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The Isoperimetric Inequality for Isotropic Unimodal Lévy Processes

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1. Introduction

Let $(\Omega X_t P_x)$ be a symmetric n-dimensional Lévy process. Let $P_t(x, dy)$ be transition function of X_t. Then, there corresponds unique (S, Φ) such that

$$\begin{aligned} &\int e^{i(x,\xi)} P_t(0, dx) = e^{-t\psi(\xi)} \\ &\psi(\xi) = \tfrac{1}{2}(S\xi, \xi) + \int_{\mathbb{R}^n} (1 - \cos(\xi, y))\, \Phi(dy), \end{aligned} \tag{1.1}$$

where S is a nonnegative symmetric matrix and Φ is a symmetric measure on $\mathbb{R}^n$ with $\Phi(\{0\}) = 0$ and $\int \frac{|x|^2}{1+|x|^2} \Phi(dx) < \infty$. The associated Dirichlet form $(\varepsilon, \mathscr{F}(\varepsilon))$ related to $L^2(\mathbb{R}^n)$ can be given by

$$\begin{aligned} &\varepsilon(u, u) = \tfrac{1}{2} D_S(u, u) + \tfrac{1}{2} J_\Phi(u, u) \\ &\mathscr{F} = \mathscr{F}(\varepsilon) = \{u \in L^2(\mathbb{R}^n) \colon \varepsilon(u, u) < \infty\}, \end{aligned} \tag{1.2}$$

where

$$D_S(u, u) = \int \sum S_{ij} \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^j} dx$$

$(S = (S_{ij}))$ and

$$J_\Phi(u, u) = \iint (u(x+y) - u(x))^2 \, \Phi(dy)\, dx.$$

(See Fukushima [4] p. 29 and Deny [3].)
Let B be any subset of $\mathbb{R}^n$. Then the α-capacity of B ($\alpha > 0$ in general and $\alpha \geqq 0$ if X_t is transient) is given by

$$\operatorname{Cap}_\alpha(B) = \inf_{A \in Q B \subset A} \operatorname{Cap}_\alpha(A) \tag{1.3}$$

where Q is the family of all open sets in $\mathbb{R}^n$ and for $A \in Q$

$$\operatorname{Cap}_\alpha(A) = \begin{cases} \inf\limits_{u \in \mathscr{F}_A^\alpha} \varepsilon_\alpha(u, u) & \text{if } \mathscr{L}_A^\alpha \neq \phi \\ \infty & \text{if } \mathscr{L}_A^\alpha \neq \phi \end{cases} \tag{1.4}$$

where $\varepsilon_\alpha(u,u)=\alpha(u,u)+\varepsilon(u,u)$

$$\mathscr{L}_A^\alpha=\{u\in\mathscr{F}: u\geqq 1 \text{ a.e. on } A\}$$
$$\mathscr{L}_A^0=\{u\in\mathscr{F}_{(e)}: u\geqq 1 \text{ a.e. on } A\}.$$

$\mathscr{F}_{(e)}$ is a completion of $\mathscr{F}$ by ε which is well-defined if X_t is transient. (See [4] p. 61.)

Definition 1.1. A measure $\mu(dx)$ on $\mathbb{R}^n$ is *isotropic* if there exists a function $\mu_0(r)$ $(0<r<\infty)$ such that $\mu(dx)=\mu_0(|x|)\,dx$ for $x\neq 0$. $\mu(dx)$ is *isotropic unimodal* if $\mu_0(r)$ is nonincreasing. ($\mu(\{0\})$ may be positive).

Definition 1.2. A symmetric Lévy process X_t is *isotropic* if transition function $p_t(0,dx)$ $(t>0)$ is isotropic and X_t is *isotropic unimodal* if $p_t(0,dx)$ is isotropic unimodal.

In this paper, we shall show the following results.

Proposition. *Let X_t be symmetric Lévy process. Then the followings are equivalent.*

(1) *X_t is isotropic unimodal.*

(2) *$G_\alpha(0,dx)$ is isotropic unimodal for $\alpha>0$ where $G_\alpha(x,dy)=\int_0^\infty e^{-\alpha t}p_t(x,dy)$.*

(3) *If (S,Φ) is given by* (1.1) *for X_t, then $S=aI$ where $a\geqq 0$ and I is the identify matrix and Φ is isotropic unimodal.*

In the following, we shall denote by $B_a(x)$ a closed ball of radius a with centre x and by B_a a closed ball of radius a if the centre is not specified.

Theorem 1. *Let X_t be an isotropic unimodal process. Then for any nonempty Borel set B, it holds that for $\alpha>0$*

$$\mathrm{Cap}_\alpha(B)\geqq \mathrm{Cap}_\alpha(B^*)=F_\alpha\left(\left(\frac{m(B)}{C_n}\right)^{\frac{1}{n}}\right).$$

($\alpha\geqq 0$ if X_t is transient), where $m(dx)$ is the n-dimensional Lebesgue measure, B^* is a closed ball with

$$m(B)=m(B^*),\qquad C_n=m(B_1)=\frac{\pi^{\frac{n}{2}}}{\Gamma\left(1+\frac{n}{2}\right)}\qquad and \quad F_\alpha(a)=\mathrm{Cap}_\alpha(B_a).$$

We can show the converse of Theorem 1 in the following two cases. Let X_t be isotropic unimodal.

