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On Convergence in Variation of the Distributions of Multivariate Point Processes

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1. Introduction

The aim of the present paper is to establish necessary and sufficient condition for convergence in variation of the distributions of multivariate point processes given by their compensators.

The main result is the following: if the limiting compensator is finite then the convergence in variation holds if and only if the variations of the differences between compensators tends to zero "in probability". The precise statement is given in Theorem 1. It extends Theorems 6 and 7 of [6] where the case of counting processes has been considered under assumption that the limiting process has a nonrandom compensator, and is therefore a process with independent increments.

The proof of the part A) of Theorem 1 is based on important inequalities which give upper bounds for the variation distance between compensators (Theorem 2). The technique used in the proof of Theorem 2 originates from the proof of Theorem 2 in [8] and based heavily on the structure and properties of a density process. (It is interesting to note that, in contrast with Theorem 1, sufficient conditions for weak convergence of finite-dimensional distributions for counting processes in terms of their compensators (see [1,8]) are not necessary. There is a counter example communicated to us by E.L. Presman and I.M. Sonin.)

As an example of an application of Theorem 2, we give an estimate of the rate of convergence of an empirical distribution function to a Poisson process.

The structure of the paper is the following. In Sect. 2 we give the statement of principle results. The Sects. 3 and 4 contain the proofs of Theorem 2 and 1. The Sect. 5 is devoted to some generalizations of Theorems 1 and 2. In Sect. 6 we give an example.

2. Main Results

Let $(\Omega, \mathscr{F}, F=(\mathscr{F}_t)_{t\geq 0})$ be a measurable space with a right-continuous filtration F, $\mathscr{F}=\bigvee_{t\geq 0}\mathscr{F}_t$, $\mathscr{P}=\mathscr{P}(F)$ be the σ-algebra of predictable sets on $\Omega\times R_+$. Let

$(E, \mathscr{E})$ be a Lusin space (i.e.: E is a Borel subset of compact metric space), Δ be an extra point, $\hat{\mathscr{P}} = \mathscr{P} \otimes \mathscr{E}$ be a σ-algebra on $\Omega \times R_+ \times E$.

Suppose that a multivariate point process $\Pi = (T_k, X_k)_{k \geqq 1}$ is given on Ω ([4]). Here $(T_k)_{k \geqq 1}$ is an increasing sequence of strictly positive stopping times with respect to F such that $T_k < T_{k+1}$ on $(T_k < \infty)$, $k = 1, 2, \ldots$. Random variables X_k with values in $(E \cup \{\Delta\}, \mathscr{E} \vee \Delta)$ are $\mathscr{F}_{T_k}$-measurable and $X_{T_k}(\omega) = \Delta$ if and only if $T_k(\omega) = \infty$. Put $T_\infty = \lim_k T_k$.

Let $\mu = \mu(\omega; dt, dx)$ be an integer-valued random measure on $(R_+ \times E, \mathscr{B}(R_+) \otimes \mathscr{E})$ associated with Π. By definition

$$\mu(\omega; dt, dx) = \sum_{k \geqq 1} \varepsilon_{(T_k(\omega), X_k(\omega))}(dt, dx),$$

where ε_a denotes the Dirak measure located at point a.

Let P be a probability measure on $(\Omega, \mathscr{F})$. There exists (unique up to a modification on a P-null set) random measure $v = v(\omega; dt, dx)$ called the compensator (or dual predictable projection) of μ such that for any $H \in \hat{\mathscr{P}}$, $H \geqq 0$, the process $H * v = (H * v_t)_{t \geqq 0}$ is predictable and $E(H * v_\infty) = E(H * \mu_\infty)$. Throughout the paper we use the brief notation for Lebesgue integrals, e.g.

$$H * v_t = \int_{[0,t] \times E} H(s, x)\, v(ds, dx), \qquad 1 * v_t = \int_{[0,t] \times E} v(ds, dx).$$

If E reduces to one point then the multivariate point process is an ordinary point process which is completely described by the counting process N and its compensator A where $N_t = \mu([0, t] \times E)$ and $A_t = v([0, t] \times E)$.

If $m = m(dt, dx)$ is a measure on $(R_+ \times E, \mathscr{B}(R_+) \otimes \mathscr{E})$ then $\mathrm{Var}_T(m)$ is the total variation of m restricted to $[0, T] \times E$.

For a random measure η, the measure M_η^P on $\hat{\mathscr{P}}$ is defined by $M_\eta^P(H) = E(H * \eta_\infty)$ where $H \in \hat{\mathscr{P}}$, $H \geqq 0$.

Denote by $\mathscr{T}(F)$ the set of all stopping times with respect to F. For $S \in \mathscr{T}(F)$, let $P_S = P | \mathscr{F}_S$ (restriction to the σ-algebra $\mathscr{F}_S$) and $\eta^S = I_{[\![0, S]\!]}\eta$.

