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# Enhancing of Semigroups*

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**Summary.** Given a semigroup $\mathbf{T}_t$ and an excessive measure $v$ a new semigroup $\tilde{\mathbf{T}}_t$ is constructed in such a way that $v$ is invariant with respect to $\tilde{\mathbf{T}}_t$ and $\tilde{\mathbf{T}}_t$ is larger than the original semigroup. A sufficient condition for $\tilde{\mathbf{T}}_t$ to be uniquely determined by $\mathbf{T}_t$ and $v$ is given.

## 1. Introduction

1.1. Let $D$ be a body in a three-dimensional space $E$ and suppose that this body is heated at each point $x$ to a certain temperature $h(x)$. Suppose that we observe the process of dissipation of heat and notice that the temperature decreases at each point $x$. The question is whether we can impose such boundary conditions that the original distribution of the temperature $h(x)$ is preserved. Physical intuition suggests the following solution. We have to look at those points of the boundary where the heat dissipates into outer space and put there reflectors which redistribute the heat over $D$ proportionally to the rate of heat loss.

We shall show that the construction similar to the one suggested by physical intuition can be used in a more general situation.

1.2. In the situation described above let $f(t,x)$ stand for the temperature at the point $x$ at time $t$. It is known that $f(t,x)$ can be obtained as a solution of the following systems

$$\frac{\partial f}{\partial t}=Lf(t,\cdot), \tag{1.2.1}$$

$$Hf(t,\cdot)=0, \tag{1.2.2}$$

$$f(0,x)=h(x).$$

Here $L$ is an elliptic differential operator of the second order in the space $D$, and $H$ is a linear operator in the space of functions on $D$ (this operator

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corresponds to the boundary conditions; see [1]). Let $\mathscr{D}$ be the space of twice continuously differentiable functions $g$ on $D$ such that $Hg=0$ and let $\mathbf{Q}_t$ be the semigroup whose generator coincides with $L$ on $\mathscr{D}$. Then $f(t,x)=\mathbf{Q}_t h(x)$. The inequality $f(t,x)\leqq h(x)$ shows that $h(x)$ is an excessive function with respect to $\mathbf{Q}_t$. This is equivalent to $v(dx)=h(x)\,dx$ being an excessive measure with respect to the conjugate semigroup $\mathbf{T}_t$.

Placing reflectors at the boundary corresponds to changing the operator $L$ in (1.2.1) into $L+\mathscr{L}$, where $\mathscr{L}$ is an integral operator, and changing boundary conditions, that is, replacing $H$ in (1.2.2) by a new operator $\tilde{H}$ (the explicit expressions for $\mathscr{L}$ and $\tilde{H}$ in the one-dimensional case can be found in [5], Sects. 18 and 22). Let $\tilde{\mathbf{Q}}_t$ be the new semigroup corresponding to the solution of the new system and $\tilde{\mathbf{T}}_t$ its conjugate. If under the new boundary conditions the temperature $h(x)$ is preserved, then $v$ is an invariant measure with respect to $\tilde{\mathbf{T}}_t$. It is easy to see that for any measurable function $g$ on $D$

$$\tilde{\mathbf{T}}_t g(x)\geqq \mathbf{T}_t g(x). \tag{1.2.3}$$

Under mild conditions in a more general situation we shall show that, given a semigroup $\mathbf{T}_t$ and an excessive with respect to $\mathbf{T}_t$ measure $v$, one can find $\tilde{\mathbf{T}}_t$ satisfying (1.2.3) for which $v$ is invariant.

If two semigroups $\tilde{\mathbf{T}}_t$ and $\mathbf{T}_t$ satisfy (1.2.3) then we say that $\tilde{\mathbf{T}}_t$ is larger than $\mathbf{T}_t$, or $\tilde{\mathbf{T}}_t$ is an enhancing of $\mathbf{T}_t$.

We write $\mathbf{T}_t=\tilde{\mathbf{T}}_t$, a.e. $\mu$, if for each measurable function $g$ $\mathbf{T}_t g(x)=\tilde{\mathbf{T}}_t g(x)$ for $\mu$-almost every $x$.

In this paper we deal only with preserving positivity contraction normal semigroups, that is, semigroups $\mathbf{T}_t$ satisfying 1.2.A–1.2.C below.

1.2.A. If $g(x)\geqq 0$ then $\mathbf{T}_t g(x)\geqq 0$ for each $t>0$.

1.2.B. For each $x\in D$

$$\mathbf{T}_t 1(x)\leqq 1 \quad \text{and} \quad \lim_{t\downarrow 0}\mathbf{T}_t 1(x)=1.$$

1.2.C. If $f(x_0)=0$, then $T_0 f(x_0)=0$.

A measurable set $D$ is called Borel if it is isomorphic to a measurable subset of a complete separable metric space endowed with a $\sigma$-field generated by all open sets. With some abuse of notions we say that $\mathbf{T}_t$ is a semigroup on a Borel space $D$ instead of saying that $\mathbf{T}_t$ is a semigroup in the Banach space of bounded measurable functions on $D$.

The same letter will be used for a measure and an integral with respect to this measure; thus, for $\mathbf{P}$ being a probability measure and $\xi$ being a random variable, $\mathbf{P}\xi$ (or $\mathbf{P}\{\xi\}$) stands for the mathematical expectation of $\xi$. The word "function" will always stand for a nonnegative bounded measurable function.