(H.1) $S=aI$ with $a>0$ and $\Phi=0$

(H.2) There exists a strictly decreasing function $\phi(r)$ $(0<r<\infty)$ such that $\Phi(dx)=\phi(|x|)\,dx$. $S=aI$ with $a>0$ or $\Phi(\mathbb{R}^n)=\infty$.

The condition $S\neq 0$ or $\Phi(\mathbb{R}^n)=\infty$ is equivalent to $p_t(0,\{0\})=0$ $(t>0)$.

Theorem 2. *If (S,Φ) given by* (1.1) *satisfies* (H.1) *or (H.2) and consequently X_t is an isotropic unimodal process, then for any nonempty compact set K the follow-*

ings are equivalent.

(1) $\mathrm{Cap}_\alpha(K) = F_\alpha\left(\left(\frac{m(K)}{C_n}\right)^{\frac{1}{n}}\right)$.

(2) $K = B \cup \Delta$ *with* $\mathrm{Cap}_\alpha(\Delta) = 0$, *where B is a ball with radius* $\left(\frac{m(K)}{C_n}\right)^{\frac{1}{n}}$. ($\alpha > 0$ *or* $\alpha \geqq 0$ *if* X_t *is transient.*)

The (isotropic) stable process of index β satisfies (H.1) if $\beta = 2$ and (H.2) if $0 < \beta < 2$, respectively. Since 0-capacity of transient stable process of index β ($0 < \beta \leqq 2$, $\beta < n$) is Riesz capacity of index β, Theorem 1 and Theorem 2 hold for Riesz capacity. In this case $F_0(R) = C(\beta, n) R^{n-\beta}$ where $C(\beta, n)$ is a constant which depends on β and n.

2. Definition of X_t^* and Proof of Proposition

Definition 2.1. For any nonnegative Lebesque measurable function $u(x)$, we shall define

$$u^*(x) = \sup\{t: \mu(t) > C_n |x|^n\},$$

where $\mu(t) = m\{x: u(x) > t\}$.

Remark. $u^*(x)$ is a right continuous nonincreasing function of $|x|$ with

$$m\{x: u^*(x) > t\} = m\{x: u(x) > t\}$$

for any $t > 0$. It also holds that

$$\int |u^*(x)|^p \, dx = \int |u(x)|^p \, dx. \qquad (p > 0).$$

Lemma 2.1. *Let* $f(x)$ *and* $\phi(x)$ *be nonnegative measurable function on* $\mathbb{R}^n$. *If* $f(x)$ *is a nondecreasing function of* $|x|$, *then*

$$\int f(x)\,\phi(x)\,dx \geqq \int f(x)\,\phi^*(x)\,dx. \tag{2.1}$$

Proof. Let B be any measurable set and I_B is an indicator of B. Then $I_B^* = I_{B^*}$ and

$$\int f(x)\,I_B(x)\,dx \geqq \int f(x)\,I_{B^*}(x)\,dx,$$

where B^* is an open ball of centre 0 with $m(B^*) = m(B)$. Set $B(\phi, t) = \{x: \phi(x) > t\}$. Then

$$I^*_{B(\phi,t)} = I_{B(\phi,t)^*} = I_{B(\phi^*,t)},$$

$$\begin{aligned}\int f(x)\,\phi(x)\,dx &= \int_0^\infty \left(\int_{\mathbb{R}^n} f(x)\,I_{B(\phi,t)}(x)\,dx\right)dt \\ &\geqq \int_0^\infty \left(\int_{\mathbb{R}^n} f(x)\,I_{B(\phi^*,t)}(x)\,dx\right)dt \\ &= \int f(x)\,\phi^*(x)\,dx.\end{aligned}$$

Let X_t be a symmetric Lévy process which corresponds to a pair (S, Φ) in (1.1). Set

$$S^* = (\det S) I$$
$$\Phi^*(dx) = \phi_0^*(x)\,dx \tag{2.2}$$

where $\phi_0(x)$ is a density function of the absolutely continuous part of $\Phi(dx)$. By Lemma 2.1

$$\int \frac{|x|^2}{1+|x|^2} \phi_0^*(x)\,dx \leqq \int \frac{|x|^2}{1+|x|^2} \phi_0(x)\,dx \leqq \int \frac{|x|^2}{1+|x|^2} \Phi(dx) < \infty.$$

Therefore, there exists a symmetric Lévy process X_t^* corresponding to (S^*, Φ^*).

Definition 2.2. For symmetric Lévy process corresponding to $(S. \Phi)$, X_t^* is a symmetric Lévy process corresponding to (S^*, Φ^*) given by (2.2).

Note that (3) in Proposition is equivalent to (3)′ $X_t = X_t^*$.