Denote by F^P the filtration obtained from F by usual completion with respect to P. Put $\mathscr{F}_t^\mu = \sigma(\mu([0, s] \times \Gamma), s \leqq t, \Gamma \in \mathscr{E})$, $F^\mu = (\mathscr{F}_t^\mu)_{t \geqq 0}$.

Let $(P^n)_{n \geqq 1}$ be a sequence of probability measures on $(\Omega, \mathscr{F})$, v^n be a compensator of μ with respect to P^n. To formulate our result, we introduce the following conditions:

(a) $\lim_n \mathrm{Var}(P_T^n - P_T) = 0$;

(b) $\lim_n P^n(\mathrm{Var}_T(v^n - v) \geqq \varepsilon) = 0, \quad \forall \varepsilon > 0$;

(c) $1 * v_T < \infty$ P-a.s.

Theorem 1. *Assume $F = F^\mu$. Then for any $T \in \mathscr{T}(F)$ we have*

A) (b) $\Rightarrow$ (a),

B) (a), (c) $\Rightarrow$ (b).

The next theorem gives estimates of the rate of convergence and will be used to prove part A) of Theorem 1.

Let $\tilde{P}$ be a probability measure on $(\Omega, \mathscr{F})$ and $\tilde{\nu}$ be the compensator of μ corresponding to $(F, \tilde{P})$. We shall denote by $\tilde{E}$ the expectation with respect to $\tilde{P}$.

Theorem 2. *Let $F=F^\mu$ and $T\in\mathscr{T}(F)$. Then*

$$\operatorname{Var}(P_T-\tilde{P}_T)\leqq 2(E\operatorname{Var}_T(\nu-\tilde{\nu})\wedge \tilde{E}\operatorname{Var}_T(\nu-\tilde{\nu})), \tag{1}$$

$$\operatorname{Var}(P_T-\tilde{P}_T)\leqq 3\,\varepsilon+2(P(\operatorname{Var}_T(\nu-\tilde{\nu})\geqq\varepsilon)\wedge \tilde{P}(\operatorname{Var}_T(\nu-\tilde{\nu})\geqq\varepsilon)) \tag{2}$$

for any $\varepsilon>0$.

Remark 1. The assertion of Theorem 2 is still true if $F=F^{\mu,Q}$ with

$$Q=(1/2)(P+\tilde{P}).$$

Remark 2. The inequality (1) has been proven independently by J. Mémin for the case of counting processes with bounded compensators (see [9]).

3. Proof of Theorem 2

1°. We start from some auxiliary results.

Lemma 1. *Let $(\Omega,\mathscr{F})$ be a measurable space with right-continuous filtration F and two probability measures P and $\tilde{P}$, $\mathscr{F}=\bigvee_{t\geqq 0}\mathscr{F}_t$. Let $(S_n)_{n\geqq 1}$ be an increasing sequence of F-stopping times. Suppose that one of the following conditions holds:*

(i) *$S_n\uparrow\infty$ P-a.s. (or $\tilde{P}$-a.s.)*

(ii) *$F=F^\mu$ and $S_n\uparrow T_\infty$ P-a.s. (or $\tilde{P}$-a.s.).*

Then

$$\lim_n \operatorname{Var}(P_{S_n}-\tilde{P}_{S_n})=\operatorname{Var}(P-\tilde{P}). \tag{3}$$

Proof. Put $Q=(1/2)(P+\tilde{P})$. In accordance with § 3 of [7] there exists right-continuous (F^Q, Q)-martingales Z and $\tilde{Z}$ such that for any $S\in\mathscr{T}(F^Q)$

$$Z_S=dP_S/dQ_S,\quad \tilde{Z}_S=d\tilde{P}_S/dQ_S,\quad Z\geqq 0,\tilde{Z}\geqq 0,\quad Z+\tilde{Z}=2\quad Q\text{-a.s.}$$

The definition of variation implies that

$$\operatorname{Var}(P_{S_n}-\tilde{P}_{S_n})=E_Q|Z_{S_n}-\tilde{Z}_{S_n}|,\quad \operatorname{Var}(P-\tilde{P})=E_Q|Z_\infty-\tilde{Z}_\infty| \tag{4}$$

where E_Q denotes expectation with respect to Q.