1.3. Our main tool is the theory developed in [11] where we deal with a stationary Markov process $(x_t, \bar{\mathbf{P}})$, $t\in T=]-\infty,+\infty[$ with the state space $E=D\cup V$ such that for each $t$

$$\bar{\mathbf{P}}\{x_t\in V\}=0.$$

Suppose that $(x_t, \bar{\mathbf{P}})$ is conservative, that is $\bar{\mathbf{P}}$ is a probability measure. Let $\bar{p}(t, x; \Gamma)$ be the transition function of $(x_t, \bar{\mathbf{P}})$, $\bar{\mathbf{P}}_x$ be the transition probabilities and (assuming that the first hitting time of $V$ is measurable)

$$\nu(\Gamma) = \bar{\mathbf{P}}\{x_t \in \Gamma\}$$
$$p(t, x; \Gamma) = \bar{\mathbf{P}}_x\{x_t \in \Gamma,\ x_s \bar{\in} V \text{ for all } s \leqq t\}.$$

It is possible to construct a stationary Markov process $(w(s), \mathbf{P})$ in the state space $D$ with transition function $p$ and one-dimensional distribution $\nu$. The process $w(\cdot)$ has random times of birth and death and the measure $\mathbf{P}$ may be infinite. (Note that the measure $\nu$ is invariant with respect to the transition function $\bar{p}$ and is excessive with respect to $p$). The process $(w(s), \mathbf{P})$ is called a subprocess in $D$ of $(x_t, \bar{\mathbf{P}})$ and $(x_t, \bar{\mathbf{P}})$ is a covering process for $(w(s), \mathbf{P})$.

In [11] the inverse problem was considered. Given a stationary Markov process $(w(s), \mathbf{P})$ in the state space $D$, when is it possible to find a set $V$ and a conservative stationary Markov process $(x_t, \bar{\mathbf{P}})$ in the state space $D \cup V$ in such a way that $(x_t, \bar{\mathbf{P}})$ is a covering process for $(w(s), \mathbf{P})$. It is proved (see [11] Th. 1) that it is possible to do this iff the one-dimensional distribution of $\mathbf{P}$ is a probability measure. Moreover, in this case $V$ can be taken as a single-point set.

If in addition $\mathbf{P}$ is an extreme measure in the class of all Markov measures with transition function $p$ then the covering process $(x_t, \bar{\mathbf{P}})$ is unique in the sense that the finite-dimensional distributions of $\bar{\mathbf{P}}$ are uniquely determined by the measure $\mathbf{P}$.

1.4. Processes with independent increments will be often used in the sequel.

Let $y_t$ be a right-continuous increasing process with independent increments and $\Pi$ be a measure on $]0, \infty[$ and $\alpha$ be a nonnegative constant. We say that $y_t$ has Levy's measure $\Pi$ and translation constant $\alpha$ (or $y_t$ is $(\alpha, \Pi)$ process) if

$$y_t - y_s = \alpha(t-s) + \sum_{u \in J,\, s < u \leqq t} (y_u - y_{u-}) \qquad \text{a.s. } \mathbf{Q}_x,$$
$$\mathbf{Q}_x\{\sum_{u \in J,\, s < y \leqq t} f(y_u - y_{u-})\} = (t-s)\,\Pi(f).$$

(Here $J = \{u\colon y_u \neq y_{u-}\}$ and $\mathbf{Q}_x$ are the transition probabilities of $y_t$.) For $\Gamma \subset T$ and $f$ a function $T \times T$ put

$$\Pi(x; \Gamma) = \Pi(\Gamma - x),$$
$$\Pi_x(f) = \int_T f(x, y)\,\Pi(x; dy).$$

We shall use the following lemma, the proof of which is well known.

**Lemma 1.4.1.** *For any function $f$ on $T \times T$*

$$\mathbf{P}_b\{\sum_{t \in J} f(y_{t-}, y_t)\} = \int \Pi_x(f)\,\lambda_b(dx) \tag{1.4.1}$$

*where*

$$\lambda_b(\Gamma) = \mathbf{P}_b \int_0^\infty 1_\Gamma(y_t)\,dt \qquad \Gamma \subset T \tag{1.4.2}$$

## 2. Formulation of Results. Proof of Theorem of Existence

2.1. The main results of the paper are given by Theorems 1 and 2.

**Theorem 1.** *Let $\mathbf{T}_t$ be a semigroup on a Borel space D such that*

2.1.A. $\mathbf{T}_t 1(x)$ *is a continuous function of t for every x.*

*If $\nu$ is a finite excessive (with respect to $\mathbf{T}_t$) measure on D, then there exists a semigroup $\tilde{\mathbf{T}}_t$ which is larger than $\mathbf{T}_t$ and for which $\nu$ is invariant.*

**Theorem 2.** *Suppose that $\mathbf{T}_t$ and $\nu$ satisfy the conditions of Theorem 1. If in addition $\nu$ is an extreme excessive measure, then $\tilde{\mathbf{T}}_t$ is unique up to the measure $\nu$.*

To prove Theorem 1 we consider a Markov transition function $p(t, x; \Gamma)$ such that $\mathbf{T}_t f(x) = p(t, x; f)$, the existence of which is proved in [3], Chap. 2, Theorem 2.1.

Then we consider a stationary Markov process generated by $p$ and $\nu$. We apply the main result of [11] and construct a conservative, covering stationary Markov process. We show that the semigroup we are looking for corresponds to the transition function of the covering process.