Proof of Proposition. The implication (3)→(1) is easily proved by Theorem 4 in [6], and Theorem 8.3 and Theorem 8.8 in [9]. The implication (1)→(2) is clear, since G_α is the Laplace transform of P_t. For any continuous function ψ on $\mathbb{R}^n - \{0\}$ with compact support

$$\lim_{\alpha\to\infty} \alpha^2 \int G_\alpha(0, dy)\, \psi(y) = \int \psi(x)\, \Phi(dx).$$

(See [4] Theorem 2.2.1.) Therefore $\Phi(dx)$ is an isotropic measure. Put for $0 < \delta < a$ and $0 < \beta$

$$\tilde{\phi}_a^{\delta\beta}(x) = \begin{cases} 0 & |x| \leqq a - \delta \\ \dfrac{|x| - a + \delta}{\delta} & a - \delta < |x| \leqq a \\ 1 & a < |x| \leqq \beta \\ \beta + 1 - |x| & \beta < |x| \leqq \beta + 1 \\ 0 & \beta + 1 < |x| \end{cases}$$

and $\phi_a^{\delta\beta}(x) = \dfrac{1}{x^{n-1}} \tilde{\phi}_a^{\delta\beta}(x)$. Since $G_\alpha(0, dx)$ is isotropic unimodal

$$\int (\phi_a^{\delta\beta} + \phi_b^{\delta\beta} - 2\phi_{\frac{a+b}{2}}^{\delta.\beta})\, \Phi(dx)$$
$$= \lim_{\alpha\to\infty} \alpha^2 \int (\phi_a^{\delta\beta} + \phi_b^{\delta\beta} - 2\phi_{\frac{a+b}{2}}^{\delta\beta})\, G_\alpha(0, dx) \geqq 0.$$

Tending $\delta \to 0$ and $\beta \to \infty$, we can show $\int_{|x| \geqq a} \frac{1}{|x|^{n-1}} \Phi(dx)$ is a convex function in a. Therefore the isotropic measure $\Phi(dx)$ is unimodal.

3. Proof of Theorem 1′

In this section we shall state and prove Theorem 1′, which is a generalization of Theorem 1.

Lemma 3.1 (Hardy-Littlewood-Polya [5] Theorem 379 and Brascamp Lieb and Luttingen [2]). *Let $u(x)$, $v(x)$ and $h(x)$ be any nonnegative Lebesgue measurable functions. Then it holds that*

$$(u, v, h) \leqq (u^*, v^*, h^*)$$

where $(u, v, h) = \iint u(x)\, v(y)\, h(x-y)\, dx\, dy$.

Lemma 3.2′. *Let $u \in \mathscr{F}(\varepsilon)$ and $u \geqq 0$, then $u^* \in \mathscr{F}(\varepsilon^*)$ and*

$$D_S(u, u) \geqq D_{S^*}(u^*, u^*), \tag{3.1}$$

$$J_\Phi(u, u) \geqq J_{\Phi^*}(u^*, u^*), \tag{3.2}$$

$$\varepsilon(u, u) \geqq \varepsilon^*(u^*, u^*) \tag{3.3}$$

where ε^* is the form corresponding to (S^*, Φ^*).

Proof. Let ε_S be the form corresponding to $(S, 0)$. First we shall show if $u \geqq 0$ and $u \in \mathscr{F}(\varepsilon_I)$, then

$$D_I(u, u) \geqq D_I(u^*, u^*). \tag{3.4}$$

Put $P_t(x) = \frac{1}{(2\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{2t}}$ and $T_t u(x) = \int P_t(y-x)\, u(y)\, dy$. Then by Lemma 3.1 for any $t > 0$

$$\begin{aligned}
\varepsilon^{(t)}(u, u) &= \frac{1}{t}(u - T_t u, u) \\
&= \frac{1}{t}\{(u, u) - (u, u, P_t)\} \\
&\geqq \frac{1}{t}\{(u^*, u^*) - (u^*, u^*, P_t)\} \\
&= \varepsilon^{(t)}(u^*, u^*).
\end{aligned}$$

By Lemma 1.3.4 in [4], we have $u^* \in \mathscr{F}(\varepsilon_I)$ and

$$D_I(u, u) = \lim_{t \to 0} 2\varepsilon^{(t)}(u, u) \geqq \lim_{t \to 0} 2\varepsilon^{(t)}(u^*, u^*) = D_I(u^*, u^*).$$

Next, we shall show if $u \geqq 0$ and $u \in \mathscr{F}(\varepsilon_S)$ then

$$D_S(u, u) \geqq D_{S^*}(u^*, u^*). \tag{3.5}$$

If $\det S = 0$, (3.5) is trivially true. If $\det S > 0$, then there exists a matrix σ such that $\det \sigma = 1$ and $S = (\det S)\, \sigma({}^t\sigma)$. Put $\hat{u}(x) = u(\sigma x)$, then $\hat{u} \in \mathscr{F}(\varepsilon_I)$ and $(\hat{u})^* = u^*$. Therefore by (3.4)

$$\begin{aligned}
D_S(u, u) &= \det S\, D_I(\hat{u}, \hat{u}) \geqq \det S\, D_I(u^*, u^*) \\
&= D_{S^*}(u^*, u^*).
\end{aligned}$$

In general, assume $u \geqq 0$ and $u \in \mathscr{F}(\varepsilon)$. Then $u \in \mathscr{F}(\varepsilon_S)$ and (3.1) holds. Let $\Phi_0(dx) = \Phi_0(x)\,dx$ be the absolutely continuous part of $\Phi(dx)$, and $\Phi_0^N(dx) = (N \wedge \phi_0(x))\,dx$. Then by Lemma 3.1,

$$\begin{aligned} J_{\Phi_0^N}(u,u) &= 2(u,u)\,\Phi_0^N(\mathbb{R}^n) - 2(u,u,N \wedge \phi_0) \\ &\geqq 2(u^*,u^*)\,(\Phi_0^*)^N(\mathbb{R}^n) - 2(u^*,u^*,N \wedge \phi_0^*) \\ &= J_{(\Phi_0^*)^N}(u^*,u^*). \end{aligned}$$

Tending $N \to \infty$, we have (3.2). By (3.1) and (3.2) $u \in \mathscr{F}(\varepsilon^*)$ and $\varepsilon(u,u) \geqq \varepsilon^*(u^*,u^*)$ is proved.