Put $S=\lim_n S_n$, $\Gamma=\bigcap_n (S_n<S)$, $\eta=\lim_n Z_{S_n}$, $\tilde{\eta}=\lim_n \tilde{Z}_{S_n}$. Assume (i), i.e. $P(S=\infty)=1$. We prove that

$$\eta=Z_\infty=0\quad\text{on } (S<\infty)\quad Q\text{ a.s.}, \tag{5}$$

$$\tilde{\eta}=\tilde{Z}_\infty=2\quad\text{on } (S<\infty)\quad Q\text{ a.s.} \tag{6}$$

Let $\mathscr{F}' = \bigvee_n \mathscr{F}_{S_n}$, $P', \tilde{P}', Q'$ be the restrictions of $P, \tilde{P}, Q$ to $\mathscr{F}'$. Then $\eta = dP'/dQ'$, $\tilde{\eta} = d\tilde{P}'/dQ'$ and $\Gamma \cap (S<\infty)$, $\bar{\Gamma} \cap (S<\infty) \in \mathscr{F}'$. From (i) we have

$$E_Q \eta I_{\Gamma \cap (S<\infty)} = P(\Gamma \cap (S<\infty)) = 0.$$

Thus, $\eta = Z_{S-} = 0$ on $\Gamma \cap (S<\infty)$ Q-a.s. Analogously $\eta = Z_S = 0$ on $\bar{\Gamma} \cap (S<\infty)$ Q-a.s. The well known property of non-negative martingales implies

$$(Z_{(S \wedge T)-} = 0) \subseteq (Z_{S \wedge T} = 0) \subseteq (Z_T = 0) \qquad Q\text{-a.s.}$$

Hence, (5) holds. But (6) is a consequence of (5) in virtue of the identity $Z + \tilde{Z} = 2$ Q-a.s.

Evidently, (3) follows from (4)–(6).

To prove (3) under (ii) we remark that $Z_{T_\infty -} = Z_{T_\infty}$, $Z^{T_\infty} = Z$, $\tilde{Z}_{T_\infty -} = \tilde{Z}_{T_\infty}$, $\tilde{Z}^{T_\infty} = \tilde{Z}$ Q-a.s. The same considerations as above show that $\eta = Z_{T_\infty} = 0$, $\tilde{\eta} = \tilde{Z}_{T_\infty} = 2$ on $(S < T_\infty)$ Q-a.s. Thus, (3) is valid under (ii) also.

Lemma 2. *Assume $F = F^\mu$. Let ν be the compensator of μ with respect to (F, Q), $\mu^S = I_{[\![0,S]\!]}\mu$, $S \in \mathscr{F}(F)$.*

Then

(i) $\mathscr{F}^{\mu^S} = \mathscr{F}_S^\mu$,

(ii) *ν^S is the compensator of μ^S with respect to (F^{μ^S}, P_S).*

Proof. (i) Obviously, $\mathscr{F}^{\mu^S} \subseteq \mathscr{F}_S$. From the other hand, (3.40) of [4] implies that the family of sets $((T_k, X_k) \in \Gamma) \cap (T_n \leqq S \leqq T_{n+1})$, $k \leqq n < \infty$, $((T_k, X_k) \in \Gamma) \cap (T_\infty \leqq S)$, $k < \infty$, $\Gamma \in \mathscr{B}(R_+) \otimes \mathscr{E}$ generates $\mathscr{F}_S^\mu$. It is easily seen that any set of this family belongs to $\mathscr{F}^{\mu^S}$. Thus, $\mathscr{F}_S^\mu = \mathscr{F}^{\mu^S}$.

(ii) It is sufficient to prove the implication

$$X \in \mathscr{P}(F^\mu) \;\Rightarrow\; X^S \in \mathscr{P}(F^{\mu^S}).$$

Let $X \in \mathscr{P}(F^\mu)$ be left-continuous. For any t $X_{S \wedge t}$ is $\mathscr{F}_{S \wedge t}^\mu$-measurable. But in virtue of (i) $\mathscr{F}_{S \wedge t}^\mu = \mathscr{F}^{\mu^{S \wedge t}} = \mathscr{F}_t^{\mu^S}$. Hence, $X^S \in \mathscr{P}(F^{\mu^S})$. The general case follows by monotone class arguments.

Lemma 3. *Let P and $\tilde{P}$ be probability measures defined on a measurable space $(\Omega, \mathscr{F})$ with $\mathscr{F} = \bigvee_{t \geqq 0} \mathscr{F}_t$. Let ν and $\tilde{\nu}$ be the F^μ-compensators of μ with respect to P and $\tilde{P}$, $S \in \mathscr{T}(F^\mu)$. Suppose $\nu^S = \tilde{\nu}^S$ P-a.s. Then $P_S = \tilde{P}_S$.*

Proof. The assertion is a corollary of the uniqueness Theorem 3.4 of [4] applied to μ^S and of Lemma 2.