2.2. We may assume without loss of generality that $\nu$ is a probability measure on $D$, i.e.

$$\nu(D) = 1. \tag{2.2.1}$$

Consider a stationary Markov process $(w(s), \mathbf{P})$, $s \in T = ]-\infty, +\infty[$ with the state space $D$ and the one-dimensional distribution $\nu$. The existence of such a process was proved in [8]. We can take the space $W$ of all paths in $D$ with random birth time $\alpha$ and death time $\beta$ as a sample space of this process. (Note that $\mathbf{P}\{W\}$ may be equal to infinity.) The condition (2.2.1) is just the same as 1.2.A in [11]. By virtue of the main result of [11] there exists a process $(x_t, \bar{\mathbf{P}})$ in a state space $D \cup V$, $V$ being a singleton, for which $(w(s), \mathbf{P})$ is a subprocess in $D$. To construct the semigroup $\tilde{\mathbf{T}}_t$ we have to find the transition function $\bar{p}$ of the process $(x_t, \bar{\mathbf{P}})$.

Consider the entrance law $\nu_s$ (with respect to the transition function $p$) such that

$$\nu = \int_0^\infty \nu_s \, ds.$$

The existence of this entrance law was proved in [4]. Let $\mathbf{P}_t^*$ be the Markov measure on $W$ with transition function $p$ and the one-dimensional distribution at time $s$ equal to $\nu_{s-t}$ (we assume that $\nu_s \equiv 0$ for $s \leqq 0$, therefore $\mathbf{P}_t^* \{\alpha \neq t\} = 0$).

Let

$$\Pi(t; \Gamma) = \mathbf{P}_t^* \{\beta \in \Gamma\}, \qquad \Gamma \subset T; \; \Pi(\Gamma) = \Pi(0; \Gamma).$$

Consider an increasing process $y_s$ on $T$ with independent increments with the translation constant 0 and the Levy measure $\Pi$. (See [10] for a more detailed discussion.) Let $\mathbf{Q}_y$, $y \in T$ be the transition probabilities of this process and let $\sigma_\ell = \inf\{t: y_t > \ell\}$. For $\Gamma \subset D$, $x \in D$, $t > 0$ put

$$\bar{p}(t, x; \Gamma) = p(t, x; \Gamma) + \int_0^t \mathbf{P}_x\{\beta \in dy\}\, \mathbf{Q}_y \left\{ \int_0^{\sigma_t} \mathbf{P}^*_{y_s}\{w(t) \in \Gamma\}\, ds \right\}$$

$$\bar{p}(t, x; V) = 0. \tag{2.2.2}$$

Here $\mathbf{P}_x$ is the transition probability of $(w(s); \mathbf{P})$.

We shall show that $\bar{p}$ defined by (2.2.2) is a transition function of $(x_t, \bar{\mathbf{P}})$. To understand this formula intuitively, assume that $x_t$ is a strong Markov process with all kinds of nice properties, so that there exists an exit system at point $V$ (see [9]) and using it we can write the so called "last exit decomposition" (see [6, 7, 9]). Let $M = \{t\colon x_t = V\}$ be closed a.s. $\bar{\mathbf{P}}$, and the complement of $M$ be a.s. a union of countable number of open intervals $]\gamma, \delta[$. Let $\tau = \inf\{t\colon x_t \in V\}$. We can write

$$\bar{p}(t, x; \Gamma) = \bar{\mathbf{P}}_x\{x_t \in \Gamma\} = \bar{\mathbf{P}}_x\{x_t \in \Gamma,\ \tau \geqq t\} + \bar{\mathbf{P}}_x\{x_t \in \Gamma,\ \tau < t\}.$$

The first term in the right-hand side of the above formula is equal to $p(t, x; \Gamma)$ and the second can be written as

$$\bar{\mathbf{P}}_x\{\sum_\gamma 1_{\gamma < t < \delta} 1_\Gamma(x_t)\}.$$

This can be written as

$$\bar{\mathbf{P}}_x \left\{ \int_0^t \mathbf{Q}^*\{w(t-s) \in \Gamma\}\, d\xi_s \right\}$$

where $\xi_t$ is the local time of $x_t$ at the point $V$ and $\mathbf{Q}^*$ is a Markov measure with transition function $p$ (see [6]). It is known that the inverse of local time at a point is a process with independent increments with Levy's measure $\Pi(\Gamma) = \mathbf{Q}^*\{\beta \in \Gamma\}$. The initial distribution of this process coincides with the distribution of $\tau$. It is easy to show that the distribution of $\tau$ under $\bar{\mathbf{P}}_x$ is the same as the distribution of $\beta$ under $\mathbf{P}_x$. In our case $\mathbf{Q}^*$ is equal to $\mathbf{P}^*_0$ (though it is not seen directly from the way the process $(x_t, \bar{\mathbf{P}})$ is constructed) and that makes (2.2.2) just one of the forms of usual "last exit decomposition".

**Theorem 2.2.1.** *The kernel $\bar{p}$, given by* (2.2.2), *is a conservative transition function of the process* $(x_t, \bar{\mathbf{P}})$.

For the proof of the theorem we have to verify the following relations.

2.2.A. For any $\Gamma, \Delta \subset D$

$$\bar{\mathbf{P}}\{x_s \in \Gamma,\ x_t \in \Delta\} = \int_\Gamma \nu(dx)\, \bar{p}(t-s, x, \Delta). \tag{2.2.3}$$

2.2.B. For each $x \in D$ and each $t > 0$

$$\bar{p}(t, x; D) = 1. \tag{2.2.4}$$

2.2.C. For each $t, s > 0$ and $\Gamma \subset D$

$$\int_D \bar{p}(s, x; dy)\, \bar{p}(t, y; \Gamma) = \bar{p}(s+t, x; \Gamma). \tag{2.2.5}$$

The formula (2.2.3) is equivalent to the formula (5.7.3) of [11], which was proved in the case of an extreme measure $v$. However, similar computations show that (5.7.3) of [11] is true for the covering process $(x_t, \bar{\mathbf{P}})$ constructed in Sect. 3 of [11], with which we deal now (see also [12] Lemma 3.2.7, where the proof of 2.2.A is given in a more general situation).

