\mathfrak{B}_\sigma].$$

The portions of these terms corresponding to $e_m=\tau\leqq s$ clearly vanish, by the definition of ψ. Hence we have only to limit the portions on which $e_{m-1}<s\leqq e_m$.

If we choose N so large that $|y-y'|<2^{-N}$ and $0<r\leqq t$ implies $|\psi(r,y)-\psi(r,y')|<\epsilon$, then for $n>N$, the first term is less than ϵ. The second term is less than $s_n=M.\ E[e_m^n-s,\ e_{m-1}^n<s\leqq e_m^n|\mathfrak{B}_\sigma]$. Since the $e_{m(n)}^n$ form a decreasing sequence, the s_n form a decreasing sequence of positive terms, and hence has a limit $\geqq 0$. It equals zero, since we have, for all k, $E[e_k^n-e_{k-1}^n|\mathfrak{B}_{k-1}]\downarrow 0$ as $n\to\infty$, since, by using (8)

$$E[e_k^n-e_{k-1}^n|\mathfrak{B}_{k-1}]<\max_{0\leqq q<2^n}\int_{a+q(b-a)2^{-n}}^{a+(q+1)(b-a)2^{-n}} m(a+q(b-a)2^{-n},y]\,dy$$
$$<K\cdot 2^{-n},$$

where $K=m(a,b]\cdot(b-a)$.

Hence, it remains to be proved that (9) tends to zero as n tends to ∞. We note that

$$E\left[\sum_{k=1}^{\infty}|\psi(t-e_k^n,x_k^n)-\psi(t-e_{k-1}^n,x_{k-1}^n)||\mathfrak{B}_\sigma\right]$$
$$\leqq E\left[\sum_{k=1}^{m(n)}M\cdot(e_k^n-e_{k-1}^n)+\sum_{k=1}^{m(n)}M\cdot\frac{1}{2^n}\middle|\mathfrak{B}_\sigma\right]$$
$$=E\left[M\cdot(e_{m(n)}^n-e_0^n)+\frac{M}{2^n}\cdot m(n)|\mathfrak{B}_\sigma\right]$$
$$<\infty.$$

(9) is thus absolutely convergent, and so may be rewritten

$$\sum_{k=1}^{\infty}E[\psi(t-e_k,x_k)-\psi(t-e_{k-1},x_{k-1})|\mathfrak{B}_\sigma].$$

Bearing in mind the usual Taylor series, and its simply proved analogue

$$f(b)=f(a)+(b-a)D^+f(a)+\left(\int_a^b m(a,y]\,dy\right)\cdot D_mD^+f(x)$$

for some x between a and b, we have

$$\psi(t-e_k,x_k)-\psi(t-e_{k-1},x_{k-1})=[\psi(t-e_k,x_k)-\psi(t-e_{k-1},x_k)]+$$
$$+[\psi(t-e_{k-1},x_k)-\psi(t-e_{k-1},x_{k-1})]$$
$$=-[\psi_1(t-\tilde{e}(k),x_k)(e_k-e_{k-1})]+$$
$$+[\psi^+(t-e_{k-1},x_{k-1})(x_k-x_{k-1})]+$$
$$+[D_mD^+\psi(t-e_{k-1},\tilde{x}(k))\cdot\int_{x_{k-1}}^{x_k}m(x_{k-1},y]\,dy],$$

where $e_{k-1} \leqq \tilde{e}(k) \leqq e_k$, and $x(k)$ falls between x_{k-1} and x_k,

$$= \psi^+(t - e_{k-1}, x_{k-1})(x_k - x_{k-1})$$

$$- \psi_1(t - e_{k-1}, x_{k-1})\left[(e_k - e_{k-1}) - \int_{x_{k-1}}^{x_k} m(x_{k-1}, y]\, dy\right]$$

$$- [\psi_1(t - \tilde{e}(k), x_k) - \psi_1(t - e_{k-1}, x_{k-1})](e_k - e_{k-1})$$

$$+ [\psi_1(t - e_{k-1}, \tilde{x}(k)) - \psi_1(t - e_{k-1}, x_{k-1})] \cdot \int_{x_{k-1}}^{x_k} m(x_{k-1}, y]\, dy,$$

since $\psi_1 = D_m \psi^+$, by assumption,

$$= T_{1k} + T_{2k} + T_{3k} + T_{4k}, \text{ say.}$$

Since $E[x_k - x_{k-1} | \mathfrak{B}_{k-1}] = 0$, we see that $\sum_{k=1}^{\infty} E[T_{1k} | \mathfrak{B}_\sigma] = 0$ by conditioning the k^{th} term first with respect to $\mathfrak{B}_{k-1}$, and then with respect to $\mathfrak{B}_\sigma$.