2°. *Proof of Theorem 2.* At first we remark that it is sufficient to prove (1) for the case $T = \infty$. Indeed, let P' and $\tilde{P}'$ be the probability measures on $(\Omega, \mathscr{F})$ corresponding to the compensators $\nu' \equiv \nu^T$ and $\tilde{\nu}' \equiv \tilde{\nu}^T$, $T \in \mathscr{T}(F)$. By virtue of Lemma 3 we have $P_T' = P_T$ and $\tilde{P}_T' = \tilde{P}_T$. Hence, with obvious notations,

$$\operatorname{Var}(P_T - \tilde{P}_T) \leqq \operatorname{Var}(P' - \tilde{P}'),$$

$$E' \operatorname{Var}_\infty(\nu' - \tilde{\nu}') = E \operatorname{Var}_T(\nu - \tilde{\nu})$$

and

$$\tilde{E}' \operatorname{Var}_\infty(\nu' - \tilde{\nu}') = \tilde{E} \operatorname{Var}_T(\nu - \tilde{\nu}).$$

These relations together with inequality (1) for P', $\tilde{P}'$ and $T=\infty$ imply that (1) holds for P, $\tilde{P}$ and $T \in \mathscr{T}(F)$.

From now on we suppose that $T=\infty$.

Let us show that it is sufficient to varify the validity of inequality

$$\operatorname{Var}(P - \tilde{P}) \leqq 2(E \operatorname{Var}_\infty(\nu - \tilde{\nu}) \wedge \tilde{E} \operatorname{Var}_\infty(\nu - \tilde{\nu})) \tag{7}$$

under assumption

$$1 * \mu_\infty + 1 * \nu_\infty + 1 * \tilde{\nu}_\infty \leqq c = \text{const.} \tag{8}$$

In fact, assume that (7) is true under (8). Let consider the probability measures P^n and $\tilde{P}^n$ corresponding to the compensators $\nu^n \equiv \nu^{S_n}$ and $\tilde{\nu}^n \equiv \tilde{\nu}^{S_n}$ where $S_n = \inf(t: 1 * \nu_t + 1 * \tilde{\nu}_t \geqq n) \wedge T_n$. By assumption (8) we have

$$\operatorname{Var}(P^n - \tilde{P}^n) \leqq 2(E^n \operatorname{Var}_\infty(\nu^n - \tilde{\nu}^n) \wedge \tilde{E}^n \operatorname{Var}_\infty(\nu^n - \tilde{\nu}^n)) \tag{9}$$

where $E^n, \tilde{E}^n$ denote the expectations with respect to $P^n, \tilde{P}^n$. From this and Lemma 3 we easily obtain that

$$\operatorname{Var}(P_{S_n} - \tilde{P}_{S_n}) \leqq 2(E \operatorname{Var}_\infty(\nu - \tilde{\nu}) \wedge \tilde{E} \operatorname{Var}_\infty(\nu - \tilde{\nu})) \tag{10}$$

If $P(S = T_\infty) = 1$ or $\tilde{P}(S = T_\infty) = 1$ where $S = S_\infty$ then (10) implies (7) by virtue of Lemma 1. But if

$$P(S = T_\infty) < 1 \quad \text{and} \quad \tilde{P}(S = T_\infty) < 1 \tag{11}$$

then (7) is trivial because the right-hand side of (7) is equal to $+\infty$ in this case. Indeed, it follows from the definition of the compensator that $\inf(t: 1 * \nu_t = \infty) = T_\infty$ P-a.s. and $\inf(t: 1 * \tilde{\nu}_t = \infty) = T_\infty$ $\tilde{P}$-a.s. Hence, $1 * \nu_S < \infty$ and $1 * \tilde{\nu}_S = \infty$ on $(S < T_\infty)$ P-a.s. and the first inequality of (11) implies that

$$E \operatorname{Var}_\infty(\nu - \tilde{\nu}) \geqq E |1 * \nu_S - 1 * \tilde{\nu}_S| = +\infty.$$

Analogously, the second inequality of (11) implies that

$$\tilde{E} \operatorname{Var}_\infty(\nu - \tilde{\nu}) = +\infty.$$

So, from now on we shall suppose (8). Under the assumption $\tilde{P} \ll P$ we prove the inequality

$$\operatorname{Var}(P - \tilde{P}) \leqq 2 \tilde{E} \operatorname{Var}_\infty(\nu - \tilde{\nu}). \tag{12}$$

By the theorem on absolute continuity of measures for multivariate point processes ([4], Th. 4.1, [7], Th. 21) there is $\hat{\mathscr{P}}$-measurable non-negative finite function $Y = Y(t, X)$ such that $\tilde{\nu} = Y \nu$ $\tilde{P}$-a.s. Moreover, $\tilde{P}(\exists t: \nu(\{t\} \times E) = 1 \nRightarrow \tilde{\nu}(\{t\} \times E) = 1) = 0$.