Lemma 3.3. *Let X_t be an isotropic Lévy process, and B be a closed ball. Then*

$$\operatorname{Cap}_\alpha B = \operatorname{Cap}_\alpha \dot{B}. \tag{3.6}$$

Moreover, if $P_t(0,\{0\}) = 0$ $(t>0)$, then

$$B = (\dot{B})^r \quad \text{and} \quad (\overline{B^c}) = (B^c)^r. \tag{3.7}$$

Proof. If x is in $\partial B = B - \dot{B}$ and $x \notin \dot{B}^r$ (or $x \notin B^c)^r$). Then for any rotation T_x around x, we have $x \notin (T_x \dot{B})^r$ (or $x \notin (T_x B^c)^r$). Therefore $x \notin (\mathbb{R}^n - \{x\})^r$ and $\{x\}$ is a finely open set. If $\dot{B}^r = B$, then (3.6) holds obviously. If there exists a point x in $B - \dot{B}^r$, then x is finely open and ∂B is also finely open, for X_t is isotropic. Since a finely open set with m-measure zero has no positive capacity (Lemma 4.2.4 in [4]), $\operatorname{Cap}_\alpha(\partial B) = 0$ and (3.6) also holds. If $P_t(0,\{0\}) = 0$ $(t>0)$, one point set can be finely open. (3.7) follows immediately from the above argument.

Lemma 3.4. *Let X_t be isotropic Lévy process. Put*

$$F_\alpha(a) = \operatorname{Cap}_\alpha(B_a) \quad \textit{for any } a \geqq 0 \quad \textit{and} \quad \alpha > 0$$

($\alpha \geqq 0$ if X_t is transient). Then $F_\alpha(a)$ is strictly increasing and continuous function of a and $\lim_{a\to\infty} F_\alpha(a) = \infty$.

Proof. It is clear that $F_\alpha(a)$ is strictly increasing and right continuous. The left continuity of $F_\alpha(a)$ follows from (3.6) in Lemma 3.3. Next suppose $\lim_{a\to\infty} F_\alpha(a) < \infty$. Then $\operatorname{Cap}_\alpha(\mathbb{R}^n) < \infty$ and $I_{\mathbb{R}^n} \in \mathscr{F}$ ($I_{\mathbb{R}^n} \in \mathscr{F}(e)$ if $\alpha = 0$ and X_t is transient). This is a contradiction.

Now we shall state and prove Theorem 1′. Notations are the same as in Theorem 1.

Theorem 1′. *Let X_t be a symmetric Lévy process and X_t^* be the process given in Definition 2.2 for X_t. Then for any nonempty Borel subset B of $\mathbb{R}^n$*

$$\operatorname{Cap}_\alpha^{X_t}(B) \geqq \operatorname{Cap}_\alpha^{X_t^*}(B^*) = F_\alpha\left(\left(\frac{m(B)}{C_n}\right)^{\frac{1}{n}}\right)$$

for $\alpha > 0$ ($\alpha \geqq 0$ if X_t is transient), when $F_\alpha(a) = \operatorname{Cap}_\alpha^{X_t^*}(B_a)$.

Remark. If X_t is isotropic unimodal, by Proposition $X_t = X_t^*$. Theorem 1 is a corollary of Theorem 1′.

Proof of Theorem 1′. Since $\mathrm{Cap}_\alpha(B) = \sup_{K \subset B,\, K:\text{compact}} \mathrm{Cap}_\alpha(K)$, by Lemma 3.4 we have only to prove the theorem when B is compact. Let K be any compact set. Then by problem 3.3.2 in [4]

$$\mathrm{Cap}_\alpha(K) = \inf_{u \in \mathscr{D}_K(\varepsilon)} \varepsilon_\alpha(u, u)$$

where $\mathscr{D}_K(\varepsilon) = \{u \in \mathscr{F} \cap C_0(\mathbb{R}^n) : u \geqq 1 \text{ on } K\, u \geqq 0 \text{ on } \mathbb{R}^n\}$

and $C_0(\mathbb{R}^n)$ is a set of all continuous functions on $\mathbb{R}^n$. Let a be the radius of K^*. If $u \in \mathscr{D}_K(\varepsilon)$, then by Lemma 3.2 $u^* \in \mathscr{D}_{B_a(0)}(\varepsilon^*)$ and

$$\begin{aligned}\inf_{u \in \mathscr{D}_K(\varepsilon)} \varepsilon_\alpha(u, u) &\geqq \inf_{u \in \mathscr{D}_K(\varepsilon)} \varepsilon_\alpha^*(u^*, u^*) \\ &\geqq \inf_{v \in \mathscr{D}_{B_a(0)}(\varepsilon^*)} \varepsilon_\alpha^*(v, v) = \mathrm{Cap}_\alpha^{X_t^*}(B_a(0)) = \mathrm{Cap}_\alpha^{X_t^*}(K^*).\end{aligned}$$

Therefore Theorem 1′ is proved for compact set K.