To prove 2.2.B, we need

**Lemma 2.2.2.** *Put*

$$\varphi(z) = \mathbf{Q}_z \int_0^{\sigma_t} \mathbf{P}^*_{y_s}\{w(t) \in D\}\, ds, \qquad z \in T.$$

*Then $\varphi(z)$ is equal to 1 for all $z \in T$ except for a countable number of points.*

For intuitive understanding of this lemma let us assume again that the last exit decomposition is true. In this case $\varphi(z)$ is nothing but $\bar{p}(t-z, V; D)$ and the assertion that $\varphi(z)=1$ for all $z$, except a countable number of points, is equivalent to $\bar{p}(u, V; V)$ being equal to zero for all $u$, except a countable number of points.

*Proof of Lemma 2.2.2.* Let $T^a = ]a, \infty[$. Compute

$$\begin{aligned}\varphi(z) &= \mathbf{Q}_z \int_0^{\sigma_t} \mathbf{P}^*_{y_s}\{\beta > t\}\, ds = \mathbf{Q}_z \int_0^{\sigma_t} \Pi(y_s; T^t)\, ds \\ &= \mathbf{Q}_0 \int_0^{\sigma_{t-z}} \Pi(y_s; T^{t-z})\, ds. \qquad (2.2.6)\end{aligned}$$

Put $Y(u) = y_{\sigma_u}$, $Y(u-) = y_{\sigma_{u-}}$. Let $\lambda_b$ be defined by formula (1.4.2). Put $u = t - z$. By virtue of Lemma 1.4.1 the right side of (2.2.6) is equal to

$$\mathbf{Q}_0\{Y(u-) < u,\ Y(u) > u\}.$$

Since $Y(u-) \leqq u$ and $Y(u) \geqq u$, then

$$1 - \varphi(z) \leqq \mathbf{Q}_0\{Y(u-) = u\} + \mathbf{Q}_0\{Y(u) = u\}.$$

Let $\Lambda_1$ be the (countable) set of atoms of the measure $\lambda_0$; and $\Lambda_2$ be the (countable) set of atoms of the measure $\Pi$. Put $\Lambda = \Lambda_1 + \Lambda_2$. By Lemma 1.4.1

$$\begin{aligned}\mathbf{Q}_0\{Y(u) = u\} &= \int_0^u \Pi(x; \{u\})\, \lambda_0(dx) \\ &= \int_0^u \Pi\{u - x\}\, \lambda_0(dx) = \sum_{\substack{x \in \Lambda_1 \\ x < u}} \lambda_0\{x\}\, \Pi\{u - x\}. \qquad (2.2.7)\end{aligned}$$

The right side of (2.2.7) differs from zero only for $u \in \Lambda$, therefore, for a countable number of $u$. Similarly

$$\mathbf{Q}_0\{Y(u-) = u\} = \lambda_0\{u\}\, \Pi\{T^0\}. \qquad (2.2.8)$$

The right side of (2.2.8) differs from zero only for $u \in \Lambda_1$. The lemma is proved.

Now show 2.2.B. By (2.1.A) for each $u>0$

$$\mathbf{P}_x\{\beta=u\}=\lim_{s\uparrow u}\mathbf{T}_s 1(x)-\mathbf{T}_u 1(x)=0.$$

Therefore, $\mathbf{P}_x\{\varphi(\beta)\neq 1\}=0$, and by (2.2.2)

$$\begin{aligned}\bar{p}(t,x,D)&=\mathbf{P}_x\{\beta>t\}+\mathbf{P}_x\{1_{\beta\leq t}\varphi(\beta)\}\\&=\mathbf{P}_x\{\beta>t\}+\mathbf{P}_x\{\beta\leq t\}=1.\end{aligned}$$

The following lemma is essential for the proof that $\bar{p}$ satisfies the Chapman-Kolmogorov equation.

**Lemma 2.2.3.** *For any $\Gamma\subset D$ and any $r<u<v$*

$$\begin{aligned}&\mathbf{Q}_r\left\{\int_0^{\sigma_u} ds\int_D \mathbf{P}^*_{y_s}\{w(u)\in dz\}\int_0^{v-u}\mathbf{P}_z\{\beta\in d\ell\}\,\mathbf{Q}_\ell\left\{\int_0^{\sigma_{v-u}} dm\,\mathbf{P}^*_{y_m}\{w(v-u)\in\Gamma\}\right\}\right\}\\&=\mathbf{Q}_r\int_{\sigma_u}^{\sigma_v} ds\,P^*_{y_s}\{w(v)\in\Gamma\}.\end{aligned}\tag{2.2.9}$$

For heuristic interpretation of (2.2.9) we must again refer to the "last exit decomposition". Assume for simplicity that $r=0$. Then the right side of (2.2.9) is equal to

$$\bar{\mathbf{P}}_V\{x_v\in\Gamma,\ x_s\in V \text{ for some } u<s<v\}.$$

The left side of (2.2.9) equals

$$\int_D \bar{\mathbf{P}}_V\{x_u\in dz\}\int_0^{v-u}\bar{\mathbf{P}}_z\{\tau\in d\ell\}\,\bar{\mathbf{P}}_V\{x_{v-u-\ell}\in\Gamma\}.$$

Applying the strong Markov property, we see that the right side of (2.2.9) equals the left side.