Similarly, (8) implies that $\sum_{k=1}^{\infty} E[T_{2k} | \mathfrak{B}_\sigma] = 0$.

Now pick $\varepsilon > 0$, and N so large and δ so small that for t, $t' > 0$, with $|t - t'| < \delta$ and $|x - x'| < 2^{-N}$, we have $|\psi_1(t, x) - \psi_1(t', x')| < \varepsilon$. Then, for $n > N$, noting that $e_k - e_{k-1} \geqq 0$, we obtain

$$\sum_1^{\infty} E[|T_{3k}| | \mathfrak{B}_\sigma] = \sum_1^{\infty} E[|T_{3k}|, e_k^n - e_{k-1}^n > \delta | \mathfrak{B}_\sigma]$$

$$+ \sum_1^{\infty} E[|T_{3k}|, e_k^n - e_{k-1}^n \leqq \delta | \mathfrak{B}_\sigma]$$

$$= s_1 + s_2,$$

$$s_2 < \varepsilon \sum_1^{\infty} E[e_k^n - e_{k-1}^n, e_k^n - e_{k-1}^n \leqq \delta | \mathfrak{B}_\sigma]$$

$$< \varepsilon \sum_1^{\infty} E[e_k^n - e_{k-1}^n | \mathfrak{B}_\sigma]$$

$$< \varepsilon\, E[\tau(\sigma, a, b) | \mathfrak{B}_\sigma],$$

$$s_1 = \sum_1^{\infty} E[|T_{3k}|, e_k^n - e_{k-1}^n > \delta | \mathfrak{B}_\sigma]$$

$$< 2M \cdot \sum_1^{\infty} E[e_k^n - e_{k-1}^n, e_k^n - e_{k-1}^n > \delta | \mathfrak{B}_\sigma]$$

$$< 2M \cdot E[e_m^n - e_0^n, \max_k (e_k^n - e_{k-1}^n) > \delta | \mathfrak{B}_\sigma].$$

To prove that this goes to zero as $n \to \infty$, we show that $P(\Delta_n | \mathfrak{B}_\sigma) \downarrow 0$ as $n \to \infty$, $\Delta_n = \{\max_k (e_k^n - e_{k-1}^n) > \delta\}$. But this amounts to showing X has, with probability 1, no interval of constancy. Since such an interval of constancy must contain a rational t_1, and since there are only denumerably many of these, it suffices to prove that

$$P[x(t) \equiv x(t_1), t_1 \leqq t \leqq t_2] = 0$$

for any given pair $t_1 < t_2$. Now,

$$\begin{aligned}
P[x(t) &\equiv x(t_1), t_1 \leqq t \leqq t_2] \\
&= \lim_n \sum_k P[k2^{-n} \leqq x(t) < (k+1)2^{-n}, t_1 \leqq t \leqq t_2, k = [x(t_1)2^n]] \\
&= \lim_n \sum_k E(P[k2^{-n} \leqq x(t) < (k+1)2^{-n}, t_1 \leqq t \leqq t_2 | \mathfrak{B}_{t_1}], k = [x(t_1)2^n]] \\
&= \lim_n \sum_k E\left(\frac{E(e^{I_k}|\mathfrak{B}_{t_1})}{t_2 - t_1}, k = [x(t_1)2^n]\right) = 0
\end{aligned}$$

where I_k is the interval of length $2^{-(n-1)}$ centered at $(k + 1/2)2^{-n}$, e^{I_k} is the exit time from this interval, and we know that $E(e^{I_k}|\mathfrak{B}_{t_1}) \downarrow 0$, by our hitting assumptions. Thus

$$\sum_{k=1}^{\infty} E[T_{3k}|\mathfrak{B}_\sigma] \to 0 \quad \text{as} \quad n \to \infty.$$

Noting now that $\int_x^{x'} m(x, y]\, dy$ is always non-negative, we have, for $n > N$,

$$\begin{aligned}
\sum_{k=1}^{\infty} E[|T_{4k}||\mathfrak{B}_\sigma] &\leqq \varepsilon E\left[\sum_{k=1}^{\infty} E\left[\int_{x_{k-1}}^{x_k} m(x_{k-1}, y]\, dy | \mathfrak{B}_{k-1}\right] \bigg| \mathfrak{B}_\sigma\right] \\
&\leqq \varepsilon E[\tau(\sigma, a, b)|\mathfrak{B}_\sigma].
\end{aligned}$$