Let $Z=(Z_t)_{t\geqq 0}$ be a cadlag version of (P, F)-martingale $E(d\tilde{P}/dP|\mathscr{F}_t)$. Then

$$Z_t=1+Z_-(Y-1+(1-a)^{\oplus}(\hat{Y}-a))*(\mu-\nu)_t \tag{13}$$

where $a_t=\nu(\{t\}\times E)$, $\hat{Y}_t=\int_E Y(t,x)\,\nu(\{t\},dx)$, $b^{\oplus}=b^{-1}$ if $b\neq 0$ and $b^{\oplus}=0$ if $b=0$. By virtue of the assumption $\tilde{P}\ll P$ we have

$$\operatorname{Var}(P-\tilde{P})=E|1-Z_\infty|. \tag{14}$$

Put $\Gamma_1=((\omega,t)\colon a_t(\omega)=0)$, $\Gamma_2((\omega,t)\colon 0<a_t(\omega)<1)$, $\xi=I_{(a=1)}*(\mu-\nu)_\infty$. Evidently $\xi\leqq 0$, $E\xi=0$. Hence $\xi=0$ P-a.s.

It follows that (P-a.s.)

$$|1-Z_\infty|\leqq J_1+J_2$$

where

$$J_i=|Z_-I_{\Gamma_i}(Y-1+(1-a)^{-1}(\hat{Y}-a))*(\mu-\nu)_\infty|, \qquad i=1,2.$$

It easy to see that

$$EJ_1\leqq E(Z_-I_{\Gamma_1}|Y-1|*(\mu+\nu)_\infty)=2E(Z_-I_{\Gamma_1}|Y-1|*\nu_\infty). \tag{15}$$

Let us show that analogous inequality holds for EJ_2 as well:

$$EJ_2\leqq 2E(Z_-I_{\Gamma_2}|Y-1|*\nu_\infty). \tag{16}$$

On Γ_2 we have

$$\begin{aligned}&\int_E (Y(s,x)-1+(1-a_s)^{-1}(\hat{Y}_s-a_s))\,\nu(\{s\},dx)\\ &\quad=\hat{Y}_s-a_s+(1-a_s)^{-1}(\hat{Y}_s-a_s)\,a_s,\end{aligned}$$

thus

$$\begin{aligned}J_2=&|\sum_{s>0} Z_{s-}I_{\Gamma_2}(s)(\int_E (Y(s,x)-1)\,\mu(s,dx)\\ &-(1-a_s)^{-1}(\hat{Y}_s-a_s)(1-\mu(\{s\}\times E))|.\end{aligned}$$

This representation implies that

$$EJ_2\leqq E(Z_-I_{\Gamma_2}|Y-1|*\mu_\infty)+E(\sum_{s>0} Z_{s-}I_{\Gamma_2}(s)|\hat{Y}_s-a_s|).$$

To finish the proof of (16) note that

$$|\hat{Y}_s-a_s|=|\int_E (Y(s,x)-1)\,\nu(\{s\},dx)|\leqq\int_E |Y(s,x)-1|\,\nu(\{s\},dx).$$

Hence

$$E\sum_{s>0} Z_{s-}I_{\Gamma_2}(s)|\hat{Y}_s-a_s|\leqq E(Z_-I_{\Gamma_2}|Y-1|*\nu_\infty).$$

It follows from (15) and (16) that

$$E|1-Z_\infty|\leqq 2E(Z_-|Y-1|*\nu_\infty). \tag{17}$$

By virtue of IV-T-47, V-T-27 ([3]) (see, also, Th. 1.47 of [4])

$$E(Z_- |Y-1| * \nu_\infty) = EZ_\infty(|Y-1| * \nu_\infty) = \tilde{E}(|Y-1| * \nu_\infty) = \tilde{E}\,\mathrm{Var}(\nu - \tilde{\nu}).$$

The desirable inequality (12) follows from this and from (14), (17).

Let us show that (12) is valid without assumption $\tilde{P} \ll P$. Put $\nu_\varepsilon = \varepsilon\tilde{\nu} + (1-\varepsilon)\nu$. Let P^ε be the probability measure on $(\Omega, \mathscr{F})$ such that ν_ε is the compensator of μ with respect to (F, P^ε). Existence of P^ε follows from (3.6) in [5]. Under (8) P and $\tilde{P}$ are absolute continuous with respect to P^ε by Th. 21 of [7]. As we proved

$$\mathrm{Var}(P - P^\varepsilon) \leqq 2E\,\mathrm{Var}_\infty(\nu - \nu_\varepsilon),$$
$$\mathrm{Var}(\tilde{P} - P^\varepsilon) \leqq 2\tilde{E}\,\mathrm{Var}_\infty(\nu - \nu_\varepsilon).$$

Since $\mathrm{Var}_\infty(\nu - \nu_\varepsilon) = \varepsilon\,\mathrm{Var}_\infty(\nu - \tilde{\nu})$, $\mathrm{Var}_\infty(\tilde{\nu} - \nu_\varepsilon) = (1-\varepsilon)\,\mathrm{Var}_\infty(\nu - \tilde{\nu})$, we obtain (12) letting $\varepsilon \to 0$.

At last, the inequality (7) holds by virtue of symmetry between P and $\tilde{P}$.