The rigorous proof certainly cannot exploit these arguments because in general the process $(x_t,\bar{\mathbf{P}})$ is not strong Markov and does not have any regularity properties that would ensure the existence of an "exit system".

*Proof of Lemma 2.2.3.* Put

$$\psi_t(\ell)=\mathbf{Q}_\ell\left\{\int_0^{\sigma_t}\mathbf{P}^*_{y_s}\{w(t)\in\Gamma\}\,ds\right\}.$$

Since $\mathbf{P}^*_r$ is a Markov measure with the transition probabilities $\mathbf{P}_z$, then

$$\begin{aligned}\int_D \mathbf{P}^*_r\{w(u)\in dz\}\,\mathbf{P}_z\{\psi_{v-u}(\beta),1_{\beta<v-u}\}&=\mathbf{P}^*_r\{\psi_{v-u}(\beta-u)1_{u<\beta<v}\}\\&=\mathbf{P}^*_r\{\psi_v(\beta)1_{u<\beta<v}\}=\mathbf{P}^*_r\{\psi'(\beta)\}\end{aligned}\tag{2.2.10}$$

where $\psi'(x)=\psi_v(x)1_{u<x<v}$. Taking into account the definition of the kernel $\Pi(x;-)$ and applying (2.2.10), we can rewrite the left side of (2.2.9) as

$$\mathbf{Q}_r\left\{\int_0^{\sigma_u} ds \int_u^v \Pi(y_s, d\ell)\psi_v(\ell)\right\} = \mathbf{Q}_r\left\{\int_0^{\sigma_u} ds\, \Pi(y_s; \psi')\, \mathrm{ds}\right\}$$
$$= \int_r^u \lambda_r(dx)\, \Pi(x; \psi')$$
$$= \int_{-\infty}^{\infty} \lambda_r(dx) 1_{x<u} \Pi(x; \psi'). \qquad (2.2.11)$$

By Lemma 1.4.1, (2.2.11) is equal to

$$\mathbf{Q}_r\{\sum_{y_{t-}<y_t} 1_{y_{t-}<u}\psi'(y_t)\} = \mathbf{Q}_r\{\sum_{y_{t-}<y_t} 1_{y_{t-}<u} 1_{u<y_t<v}\psi_v(y_t)\}$$
$$= \mathbf{Q}_r\{\psi'(Y(u))\}. \qquad (2.2.12)$$

Using again Lemma 1.4.1 and the strong Markov property of $y_s$, we get that (2.2.12) is equal to

$$\mathbf{Q}_r\left\{1_{Y(u)<v}\left\{\mathbf{Q}_{Y(u)}\left\{\int_0^{\sigma_v} \mathbf{P}^*_{y_s}\{w(v)\in\Gamma\}\, ds\right\}\right\}\right\} = \mathbf{Q}_r\left\{\int_{\sigma_u}^{\sigma_v} \mathbf{P}^*_{y_s}\{w(v)\in\Gamma\}\, ds\right\},$$

and the lemma is proved.

Now we are able to verify 2.2.C. Consider

$$\int_D \bar{p}(t, x; dy)\, \bar{p}(s, y; \Gamma) = \int_D p(t, x; dy)\, p(s, y; \Gamma)$$
$$+ \int_D p(t, x; dy) \int_0^s \mathbf{P}_y\{\beta\in du\}\, \mathbf{Q}_u\left\{\int_0^{\sigma_s} dr\, \mathbf{P}^*_{y_r}\{w(s)\in\Gamma\}\right\}$$
$$+ \int_0^t \mathbf{P}_x\{\beta\in dy\}\, \mathbf{Q}_y\left\{\int_0^{\sigma_t} dr \int_D \mathbf{P}^*_{y_r}\{w(t)\in dz\}\, p(s, z; \Gamma)\right\}$$
$$+ \int_0^t \mathbf{P}_x\{\beta\in dy\}\, \mathbf{Q}_y\left\{\int_0^{\sigma_t} dr \int_D \mathbf{P}^*_{y_r}\{w(t)\in dz\} \int_0^s \mathbf{P}_z\{\beta\in d\ell\}\right.$$
$$\left.\cdot\, \mathbf{Q}_\ell\left\{\int_0^{\sigma_s} du\, \mathbf{P}^*_{y_u}\{w(s)\in\Gamma\}\right\}\right\}. \qquad (2.2.13)$$

By virtue of the Chapman-Kolmogorov equation for $p$ the first term in the right side of (2.2.13) is equal to $p(s+t, x; \Gamma)$. Since $\mathbf{P}^*_r\{w(s)\in\Gamma\} = \mathbf{P}^*_{r+t}\{w(t+s)\in\Gamma\}$, then

$$\mathbf{Q}_u\left\{\int_0^{\sigma_s} \mathbf{P}^*_{y_r}\{w(s)\in\Gamma\}\, dr\right\} = \mathbf{Q}_{u+t}\left\{\int_0^{\sigma_{s+t}} \mathbf{P}^*_{y_r}\{w(t+s)\in\Gamma\}\, dr\right\}.$$

Together with the Markov property of $(w(\cdot), \mathbf{P})$ this relation yields that the second term in the right side of (2.2.13) is equal to

$$\int_t^{t+s} \mathbf{P}_x\{\beta\in dy\}\, \mathbf{Q}_y\left\{\int_0^{\sigma_{t+s}} \mathbf{P}^*_{y_s}\{w(t+s)\in\Gamma\}\, dr\right\}. \qquad (2.2.14)$$