Thus $\sum_{k=1}^{\infty} E[T_{4k}|\mathfrak{B}_\sigma] \to 0$ as $n \to \infty$, and we may finally conclude that

$$E[(\psi(t-s, x(\sigma+s)) - \psi(t, x(\sigma))|\mathfrak{B}_\sigma] = 0.$$

Q.E.D.

IV. Martingales and diffusions with natural boundaries at $\pm\infty$

In this section we restrict ourselves to diffusions that are regular on every finite interval, having natural boundaries at $\pm\infty$ and generators $D_m D^+$. For a definition of natural boundaries, see McKean [6]. The intuitive meaning of natural boundaries is that "it takes a particle moving under the diffusion infinitely long to reach them".

The classical example of such a diffusion is the Brownian motion (Wiener process) $b(t)$, determined by $m(x) = 2x$, which thus has generator

$$\frac{1}{2}\frac{d^2}{dx^2}.$$

It is normally characterized by the fact that it has gaussian increments:

$$P_x\{b(t) \in \Gamma\} = \int_\Gamma \frac{1}{\sqrt{2\pi t}} e^{-(y-x)^2/2t}\, dy.$$

Recall that a stochastic process $x(t)$ is a martingale if, for $s < t$,

$$E[x(t)|\mathfrak{B}_s] = x(s).$$

It is well known that we may associate with the Brownian motion a sequence of polynomials $u_n(t, x)$ with the property that the stochastic process $u_n(t, b(t))$, obtained by replacing x by the random variable $b(t)$, is a martingale, for each n. We thus call $u_n(t, x)$ a martingale function for $b(t)$. These polynomials are intimately related to the heat polynomials of ROSENBLOOM and WIDDER [7] (which go back at least to APPELL [1], 1892) and have the generating function

$$e^{(\sqrt{2}z)x - z^2 t} = \sum_{n=0}^{\infty} u_n(t, x) \frac{z^n}{n!}$$

Thus, up to a multiplicative constant,

$$u_0 = 1,$$
$$u_1 = x,$$
$$u_2 = x^2 - t,$$
$$u_3 = \tfrac{2}{3} x^3 - tx,$$
$$u_4 = \tfrac{1}{3} x^4 - tx^2 + \tfrac{t^2}{2}, \text{ etc.}$$

A compact expression (again up to a multiplicative constant) using the Hermite polynomials $H_n(y)$ is

$$u_n(t, x) = \left(\frac{t}{2}\right)^{n/2} H_n\left(\frac{x}{\sqrt{2t}}\right).$$

We recall that LÉVY's theorem tells us that if a continuous stochastic process has both u_1 and u_2 as martingale functions, then it is the Brownian motion.

We thus ask: Given the diffusion $x(t)$ with natural boundaries at $\pm\infty$, does there exist a sequence of martingale functions $\pi_0, \pi_1, \pi_2, \ldots$ for $x(t)$, reducing naturally to $u_0, u_1, u_2, \ldots,$ for $b(t)$? Furthermore, if a continuous stochastic process has π_1 and π_2 as martingale functions, must it then *be* the diffusion $x(t)$? The answer to both questions is in the affirmative.

We note that

$$e^{(\sqrt{2}z)x} = \sum_{n=0}^{\infty} z^n \left(\frac{2^{n/2} x^n}{n!}\right)$$

and that the sequence

$$p_n(x) = \frac{2^{n/2} x^n}{n!}$$

has the property that

$$\frac{1}{2} \frac{d^2}{dx^2} p_{n+2}(x) = p_n(x).$$

We generalize this sequence for the generator $D_m D^+$ by the defining relations:

$$p_0 = 1, \quad p_1 = x,$$
$$p_n = \int_0^x d\eta \int_0^\eta p_{n-2}(\xi)\, dm(\xi)$$

so that we do indeed have $D_m D^+ p_{n+2} = p_n$.

It can be verified by induction separately on odd and even n that $p_n(x)$ is positive for n even, and has the sign of x for n odd.