By Lemma 3.4 and Lemma 3.1.5 in [4], we immediately have the following Corollary.

Corollary. *Under the same assumption as in Theorem* 1′

$$F_\alpha\left(\left(\frac{m(\{x : |u(x)| > t\})}{c_n}\right)^{\frac{1}{n}}\right) \leqq \frac{1}{t^2} \varepsilon_\alpha(u, u) \tag{3.8}$$

for $u \in \mathscr{F}$ ($u \in \mathscr{F}_{(e)}$ *if* $\alpha = 0$ *and* X_t *is transient*) *and* $0 \leqq t < \sup_{x \in \mathbb{R}^n} |u(x)|$.

4. Proof of Theorem 2

Lemma 4.1. *Let* X_t *be an isotropic unimodal Lévy process and* $P_t(0, \{0\}) = 0$. *Then for any* $\alpha > 0$, $\dfrac{G_\alpha(0, dx)}{dx}$ *is continuous at* $x \neq 0$.[1]

Proof. Since X_t is isotropic unimodal, there exists a nonincreasing function $g(r)$ $(0 < r < \infty)$ such that $G_\alpha(0, dx) = g(|x|)\, dx$. Assume $g(r)$ is not continuous at r_0. On the other hand, we can take α-excessive density function $\tilde{g}(x)$ with $g(x) = \tilde{g}(x)$ a.e. ([1] VI). Since $\tilde{g}(x) = \lim_{t \downarrow 0} \int p_t(x, dy)\, \tilde{g}(y) = \lim_{t \to 0} \int p_t(x, dy)\, g(y)$ and X_t is right continuous,

$$\tilde{g}(x) \geqq g(r_0 -) \quad \text{if } |x| < r_0$$

and

$$\tilde{g}(x) \leqq g(r_0 +) \quad \text{if } |x| > r_0.$$

By Lemma 3.3, we know

$$\{|x| < r_0\}^r = \{|x| \leqq r_0\} \quad \text{and} \quad \{|x| > r_0\}^r = \{|x| \geqq r_0\}.$$

[1] If X_t is transient, Lemma 4.1 also holds for $\alpha \geqq 0$

Therefore by Proposition 2.10 in [1], if $|x_0|=r_0$

$$\tilde{g}(x_0)\geqq \inf_{|x|<r_0} \tilde{g}(x)\geqq g(r_0-)$$

and

$$\tilde{g}(x_0)\leqq \sup_{|x|>r_0} \tilde{g}(x)\leqq g(r_0+)$$

which is a contradiction.

Lemma 4.2. *Let X_t be an isotropic unimodal Lévy process and transient. If $u\in\mathscr{F}_{(e)}$, $u\geqq 0$ and $\sup_x u(x)<\infty$, then $u^*\in\mathscr{F}_{(e)}$ and $\varepsilon(u,u)\geqq\varepsilon(u^*,u^*)$.*

Proof. For any $t>0$, put $u_t=u-u\wedge t$. Then by Lemma 3.4 and (3.8), we have $m\{x: u_t(x)>0\}<\infty$, so u_t is in $L^p(R)$ for any $p\geqq 1$. Since u_t is a normal contraction of u, u_t is in $\mathscr{F}_{(e)}\cap L^2=\mathscr{F}$. So by Lemma 3.2 $(u_t)^*=(u^*)_t\in\mathscr{F}$ and

$$\varepsilon(u_t,u_t)\geqq\varepsilon((u^*)_t,(u^*)_t).$$

In the same way as in proof of Theorem 1.4.2 in [4], we can show

$$u_t\to u \quad \text{and} \quad (u^*)_t\to u^* \ (t\to 0)$$

strongly in $(\varepsilon,\mathscr{F}_{(e)})$. The Lemma is proved.

Lemma 4.3. *Let X_t be isotropical unimodal. Let $u\geqq 0$ and $u\in\mathscr{F}$, (or $u\geqq 0$, $u\in\mathscr{F}_{(e)}$ and u be bounded if X_t is transient). Suppose*

$$\varepsilon(u,u)=\varepsilon(u^*,u^*),$$

then

$$\varepsilon(u_t,u_t)=\varepsilon((u^*)_t,(u^*)_t),$$
$$\varepsilon(u^t,u^t)=\varepsilon((u^*)^t,(u^*)^t)$$

where $u_t=u-u\wedge t$ and $u^t=u\wedge t$.