Since $\mathbf{P}_u^*$ is a Markov measure with the transition function $p$, then the third term in the right side of (2.2.13) is equal to

$$\int_0^t \mathbf{P}_x\{\beta \in dy\}\, \mathbf{Q}_y \left\{ \int_0^{\sigma_t} \mathbf{P}_{y_r}^*\{w(t+s) \in \Gamma\}\, dr \right\}. \tag{2.2.15}$$

By Lemma 2.2.3 the fourth term in the right side of (2.2.13) is equal to

$$\int_0^t \mathbf{P}_x\{\beta \in dy\}\, \mathbf{Q}_y \left\{ \int_{\sigma_t}^{\sigma_{s+t}} \mathbf{P}_{y_r}^*\{w(t+s) \in \Gamma\}\, dr \right\}. \tag{2.2.16}$$

Adding (2.2.14), (2.2.15) and (2.2.16) we get that (2.2.13) reduces to

$$p(s+t, x; \Gamma) + \int_0^{t+s} \mathbf{P}_x\{\beta \in dy\}\, \mathbf{Q}_y \left\{ \int_0^{\sigma_{t+s}} \mathbf{P}_{y_r}^*\{w(s+t) \in \Gamma\}\, dr \right\} = \bar{p}(s+t, x; \Gamma),$$

and that completes the proof of Theorem 2.2.1.

2.3. Put

$$\tilde{\mathbf{T}}_t f(x) = \int_D \bar{p}(t, x; dy) f(y).$$

By Theorem 2.2.1 $\bar{p}$ satisfies 2.2.A–2.2.C; therefore, $\tilde{\mathbf{T}}_t$ is a contraction semigroup such that $\tilde{\mathbf{T}}_t 1 \equiv 1$. It is obvious that $\tilde{\mathbf{T}}_t$ is an enhancing of $\mathbf{T}_t$. Inasmuch as $\bar{p}$ is the transition function of a stationary Markov process with the one-dimensional distribution $\nu$, then $\nu$ is an invariant measure with respect to $\tilde{\mathbf{T}}_t$.

*Remark.* Although the state space of the covering process $(x_t, \bar{\mathbf{P}})$ is larger than $D$, we were able to exclude the additional point $V$ from the formulation of the final result. To this end we proved that the transition function of $(x_t, \bar{\mathbf{P}})$ restricted to $D$ remains a transition function. That was possible due to the condition 2.1.A. In a general situation, when 2.1.A is not satisfied, one can prove a weaker analogue of Theorem 1 in which $\tilde{\mathbf{T}}_t$ is not a semigroup on the same space $D$ but on a larger space $E \supset D$.

## 3.1. Theorem of Uniqueness

3.1. Suppose $\tilde{\mathbf{T}}_t$ is constructed and we want to prove that $\tilde{\mathbf{T}}_t$ is unique up to the measure $\nu$. As always, consider transition functions $p$ and $\bar{p}$ such that

$$\mathbf{T}_t f(x) = p(t, x; f),$$
$$\bar{\mathbf{T}}_t f(x) = \bar{p}(t, x; f).$$

If we knew that the only way to obtain a larger semigroup is to construct a covering process then theorem of uniqueness would be an immediate consequence of Theorem 2 of [11]. We would have a process $(x_t, \bar{\mathbf{P}})$ with the state space $D+V$ (such that $\nu(V)=0$) and with subprocess in $D$ equal to $(w(s), \mathbf{P})$. The $p$-excessive measure $\nu$ is extreme iff $\mathbf{P}$ is extreme in the class of all Markov measures with the transition function $p$. The uniqueness of the two-dimensional distributions of $\bar{\mathbf{P}}$ would immediately imply that $\bar{p}(t, x; \Gamma)$ is given by (2.2.2) for $\nu-$ a.e. $x$.

Therefore, Theorem 2 of the present paper will follow from

**Theorem 3.1.1.** *Suppose that $\bar{p}$ and $p$ are two transition functions on $D$ such that $\bar{p}(t, x; \Gamma) \geqq p(t, x; \Gamma)$ for all $\Gamma \subset D$, $x \in D$ and $t>0$. If $\nu$ is an invariant measure with respect to $\bar{p}$ then there exists a stationary Markov process $(x_t^*, \mathbf{Q})$ with the state space $E = D \cup V$ such that*

3.1.A. *For each $t>s$ and $\Gamma \subset D$*

$$\mathbf{Q}\{x_t^* \in V\} = 0,$$
$$\mathbf{Q}\{x_t^* \in \Gamma\} = \nu(\Gamma),$$
$$\mathbf{Q}\{x_t^* \in \Gamma \mid x_s^*\} = \bar{p}(t-s, x; \Gamma).$$

3.1.B. *The set $M = \{t\colon x_t^* \in V\}$ is closed a.s. $Q$.*

3.1.C. *The subprocess in $D$ of $(x_t^*, \mathbf{Q})$ is a stationary Markov process with the one-dimensional distribution $\nu$ and transition function $p$.*