Then we set

$$g(x, z) = \sum_{n=0}^{\infty} z^n p_n(x)$$

so that

$$g(x, z)\, e^{-z^2 t} = \sum_{n=0}^{\infty} \pi_n(x, t) \frac{z^n}{n!}$$

is, at least formally, the generating function of the space-time functions

$$\pi_0 = 1,$$
$$\pi_1 = x,$$
$$\pi_2 = p_2(x) - t,$$
$$\pi_3 = p_3(x) - t p_1(x),$$
$$\pi_4 = p_4(x) - t p_2 + \frac{t^2}{2!}, \text{ etc.},$$

and π_n reduces to u_n after replacing $p_n(x)$ by $2^{n/2} x^n / n!$ and factoring out an unimportant $\sqrt{2}$ from each π_{2n+1}.

Theorem 2. *The π_n are martingale functions for the diffusion with natural boundaries at $\pm\infty$, and generator $D_m D^+$.*

We already know this to be true for $m = 2x$, and (as in fact the reader may check from our construction), our experience tells us that it suffices to prove

$$E_0[p_{2n}(x(t))] = \frac{t^n}{n!}$$
$$E_0[p_{2n+1}(x(t))] = 0.$$

We shall thus have our theorem as soon as we have proved Lemma 4 below.

We recall (McKEAN [6]) that, for each $\alpha > 0$, we have two positive solutions $g_1(x)$, $g_2(x)$ of $D_m D^+ g(x) = \alpha g(x)$, the first increasing, the second decreasing. We normalize them so that their Wronskian $g_1^+ g_2 - g_1 g_2^+$ is identically 1.

The Green function for $D_m D^+$ is

$$G_\alpha(\xi, \eta) = G_\alpha(\eta, \xi) = g_1(\xi) g_2(\eta) \quad \text{for} \quad \xi \leqq \eta.$$

If $p(t, y, x)$ is the fundamental solution of

$$D_m D^+ f = \frac{\partial}{\partial t} f$$

it has the property that the transition probability for the diffusion with generator $D_m D^+$ is

$$P(t; y, \Gamma) = \int_\Gamma p(t, y, x) \, m(dx).$$

Furthermore,

$$G_\alpha(y, x) = \int_0^\infty e^{-\alpha t} p(t, y, x) \, dt.$$

Lemma 2. *For a natural boundary at $+\infty$:*

(i) $$g_2(\infty) = 0,$$

(ii) $$A g_2^+(A) \to 0 \quad \text{as} \quad A \to \infty.$$

Proof. (i) See McKEAN [6] (p. 524).

(ii) Since $dg_2^+ = \alpha g_2 dm \geqq 0$, g_2 must be concave.

Thus

$$|(A - B) g_2^+(A)| \leqq g_2(B) - g_2(A).$$
$$\therefore \lim_{A \to \infty} |A g_2^+(A)| \sim g_2(B) - g_2(\infty) \to 0 \quad \text{as} \quad B \to \infty.$$

Q.E.D.

Lemma 3. *If $x(t)$ is the diffusion with generator $D_m D^+$, and natural boundaries at $\pm\infty$, then $x(t)$ is a martingale.*

Proof. We first compute the Laplace transform

$$\int_0^\infty e^{-\alpha t} E_0[x(t), x(t) \geqq 0]\, dt$$

$$= \int_0^\infty m(dx)\, x \int_0^\infty e^{-\alpha t} p(t, 0, x)\, dt$$

$$= \int_0^\infty x\, G_\alpha(0, x)\, m(dx)$$

$$= \frac{g_1(0)}{\alpha} \int_0^\infty x\, \alpha\, g_2(x)\, m(dx)$$

$$= \frac{g_1(0)}{\alpha} \int_0^\infty x\, dg_2^+ \quad (\text{since } D_m D^+ g_2 = \alpha g_2) \tag{10}$$

$$= \lim_{A \to \infty} \frac{g_1(0)}{\alpha} \left([x g_2^+(x)]_0^A - \int_0^A g_2^+(x)\, dx\right.$$

$$= \frac{g_1(0)\, g_2(0)}{\alpha} \text{ by Lemma 2}$$

$$= \frac{G_a(0,0)}{\alpha}$$

$$= \frac{1}{\alpha} \int_0^\infty e^{-\alpha t} p(t, 0, 0)\, dt$$

$$= \int_0^\infty e^{-\alpha t} \int_0^t p(s, 0, 0)\, ds\, dt.$$

Thus, by the uniqueness of the Laplace transform, we have, for almost all t,

$$E_0[x(t), x(t) \geqq 0] = \int_0^t p(s, 0, 0)\, ds.$$

Analogously, for almost all t,

$$E_0[x(t), x(t) \geqq 0] = -\int_0^t p(s, 0, 0)\, ds.$$

Thus

$$E_0[|x(t)|] = 2 \int_0^t p(s, 0, 0)\, ds \quad \text{for almost all } t.$$

That is,

$$E_0[|x(t)|] \leqq 2 \int_0^t p(s, 0, 0)\, ds \quad \text{for } \textit{all } t, \text{ by Fatou's lemma}.$$