To prove (2) assume at first that T is bounded. Let $Q = (1/2)(P + \tilde{P})$, $S = \inf(t: \mathrm{Var}_t(\nu - \tilde{\nu}) \geqq \varepsilon)$. Accordingly to VI-T-16 of [3] S is predictable stopping time with respect to F^Q and so there exists a sequence $(S_n)_{n \geqq 1}$ of stopping times with respect to F^Q such that $S_n \uparrow S$ Q-a.s. and $S_n < S$ on $(S > 0)$. Choose the sequence $(S_n)_{n \geqq 1}$ to satisfy the inequality $P(S_n \leqq S - n^{-1},\ S_n < T) \leqq n^{-1}$, $n \geqq 1$. Put $R_n = T \wedge S_n$. It follows from (1) that

$$\mathrm{Var}(P_{R_n} - \tilde{P}_{R_n}) \leqq 2E\,\mathrm{Var}_{R_n}(\nu - \tilde{\nu}) \leqq 2\varepsilon.$$

Note that $A \cap (S_n \geqq T) \in \mathscr{F}^Q_{R_n}$ if $A \in \mathscr{F}_T$.

Thus

$$\begin{aligned}\mathrm{Var}(P_T - \tilde{P}_T) &\leqq \mathrm{Var}(P_{R_n} - \tilde{P}_{R_n}) + P(S_n < T) + \tilde{P}(S_n < T)\\ &\leqq (3/2)\,\mathrm{Var}(P_{R_n} - \tilde{P}_{R_n}) + 2P(S_n < T) \leqq (3/2)\,\varepsilon + 2P(S_n < T).\end{aligned} \tag{18}$$

But

$$P(S_n < T) = P(S_n < T, S_n \leqq S - n^{-1}) + P(S_n < T, S_n > S - n^{-1}) \leqq n^{-1} + P(S < T + n^{-1}).$$

Hence

$$\overline{\lim_n}\, P(S_n < T) \leqq P(S \leqq T) = P(\mathrm{Var}_T(\nu - \tilde{\nu}) \geqq \varepsilon) \tag{19}$$

and (2) follows from (18), (19).

We obtain general case (unbounded T) applying (2) for $T \wedge n$ and letting $n \to \infty$.

4. Proof of Theorem 1

A) It is an immediate corollary of (2).

B) We need the following general statement.

Lemma 4. *Let $(\Omega, \mathscr{F}, F, P)$ be a probability space with right-continuous filtration F, $\eta = \eta^1 - \eta^2$ where η^1, η^2 are predictable random measures on $(R_+ \times E, \mathscr{B}(R_+) \otimes \mathscr{E})$ such that the measures $M^P_{\eta^i}$ on $\tilde{\mathscr{P}}$ are σ-finite.*

Then there exists a $\hat{\mathscr{P}}$-measurable function H such that $|H|=1$ and $\mathrm{Var}_\infty(\eta) = H * \eta_\infty$.

The proof is obvious.

$$\text{Let } H^n \in \hat{\mathscr{P}}, |H^n|=1 \quad \text{and} \quad \mathrm{Var}_\infty(v-v^n) = H^n * (v-v^n)_\infty \quad P^n\text{-a.s.}$$

Put $S_m = \inf(t: 1 * v_t \geqq m)$, $R_m = S_m \wedge T_m \wedge T$. The Chebyshev inequality and definition H^n imply that

$$\begin{aligned} P(\mathrm{Var}_{R_m}(v^n - v) \geqq \varepsilon) &\leqq (1/\varepsilon) E^n H^n * (v - v^n)_{R_m} \\ &= (1/\varepsilon)(E^n H^n * v_{R_m} - EH^n * v_{R_m}) + (1/\varepsilon)(EH^n * v_{R_m} - E^n H^n * v^n_{R_m}). \end{aligned} \tag{20}$$

Since $H^n * v_{R_m}$ is $\mathscr{F}_T$-measurable and $|H^n * v_{R_m}| \leqq m+1$ then

$$|E^n H^n * v_{R_m} - EH^n * v_{R_m}| \leqq (m+1) \mathrm{Var}(P^n_T - P_T). \tag{21}$$

The random variable $H^n * \mu_{R_m}$ is $\mathscr{F}_T$-measurable as well and $|H^n * \mu_{R_m}| \leqq m$. Thus

$$|EH^n * v_{R_m} - E^n H^n * v^n_{R_m}| = |EH^n * \mu_{R_m} - E^n H^n * \mu_{R_m}| \leqq m \mathrm{Var}(P_T - P^n_T). \tag{22}$$