The general outline of the proof of Theorem 3.3.1 is the following. First we construct a stationary Markov process $(x_t, \bar{\mathbf{P}})$ with the state space $D$ with the one-dimensional distribution $\nu$ and the transition function $\bar{p}$. Applying Theorem 9.3 of [4] to the transition functions $p$ and $\bar{p}$, we obtain a multiplicative functional $\alpha_t$ such that $p(s, x; \Gamma) = \bar{\mathbf{P}}_x\{1_\Gamma(x_s)\alpha_s\}$, $\bar{\mathbf{P}}_x$ being the transition probabilities of $x_t$. If $\alpha_t$ took on only 0 and 1 values that would be almost the end of the construction. In this case we would consider a family of stopping times $\sigma_s = s + \inf\{t\colon \theta_s \alpha_{t-s} = 0\}$ ($\theta_s$ is a shift operator in the sample space of $x_t$). The family $\sigma_s$ would have the same properties as the family of hitting times of a set in the state space. We would then put $x_t^* \in V$ if $\sigma_{s-} = t$ for some $s \leqq t$ and $x_t^* = x_t$ otherwise. Doing so, we would alter $x_t$ on the set of $\bar{\mathbf{P}}$-measure zero for each fixed $t$; thus, $x_t$ and $x_t^*$ would have the same finite-dimensional distributions. Since by such a construction $p(s, x; \Gamma) = \bar{\mathbf{P}}_x\{x_s \in \Gamma;\ x_t \in D \text{ for all } t<s\}$, we would get that the subprocess in $D$ of $(x_t^*, \mathbf{Q})$ has transition function $p$ (it is known a priori that the one-dimensional distribution of the subprocess is equal to $\nu$). That would imply Theorem 3.3.1.

Unfortunately $\alpha_t(\omega)$ may take on any values between zero and one, and it is necessary to use a coupling technique to overcome this difficulty. We consider a new sample space $\tilde{\Omega} = \Omega \times (T)^\infty$ and a probability measure $\mathbf{Q}$ on $\tilde{\Omega}$ with marginal distribution on $\Omega$ equal to $\bar{\mathbf{P}}$. A family of random variables $\tau_s(\tilde{\omega})$ is constructed in such a way that the conditional probability of $\tau_s$ being greater than $t$ given $\omega$ is equal to $\alpha_{t-s}(\theta_s \omega)$. The family $\tau_s$ has all the properties that $\sigma_s$ has, except $\tau_s$ is not measurable with respect to the $\sigma$-field generated by $x_t$. Nevertheless, it is possible to construct $x_t^*$ acting in the same way as if $\tau_s$ was the first hitting time after $s$ of a set in the state space. In this case $x_t^*$ has the same finite dimensional distributions as $x_t$ and the subprocess of $x_t^*$ is equal to $(w(s), \mathbf{P})$.

3.2. In this Section we shall prove Theorem 3.1.1. Let $\Omega$ be the space of all paths $\omega(t)$ in $D$, $-\infty<t<+\infty$. Let $\bar{p}(t,x;\Gamma)$ be a transition function such that $\tilde{\mathbf{T}}_t f(x)=\bar{p}(t,x;f)$. By the Kolmogorov Theorem there exists a stationary Markov measure $\overline{\mathbf{P}}$ on $\Omega$ with the one-dimensional distribution $\nu$ and the transition function $\bar{p}$. Let $\theta_t$ be a shift operator in the space $\Omega$, that is $\theta_t\omega(s)=\omega(s+t)$. Put $x_t(\omega)=\omega(t)$ and put $\mathscr{F}_t=\sigma(x_s, s<t)$, $\mathscr{F}^t=\sigma(x_s, s>t)$. Let $\overline{\mathbf{P}}_x$ be the transition probabilities of $(x_s, \overline{\mathbf{P}})$, that is $\overline{\mathbf{P}}_x$ is a measure on $\mathscr{F}^0$ such that

$$\begin{aligned}&\overline{\mathbf{P}}_x\{x_{t_1}\in dx_1, \dots, x_{t_n}\in dx_n\}\\ &\quad=\bar{p}(t_1, x; dx_1)\,\bar{p}(t_2-t_1, x_1; dx_2)\dots\bar{p}(t_n-t_{n-1}; x_{n-1}\,dx_n),\\ &\quad 0<t_1<t_2<\dots<t_n.\end{aligned}$$

By Theorem 9.3 of [3] there exists an almost homogenous, almost multiplicative functional (AHAMF) $\bar{\alpha}_t(\omega)$ such that

$$p(x,t;\Gamma)=\overline{\mathbf{P}}_x\{\bar{\alpha}_t(\omega)\,1_\Gamma(x_t)\}. \tag{3.2.1}$$

By properties of AHAMF the random variable $\bar{\alpha}_t$ is $\sigma(x_s, 0\leqq s\leqq t)$-measurable. Formula (9.13) in [3] shows that for any $t$ and for all $x\in D$

$$\bar{\alpha}_t\leqq 1 \qquad \text{a.s. } \overline{\mathbf{P}}_x.$$

**Lemma 3.2.1.** *There exists AHAMF $\alpha_t$ such that for all $x\in D$ a.s. $\overline{\mathbf{P}}_x$:*

3.2.$\alpha$. *For any $t$ $\alpha_t=\bar{\alpha}_t$.*

3.2.$\beta$. *$\alpha_t$ is a right-continuous function of $t$, $t>0$, and $\alpha_{0+}=1$.*

*Proof.* Since $\bar{\alpha}_t\leqq 1$ a.s. $\overline{\mathbf{P}}_x$, then $\bar{\alpha}_t\leqq\bar{\alpha}_s$ a.s. $\mathbf{P}_x$ for $s<t$. Let $R-\lim$ stand for the limit taken over rational points and put

$$\alpha_t(\omega)=R-\limsup_{r\downarrow t}\bar{\alpha}_r(\omega).$$

Inasmuch as $\bar{\alpha}_r$, $r$-rational, is a decreasing function of $r$ a.s. $\overline{\mathbf{P}}_x$, then $\alpha_t$ is right-continuous. It is obvious that

$$\alpha_t\leqq\bar{\alpha}_t \qquad \text{a.s. } \overline{\mathbf{P}}_x. \tag{3.2.2}$$