Proof. Since $u=u_t+u^t$, we have

$$\begin{aligned}\varepsilon(u,u)&=\varepsilon(u_t,u_t)+2\varepsilon(u_t,u^t)+\varepsilon(u^t,u^t)\\&=\varepsilon(u^*,u^*)=\varepsilon(u_t^*,u_t^*)+2\varepsilon(u_t^*,u^{t*})+\varepsilon(u^{t*},u^{t*}).\end{aligned}\tag{4.1}$$

By Lemma 3.2 and Lemma 4.2,

$$\begin{aligned}\varepsilon(u_t,u_t)&\geqq\varepsilon((u^*)_t,(u^*)_t)\\\varepsilon(u^t,u^t)&\geqq\varepsilon((u^*)^t,(u^*)^t).\end{aligned}\tag{4.2}$$

Since $u^t(x)=t$ if $u_t(x)>0$, we have

$$(u_t,u^t)=t\,\|u_t\|_{L^1}=t\,\|(u^*)_t\|_{L^1}=((u^*)_t,(u^*)^t).$$

Since $G_\alpha(0,dx)$ is isotropic unimodal by Proposition, $(G_\alpha u_t,u^t)\leqq(G_\alpha(u^*)_t, (u^*)^t)$ by Lemma 3.1. where $G_\alpha v(x)=\int G_\alpha(x,dy)f(y)$. Therefore

$$\begin{aligned}\varepsilon^{(\alpha)}(u_t,u^t)&=\alpha(u_t-\alpha G_\alpha u_t,u^t)\\&\geqq\alpha((u^*)_t-\alpha G_\alpha(u^*)_t,(u^*)^t\\&=\varepsilon^{(\alpha)}((u^*)_t,(u^*)^t)\end{aligned}$$

and

$$\begin{aligned}\varepsilon(u_t, u^t) &= \lim_{\alpha\to\infty} \varepsilon^{(\alpha)}(u_t, u^t)\\ &\geqq \lim_{\alpha\to\infty} \varepsilon^{(\alpha)}((u^*)_t, (u^*)^t) = \varepsilon((u^*)_t, (u^*)^t).\end{aligned} \tag{4.3}$$

By (4.1), (4.2) and (4.3), Lemma 4.3 is proved.

Lemma 4.4. *Let $u, \phi \geqq 0$, u, $\phi \in L^1(\mathbb{R}^n)$, u be bounded and ϕ be a strictly decreasing function of $|x|$. Suppose*

$$(u, u, \phi) = (u^*, u^*, \phi) < \infty \tag{4.4}$$

holds, then

$$(u, u, h) = (u^*, u^*, h)$$

where $h \geqq 0$, $h \in L^1(\mathbb{R}^n)$ and h is a nonincreasing function of $|x|$.

Proof. By (4.4)

$$\int_0^\infty \{(u, u, I_{B(\phi,t)}) - (u^*, u^*, I_{B(\phi,t)})\}\, dt = 0$$

where $B(\phi, t) = \{x\colon \phi(x) \geqq t\}$. So by Lemma 3.1

$$(u, u, I_{B(\phi,t)}) = (u^*, u^*, I_{B(\phi,t)})$$

for a.e. $t > 0$. Since ϕ is strictly decreasing in $|x|$

$$(u, u, I_{B_a(0)}) = (u^*, u^*, I_{B_a(0)})$$

except countably many a ($a > 0$). Since both sides of the above equality are left continuous in a,

$$(u, u, I_{B(f,t)}) = (u^*, u^*, I_{B(h,t)})$$

and (4.3) holds.

Lemma 4.5. *Let f and g be nonnegative bounded functions in $L^1(\mathbb{R}^n)$ with $g = g^*$. Then there exists a function h such that*

$$(f^*, g) = (f, h) \quad \textit{and} \quad g = h^*.$$

Proof. Set $B(\phi, t) = \{x\colon \phi(x) \geqq t\}$ for any function ϕ on $\mathbb{R}^n$. We can choose sets $B(s)$ ($s \geqq 0$) such that

(i) $B(s)$ is decreasing in s
(ii) $B(0) = B(g, 0) = \mathbb{R}^n$, $m(B(s)) = m(B(g, s))$.
(iii) For s with $B(f^*, t+) \subset B(g, s) \subset B(f^*, t)$,

$$B(f, t+) \subset B(s) \subset B(f, t).$$

Then for any s and t

$$\begin{aligned}m(B(s) \cap B(f, t)) &= m(B(s)) \wedge m(B(f, t))\\ &= m(B(g, s) \cap B(f^*, t)).\end{aligned}$$

Set

$$h(x)=\sup\{s: x\in B(s)\},$$

then

$$B(h,s+)\subset B(s)\subset B(h,s).$$

Noting $m\{B(h,s+)\}=m(B(s))=m(B(h,s))$ except countably many s, we have

$$g=h^*$$

and

$$\begin{aligned}(f,h)&=\int_0^\infty\int_0^\infty ds\,dt\Big(\int_{\mathbb{R}^n} I_{B(f,t)}I_{B(h,s)}\,dx\Big)\\&=\int_0^\infty\int_0^\infty ds\,dt\Big(\int_{\mathbb{R}^n} I_{B(f^*,t)}I_{B(g,s)}\,dx\Big)\\&=(f^*,g).\end{aligned}$$

Lemma 4.6. *Let ρ be a smooth nonnegative function with compact support and $\rho=\rho^*$. Let u and ϕ be functions given in Lemma* 4.4. *Suppose*

$$(u,u,\phi)=(u^*,u^*,\phi)<\infty$$

then

$$(\rho*u,\rho*u,\phi)=((\rho*u)^*,(\rho*u)^*,\phi). \tag{4.6}$$

Proof. By Lemma 4.4, we have

$$\begin{aligned}(\rho*u,\rho*u,\phi)&=(u,u,\rho*\rho*\phi)\\&=(u^*,u^*,\rho*\rho*\phi)=(\rho*u^*,\rho*u^*,\phi).\end{aligned}\tag{4.7}$$