We have

$$\begin{aligned} P^n(\mathrm{Var}_T(v^n - v) \geqq \varepsilon) &\leqq P^n(\mathrm{Var}_{R_m}(v^n - v) \geqq \varepsilon) \\ &+ P^n(R_m < T) \leqq P^n(\mathrm{Var}_{R_m}(v^n - v) \geqq \varepsilon) + |P^n(R_m < T) - P(R_m < T)| \\ &+ P(R_m < T) \leqq P^n(\mathrm{Var}_{R_m}(v^n - v) \geqq \varepsilon) \\ &+ (1/2) \mathrm{Var}(P^n_T - P_T) + P(S_m \wedge T_m < T). \end{aligned} \tag{23}$$

It follows from (19)–(22) that

$$P^n(\mathrm{Var}_T(v^n - v) \geqq \varepsilon) \leqq (\varepsilon^{-1}(2m+1) + 1/2) \mathrm{Var}(P^n_T - P_T) + P(S_m \wedge T_m < T).$$

But (c) implies that $\lim\limits_m P(S_m \wedge T_m < T) = 0$. Letting at first $n \to \infty$ and later $m \to \infty$ we obtain (b).

Remark. The following example shows that the implication (a)$\Rightarrow$(b) fails in general.

Let $T = \infty$, P^n and P be the distributions of the Poisson processes with mean value functions $A^n_t = t + \mathrm{arctg}(t/n)$, $A_t = t$, $t \geqq 0$. Then obviously $\mathrm{Var}_\infty(v^n - v) = \mathrm{Var}_\infty(A^n - A) = \pi/2$, but $\lim\limits_n (P^n - P) = 0$.

To prove this put $\lambda^n_t = dA^n_t / dA_t = 1 + n^{-1}(1 + (t/n)^2)^{-1}$. Then

$$c_n \equiv \int_{[0,\infty]} |\lambda^n_t - 1|^2 \, dt \to 0, \qquad n \to \infty$$

and $P^n \ll P$ (see [7]). Put $Z^n = dP^n/dP$. Easy calculation shows that $E(Z^n)^2 = \exp(c_n) \to 1$. Hence

$$\mathrm{Var}(P^n - P) = E|1 - Z^n| \leqq (E(1 - Z^n)^2)^{1/2} = (E(Z^n)^2 - 1)^{1/2} \to 0, \qquad n \to \infty.$$

5. Supplement to Theorems 1 and 2

Suppose that on $(\Omega, \mathscr{F})$ we are given the following objects:

1°. A multivariate point process Π with associated integer-valued random measure μ.

2°. A right-continuous filtration $G=(\mathscr{G}_t)_{t\geqq 0}$ such that $\mathscr{F}_t^\mu \subseteq \mathscr{G}_t \subseteq \mathscr{F}_t$ for every $t \geqq 0$.

3°. Probability measures $\mathring{P}$, $\tilde{P}$. For $S \in \mathscr{T}(\mathscr{G})$ put $\mathring{P}_S = \mathring{P} \mid \mathscr{G}_S$, for $S \in \mathscr{T}(F^\mu)$ put $\tilde{P}_S = \tilde{P} \mid \mathscr{F}_S^\mu$ and $P = \mathring{P} \mid \mathscr{F}_\infty^\mu$. Corresponding notations will be used for expectations.

Denote by $\mathring{\nu}$, $\tilde{\nu}$, ν the compensators of μ with respect to $(G, \mathring{P})$, $(F^\mu, \tilde{P})$, (F^μ, P). Analogous meanings will have notations G^n, $\mathring{P}^n$, $\mathring{\nu}^n$ and so on.

Proposition 1. *Let $T \in \mathscr{T}(F^\mu)$. Then*

$$\operatorname{Var}(P_T - \tilde{P}_T) \leqq 2 \mathring{E} \operatorname{Var}_T(\mathring{\nu} - \tilde{\nu}). \tag{24}$$

Proof. By virtue of (1) it is sufficient to show that

$$E \operatorname{Var}_T(\nu - \tilde{\nu}) \leqq \mathring{E} \operatorname{Var}_T(\mathring{\nu} - \tilde{\nu}). \tag{25}$$

Evidently, we need to consider the case $\mathring{E} \operatorname{Var}_T(\mathring{\nu} - \tilde{\nu}) < \infty$ only. If

$$1 * \nu_T \leqq \text{const.} \tag{26}$$

then for $H \in \hat{\mathscr{P}}(F^\mu)$ such that $|H| \leqq 1$ and $\operatorname{Var}(\nu - \tilde{\nu}) = H * (\nu - \tilde{\nu})$ P-a.s., we have

$$\begin{aligned} E \operatorname{Var}_T(\nu - \tilde{\nu}) &= EH * (\mu - \tilde{\nu})_T = \mathring{E} H * (\mu - \tilde{\nu})_T = \mathring{E} H * (\mathring{\nu} - \tilde{\nu})_T \\ &\leqq \mathring{E} \operatorname{Var}_T(\mathring{\nu} - \tilde{\nu}) \end{aligned} \tag{27}$$

proving (24).