More generally,

$$E_\eta[|x(t)|] \leqq \eta + 2 \int_0^t p(s, \eta, 0)\, ds \quad \text{for almost all } t,$$

and

$$E_\eta[x(t)] = \eta \quad \text{for almost all } t,$$

as may be shown by a slight elaboration of the argument above for $\eta = 0$.

By the Chapman-Kolmogorov equation

$$E_\eta[x(t)] = E_\eta[E_{x(s)}[x(t-s)]],$$

and so, since the measure ds/t on $(0, t)$ has total mass 1

$$E_\eta[x(t)] = \int_0^t \frac{ds}{t} \int_{-\infty}^{\infty} p(s, \eta, y)\, m(dy) \int_{-\infty}^{\infty} p(t-s, y, z)\, m(dz). \tag{11}$$

Let

$$\chi(s, y) = \begin{cases} 1 & \text{if} \quad E_y[x(t-s)] = y \quad \text{and} \quad E_\eta[x(s)] = \eta \\ 0 & \text{elsewhere}. \end{cases}$$

Then

$$\int_0^t \frac{ds}{t} \int_{-\infty}^{\infty} \chi(s, y)\, m(dy) = 1$$

which allows us to modify (11) on an (s, y) set of measure 0 to obtain

$$E_\eta[x(t)] = \eta \quad \text{for } all\ t,$$

and so $x(t)$ is a martingale, by stationarity. Q.E.D.

Lemma 4.

$$E_0[p_{2n}(x(t))] = \frac{t^n}{n!}$$

$$E_0[p_{2n+1}(x(t))] = 0.$$

Proof. Let

$$c_n^+ = E_0[p_n(x), x \geqq 0],$$
$$c_n^- = E_0[p_n(x), x \leqq 0],$$
$$c_n = E_0[p_n(x)].$$

Then the Laplace transform of c_n^+ is

$$\begin{aligned}
\mathscr{L} c_n^+ &= g_1(0) \int_0^\infty p_n g_2\, dm \\
&= \frac{g_1(0)}{\alpha} \int_0^\infty p_n\, dg_2^+, \quad \text{just as we derived (10)}, \\
&= \frac{g_1(0)}{\alpha} \lim_{A\to\infty} \left[p_n(A) g_2^+(A) - \int_0^A p_n^+(x)\, dg_2(x) \right] \\
&= \frac{g_1(0)}{\alpha} \lim_{A\to\infty} \left[p_n(A) g_2^+(A) - p_n^+(A) g_2(A) + \int_0^A g_2 p_{n-2}\, dm \right]
\end{aligned}$$

for $n \geqq 2$, since $p_{n-2} = D_m D^+ p_n$.

$$\therefore \mathscr{L} c_n^+ = \frac{1}{\alpha} \mathscr{L} c_{n-2}^+ + \frac{g_1(0)}{\alpha} \lim_{A\to\infty} [p_n(A) g_2^+(A) - p_n^+(A) g_2(A)]. \tag{12}$$

$$\begin{aligned}
\int_A^\infty p_n g_2\, dm &\geqq \frac{p_n(A)}{\alpha} \int_A^\infty \alpha g_2\, dm \\
&= \frac{p_n(A)}{\alpha} \int_A^\infty dg_2^+ \\
&= \frac{p_n(A)}{\alpha} (-g_2^+(A)) \\
&\geqq 0,
\end{aligned}$$

since g_2 is decreasing and $p_n(A) > 0$ for $A > 0$.

$$\therefore \qquad p_n(A)\,g_2^+(A) \to 0 \quad \text{as} \quad A \to \infty .$$