For any nonnegative bounded functions v and w with $v,w\in L^1(\mathbb{R}^n)$ and $w=w^*$, take h in Lemma 4.5 for $f=\rho*v$ and $g=w*\phi$, then by Lemma 3.1

$$\begin{aligned}((\rho*v)^*,w,\phi)&=((\rho*v)^*,w*\phi)\\&=(\rho*v,h)=(v,h,\rho)\\&\leqq(v^*,h^*,\rho)=(v^*,w*\phi,\rho)\\&=(\rho*v^*,w,\phi).\end{aligned}\tag{4.8}$$

Set $v=u$ and $w=(\rho*u)^*$ in (4.8), then

$$((\rho*u)^*,(\rho*u)^*,\phi)\leqq(\rho*u^*,(\rho*u)^*,\phi).$$

Set $v=u$ and $w=\rho*u^*$ in (4.8) again,

$$(\rho*u,(\rho*u)^*,\phi)\leqq(\rho*u^*,\rho*u^*,\phi).$$

Therefore

$$\begin{aligned}(\rho*u,\rho*u,\phi)&\leqq((\rho*u)^*,(\rho*u)^*,\phi)\\&\leqq(\rho*u^*,\rho*u^*,\phi).\end{aligned}\tag{4.9}$$

By (4.7) and (4.9), we have proved (4.6).

Lemma 4.7 (Talenti [10]). *Put $\mathscr{D}_0 = \{u: u$ is Lipschitz continuous and of compact support$\}$. Suppose $u \in \mathscr{D}_0$ and $u \geqq 0$. Then*

$$D_I(u, u) = D_I(u^*, u^*)$$

if and only if

(i) $\{x: u(x) \geqq t\}$ *is a closed ball a.e.* $t \geqq 0$

(ii) $|$*gradient* $u|$ *is constant on* $\{x: u(x) = t\}$ *a.e.* $t \geqq 0$.

Proof of Theorem 2. To prove Theorem 2, it is sufficient to show that if a nonempty compact set K satisfies $\mathrm{Cap}_\alpha(\mathrm{K}) = \mathrm{Cap}_\alpha(B_a)$ and $m(K) = m(B_a)$ for some $a > 0$, then (2) in Theorem 2 holds.

Put

$$e_K(x) = \begin{cases} E_x(e^{-\alpha\sigma_K}) & \text{if } \alpha > 0 \\ P_x(\sigma_K < \infty) & \text{if } \alpha = 0 \end{cases}$$

where σ_K is the hitting time to K. Then $\mathrm{Cap}_\alpha(K) = \varepsilon_\alpha(e_K, e_K)$. Since $m(B_a) = m(K) = m(K^r) \leqq m(e_K \geqq 1) = m(e_K^* \geqq 1)$, $e_K^* \geqq 1$ on $\dot{B}_a(0)$ a.e. and by Lemma 3.2 and Lemma 4.2

$$\mathrm{Cap}_\alpha(K) = \varepsilon_\alpha(e_K, e_K) \geqq \varepsilon_\alpha(e_K^*, e_K^*) \geqq \mathrm{Cap}_\alpha(B_a(0)).$$

Therefore by assumption

$$\begin{aligned} \varepsilon_\alpha(e_K, e_K) &= \varepsilon_\alpha(e_K^*, e_K^*) \\ e_K^* &= e_{B_a(0)} \quad \text{a.e.} \end{aligned} \tag{4.10}$$

Case I When (H.1) *Holds.* In this case, X_t is the Brownian motion. Put $u = e_K$. Since u is lower semicontinuous, we can choose s and t such that $0 < s < t < \inf_{x \in K} u(x)$.[2] Then by Lemma 4.3

$$\varepsilon_\alpha(u_s^t, u_s^t) = \varepsilon_\alpha((u_s^t)^*, (u_s^t)^*)$$

and $u_s^t \in \mathscr{D}_0$, for u is α-harmonic outside K. Therefore by Lemma 4.7 we can choose r such that $s < r < t$ and

$$\{u \geqq r\} = \{u_s^t \geqq r - s\} = B_c(x_0)$$

for some closed ball $B_c(x_0)$. On the other hand by (4.10)

$$m\{e_{B_a(x_0)} \geqq r\} = m\{e_{B_a(0)} \geqq r\} = m\{u^* \geqq r\} = m\{u \geqq r\},$$

$$u = e_{B_a(x_0)} = r \quad \text{on } \partial B_c(x_0), \quad B_c(x_0) \supset K$$

and $u = e_{B_a(x_0)}$ outside $B_c(x_0)$. Let U be the outer connected component of $\{B_a(x_0) \cup K\}^c$, then by principle of coincidence of α-harmonic function