To prove (24) without (26) we define $S_k = \inf(t\colon 1 * \nu_t \geqq k)$ and $S = \lim_k S_k$. Since $S_k \in \mathscr{T}(F^\mu)$ and $1 * \nu_{T \wedge S_k} \leqq k+1$, we have

$$E \operatorname{Var}_{T \wedge S_k}(\nu - \tilde{\nu}) \leqq \mathring{E} \operatorname{Var}_{T \wedge S_k}(\mathring{\nu} - \tilde{\nu}) \leqq \mathring{E} \operatorname{Var}_T(\mathring{\nu} - \tilde{\nu}). \tag{28}$$

It follows from $\mathring{E} \operatorname{Var}_T(\mathring{\nu} - \tilde{\nu}) < \infty$ and (28) that $E \lim_k \operatorname{Var}_{T \wedge S_k}(\nu - \tilde{\nu}) < \infty$. Since $\lim_k 1 * \nu_{T \wedge S_k} = \infty$ on $(S < T)$ P-a.s. then the previous inequality implies $\lim_k 1 * \tilde{\nu}_{T \wedge S_k} = \infty$ on $(S < T)$ P-a.s.

Thus,

$$\lim_k \operatorname{Var}_{T \wedge S_k}(\nu - \tilde{\nu}) = \operatorname{Var}_T(\nu - \tilde{\nu}) \qquad P\text{-a.s.}$$

and (25) follows from (28).

Proposition 2. *Let $T \in \mathscr{T}(F^\mu)$ and $1 * \tilde{\nu}_T < \infty$ $\tilde{P}$-a.s. Suppose that the following condition is satisfied*

$$(\tilde{b}) \lim_n \mathring{P}^n(\operatorname{Var}_T(\mathring{\nu}^n - \tilde{\nu}) \geqq \varepsilon) = 0, \qquad \forall \varepsilon > 0.$$

Then

$$\lim_n \operatorname{Var}(P_T^n - \tilde{P}_T) = 0. \tag{29}$$

Proof. Put $B_t^n = \operatorname{Var}_t(\mathring{v}^n - \tilde{v})$. Suppose that $c = \sup_n \mathring{E}^n (B_T^n)^2 < \infty$. Then from (24) we have

$$\operatorname{Var}(P_T^n - \tilde{P}_T) \leqq 2\varepsilon + 2\mathring{E}^n B_T^n I_{(B_T^n \geqq \varepsilon)} \leqq 2\varepsilon + 2c^{1/2} \mathring{P}^n (B_T^n \geqq \varepsilon)$$

and (29) is obvious.

In general case we define $R_m = T \wedge T_m \wedge \inf(t: 1 * \tilde{v}_t \geqq m)$. It is easy to see that

$$\mathring{E}^n (B_{R_m}^n)^2 \leqq \mathring{E}^n (1 * \mathring{v}_{R_m}^n + 1 * \tilde{v}_{R_m})^2 \leqq 2(2m^2 + (m+1)^2).$$

By assumption $\lim_n \tilde{P}(R_m < T) = 0$ and it remains to note that

$$\operatorname{Var}(P_T^n - \tilde{P}_T) \leqq (3/2) \operatorname{Var}(P_{R_m}^n - \tilde{P}_{R_m}) + 2\tilde{P}(R_m < T).$$

6. Example

Let $\xi_1, \ldots, \xi_n$ be independent uniformly distributed on [0, 1] random variables, $\hat{N}_t^n = \sum_{k \leqq n} I(\xi_k \leqq t)$ (the empirical distribution function multiplied by n), $N_t^n = \hat{N}_{t/n}^n$. It is well known (see, e.g., [8]) that finite-dimensional distributions of $N^n = (N_t^n)_{t \geqq 0}$ converge to finite-dimensional distributions of a Poisson process with unit rate (and the compensator $A_t = t$).

We show that, in fact, there is a convergence of the distributions in variation on any finite interval $[0, T]$ and

$$\operatorname{Var}(P_T^n - P_T) \leqq 2T^2/n \tag{30}$$

where P_T^n and P_T are the distributions N^n and the Poisson process on $[0, T]$.

The compensator $A^n = (A_t^n)_{t \geqq 0}$ of N^n with respect to the filtration generated by N^n has the following form (see [8]):

$$A_t^n = -\sum_{k \leqq n} \ln(1 - \xi_k \wedge (t/n))$$

and so

$$dA_t^n/dt = n^{-1} \sum_{k \leqq n} I(\xi_k > t/n)(1 - t/n)^{-1}.$$

Easy calculation shows that

$$E|dA_t^n/dt - 1| = E|I(\xi_1 > t/n)(1 - t/n)^{-1} - 1| = 2t/n.$$

Thus,

$$E \operatorname{Var}_T(A^n - A) = E \int_0^T |dA_t^n/dt - 1| \, dt = T^2/n$$

and estimate (30) follows obviously from Theorem 2.

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