Furthermore,

$$\int_A^B p_n g_2 \, dm \geqq g_2(B) \int_A^B p_n \, dm = g_2(B)\,[p_{n+2}^+(B) - p_{n+2}^+(A)] ;$$

$$\therefore \qquad \lim_{B\to\infty} g_2(B)\,p_{n+2}^+(B) \sim \lim_{B\to\infty} \int_A^B p_n g_2 \, dm \downarrow 0 \quad \text{as} \quad A \to \infty .$$

Thus we deduce from (12) that

$$\mathscr{L} c_n^+ = \frac{1}{\alpha} \mathscr{L} c_{n+2}^+ ,$$

$$\therefore \qquad \mathscr{L} c_{2n}^+ = \frac{1}{\alpha^n} \mathscr{L} c_0^+ = \alpha^{-n-1}(-g_2^+(0)\,g_1(0)) ,$$

$$\mathscr{L} c_{2n+1}^+ = \frac{1}{\alpha^n} \mathscr{L} c_1^+ = \alpha^{-n-1}(g_1(0)\,g_2(0)) .$$

Similarly,

$$\mathscr{L} c_{2n}^- = \alpha^{-n-1}(g_1^+(0)\,g_2(0)) ,$$

$$\mathscr{L} c_{2n+1}^- = \alpha^{-n-1}(-g_1(0)\,g_2(0)) ,$$

and we see that $E_0[|p_n(x)|] < \infty$.

Thus, recalling that $g_1^+ g_2 - g_1 g_2^+ \equiv 1$, we have

$$\mathscr{L} c_{2n} = \frac{1}{\alpha^{n+1}}$$

$$\mathscr{L} c_{2n+1} = 0 .$$

$$\therefore \qquad E_0[p_{2n}(x(t))] = \mathscr{L}^{-1}(\alpha^{-n-1}) = \frac{t^n}{n!} \qquad \text{for almost all } t ,$$

$$E_0[p_{2n}(x(t))] = 0 \qquad \text{for almost all } t .$$

The Chapman-Kolmogorov trick in the proof of Theorem 2 now permits us to replace "for almost all t" by "for all t".

Q.E.D.

We are now in a position to combine Theorem 1 and 2 to obtain the promised martingale characterization of which LÉVY's theorem is clearly a special case.

Theorem 3. *Let $x(t)$ be a stochastic process. Then in order that $x(t)$ be the diffusion with generator $D_m D^+$ and natural boundaries at $\pm\infty$, it is both necessary and sufficient that it satisfy all three conditions:*

a) *almost all sample paths of $x(t)$ are continuous,*

b) *$x(t)$ is a martingale,*

c) $\int_0^{x(t)} m(0, y]\, dy - t$ *is a martingale.*

Proof. (i) The necessity follows from Theorem 2.

(ii) Sufficiency will follow from Theorem 1, as soon as we show that the conditions a), b), c) suffice to fix the hitting probabilities and mean exit times of the process.

We want to determine

$$p(T, x_1, x_2) = P[x(T+\tau) = x_1 \,|\, \mathfrak{B}_T]$$

which yields

$$p(T, x_2, x_1) = 1 - p(T, x_1, x_2)$$

and

$$E[\tau(T, x_1, x_2) \,|\, \mathfrak{B}_T].$$

Now, if optional sampling at T and $T+\tau$ preserves "martingaleness" in b) and c), we have

$$E\left[\frac{x(T+\tau)-x_1}{x_2-x_1}\,\middle|\,\mathfrak{B}_T\right] = \frac{x(T)-x_1}{x_2-x_1}.$$

That is,

$$\frac{x_1-x_1}{x_2-x_1}\,p(T, x_1, x_2) + \frac{x_2-x_1}{x_2-x_1}\,p(T, x_2, x_1) = \frac{x(T)-x_1}{x_2-x_1}$$

i.e.

$$p(T, x_2, x_1) = \frac{x(T)-x_1}{x_2-x_1}.$$

Similarly, c) yields

$$E[\tau \,|\, \mathfrak{B}_T] = p(T, x_1, x_2)\int_0^{x_1} m(0, y]\, dy + p(T, x_2, x_1)\int_0^{x_2} m(0, y]\, dy - \int_0^{x(T)} m(0, y]\, dy,$$

and the determination is complete. To make the argument rigorous, we merely replace T by $\max(-n, \min(T, n))$ and $T+\tau$ by $\max(-n, \min(T+\tau, n))$, apply Theorem VII.11.8 of Doob [3] (p. 376) carry out the analogues of the computation above, and then let $n \to \infty$.

Q.E.D.

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(Received September 30, 1963)