$$e_K = u = e_{B_a(x_0)} \quad \text{in } U.$$

---

[2] Since u is not identically zero, it is positive everywhere

Assume $K^r \not\subset B_a(x_0)$, and put

$$A=\{X_{\sigma_K}\in K^r,\ \sigma_K<\sigma_{B_a(x_0)}\}$$

then $P_x(A)>0$ for $x\in U$.[3] Take a sample w in A, then $\lim_{t\uparrow\sigma_K} e_K(X_t(w)=1$, $\lim_{t\uparrow\sigma_K} e_{B_a(x_0)}(X_t(w))<1$ and $X_t(w)\in U$ for $0<t<\sigma_K$, which is a contradiction. Therefore $K^r\subset B_a(x_0)$ and

$$\mathrm{Cap}_\alpha(K-B_a(x_0))=0. \tag{4.11}$$

Hence $m(K-B_a(x_0))=0$ and then

$$m(K\cap B_a(x_0))=m(K)=m(B_a(x_0)).$$

Since K is compact, we can see $B_a(x_0)\subset K$. (2) in Theorem 2 is derived.

Case II When (H.2) *Holds.* Put $u=e_K$ again, then by (4.10) and Lemma 4.3 $\varepsilon_\alpha(u_t, u_t)=\varepsilon_\alpha(u_t^*, u_t^*)$ for any $t>0$. By (H.2), we can decompose Φ in such way that

$$\Phi=\Phi_1+\Phi_2$$

where Φ_i $(i=1,2)$ are isotropic unimodal and $\phi_1(x)=\dfrac{\Phi_1(dx)}{dx}$ $(x\neq 0)$ is strictly decreasing and in $L^1(\mathbb{R}^n)$. Then by (4.10)

$$J_{\Phi_1}(u_t, u_t)=J_{\Phi_1}(u_t^*, u_t^*)^4 \quad \text{or} \quad (u_t, u_t, \phi_1)=(u_t^*, u_t^*, \phi_1).$$

Let ρ be a smooth function on $\mathbb{R}^n$ such that

$$\rho=\rho^*,\ \int\rho(x)\,dx=1,\ \rho\geqq 0,\ \rho(0)>0 \tag{4.12}$$

and support $\rho\subset B_\delta(0)$ for some $\delta>0$. Then by Lemma 4.4 and Lemma 4.6

$$(\rho*u_t, \rho*u_t, h_s)=((\rho*u_t)^*, (\rho*u_t)^*, h_s)$$

where $h_s(x)=\dfrac{1}{(2\pi s)^{\frac{n}{2}}}e^{\frac{|x|^2}{2s}}$. Since in general

$$D_I(v,v)=\lim_{s\downarrow 0}\frac{1}{s}\{(v,v)-(v,v,h_s)\}$$

for v with $D_I(v,v)<\infty$, we can see

$$D_I(\rho*u_t, \rho*u_t)=D_I((\rho*u_t)^*, (\rho*u_t)^*),$$

and $\rho*u_t\in\mathscr{D}_0$. Therefore by Lemma 4.7

$$\{\rho*u_t\geqq r\} \text{ is a closed ball for a.e. } r>0.$$

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[3] Since $P_x(A)=1$ if $x\in K^r-B_a(x_0)$, it is positive on $R^n-B_a(x_0)$

[4] By the similar way as the proof of (3.2) in Lemma 3.2, we can show

$$J_{\phi_i}(u_t, u_t)\geqq J_{\phi_i}(u_t^*, u_t^*) \quad (i=1,2)$$

Therefore $B_{\rho,t}=\overline{\{\rho * u_t>0\}}$ is a closed ball. Set $B=\bigcap_{\rho,t} B_{\rho,t}$, where ρ ranges over smooth functions satisfying (4.12) and $0<t<1$. B is a closed ball. Since $S \neq 0$ or $\Phi(\mathbb{R}^n)=\infty$, $P_t\{0,\{0\}\}=0$ and by Lemma 4.1 $\frac{G_\alpha(0,dx)}{dx}$ is continuous at $x \neq 0$. Therefore u is lower semicontinuous in $\mathbb{R}^n$ and continuous in $\mathbb{R}^n-K$. Set $K_0=\{x: u(x)=1\}$ then $K^r \subset K_0$, $\mathrm{Cap}_\alpha K_0 \leqq \varepsilon_\alpha(u,u)=\mathrm{Cap}_\alpha K=\mathrm{Cap}_\alpha K^r$ and $\mathrm{Cap}_\alpha K=\mathrm{Cap}_\alpha K_0$. If $x \in K_0$, then $\lim_{y\to x} u(y)=1=u(x)$ and $x \in B$. If $x \notin K \cup K_0$, then $\lim_{y\to x} u(y)=u(x)<1$ and $x \notin B$. Therefore $K_0 \subset B \subset K \cup K_0 \subset K_0 \cup (K-K^r)$. Since $\mathrm{Cap}_\alpha(K-K^r)=0$, $\mathrm{Cap}_\alpha B=\mathrm{Cap}_\alpha K_0=\mathrm{Cap}_\alpha K=\mathrm{Cap}_\alpha B_a$. So, the radius of B is a and $m(K)=m(B)$. Therefore $K=K^r \cup \Delta_0 \subset B \cup \Delta_0$ with $\mathrm{Cap}_\alpha \Delta_0=0$. Since K is compact, $B \subset K \subset B \cup \Delta_0$ which proves the theorem.

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