By 1.2.B and the Chapman-Kolmogorov equation for $p$ we get

$$p(t,x;D)\uparrow p(s,x;D) \qquad \text{as } t\downarrow s. \tag{3.2.3}$$

By the monotone convergence theorem we have

$$R-\lim_{r\downarrow t}\overline{\mathbf{P}}_x\{\bar{\alpha}_r(\omega)\}=\overline{\mathbf{P}}_x(\bar{\alpha}_t(\omega)\}. \tag{3.2.4}$$

Combining (3.2.1), (3.2.2), (3.2.3) and (3.2.4) we get that $\alpha_t=\bar{\alpha}_t$ a.s. $\overline{\mathbf{P}}_x$. By 1.2.B

$$\lim_{s\downarrow 0}p(s,x;D)=1,$$

therefore, $\lim_{s\downarrow 0}\alpha_s=1$ a.s. $\overline{\mathbf{P}}_x$ and the lemma is proved.

Note that because $\nu$ is an extreme excessive measure then it has to be null-excessive (see [4]), and that implies for $\nu$ a.e. $x$

$$p(s,x;D)\to 0 \quad \text{as } s\to\infty.$$

(We exclude the trivial case in which $\nu$ is an invariant measure.) Therefore, for $\nu$ a.e. $x$

$$\lim_{t\to\infty}\alpha_t(\omega)=0 \quad \text{a.s. } \bar{\mathbf{P}}_x. \tag{3.2.5}$$

Put

$$\alpha_t^s(\omega)=\alpha_{t-s}(\theta_s\omega)$$

and let $\alpha^s(dy|\omega)$ stand for the measure on $T$ with the distribution function $F(\cdot)$ equal to $1-\alpha_\cdot^s(\omega)$. Inasmuch as (3.2.5) holds, $1-\alpha_\cdot^s(\omega)$ is a proper distribution function for $\bar{\mathbf{P}}$-almost all $\omega$.

Let $\tilde{\Omega}$ be a product of $\Omega$ and $(T)^\infty$ and the measurable structure $\tilde{G}$ in $\tilde{\Omega}$ generated by the product of the corresponding measurable structures. We denote by the same letter $\mathscr{F}$ the $\sigma$-field in $\tilde{\Omega}$ generated by the sets of the type $A\times(T)^\infty$ where $A\in\mathscr{F}\equiv\mathscr{F}_\infty$.

The following theorem is most important in carrying out the construction of the process $x_t^*$.

**Theorem 3.2.2.** *There exists a probability measure* $\mathbf{Q}$ *on* $\tilde{\Omega}$ *and a family of random variables* $\tau_s(\tilde{\omega})$ *such that*

3.2.A. *For each* $A\in\mathscr{F}$

$$\mathbf{Q}\{A\times(T)^\infty\}=\bar{\mathbf{P}}\{A\}.$$

3.2.B. *For each* $s\in T$ *and* $\Gamma\subset T$

$$\mathbf{Q}\{\tau_s(\tilde{\omega})\in\Gamma|\mathscr{F}\}=\alpha^s(\Gamma|\omega) \quad \textit{a.s. } \bar{\mathbf{P}}.$$

3.2.C. *The following relations hold simultaneously a.s.* $\mathbf{Q}$ *for all* $s\leqq t$:

$$\tau_s\geqq s,$$
$$\tau_s\leqq\tau_t, \quad \text{and} \quad \tau_s=\tau_t \ \textit{on the set } \{\tau_s>t\}.$$

3.2.D. *If* $s<u\leqq t<r$ *then the events* $\{\tau_s<u\}$ *and* $\{\tau_t>r\}$ *are conditionally independent given* $\mathscr{F}$.

*Proof.* 1°. Put $\tilde{\Omega}^0=\Omega$, $\tilde{\Omega}^{k+1}=\tilde{\Omega}^k\times T$. The set $\tilde{\Omega}^k$ consists of all $k+1$-tuples $(\omega,t_1t_2,\dots,t_k)$, where $\omega\in\Omega$, $t_k\in T$. To construct the measure $\mathbf{Q}$ on $\tilde{\Omega}$ it is enough to construct a sequence of probability kernels $n_1,n_2,\dots,n_k,\dots$ where $n_k$ is a kernel from $\tilde{\Omega}^{k-1}$ into $T$ and then put

$$\begin{aligned}&\mathbf{Q}\{d\omega\times dt_1\times dt_2\times\dots\times dt_m\times T\times T\times\dots\}\\&\quad=\bar{\mathbf{P}}\{d\omega\}n_1(\omega;dt_1)n_2(\omega,t_1;dt_2)\dots n_m(\omega,t_1,t_2,\dots,t_{m-1};dt_m).\end{aligned} \tag{3.2.6}$$

Let $r_1,r_2,\dots,r_k,\dots$ be a sequence of all rational numbers. For typographical purposes we write $r_k$ and $r(k)$ interchangeably. The last coordinate in $\tilde{\omega}^k$

corresponds to the value of $\tau_{r(k)}$, i.e., $\tau_{r(k)}(\tilde{\omega})=t_k$. We shall construct $n_k$ by induction in such a way that 3.2.B–3.2.C will hold for the family $\{\tau_{r(1)}, \tau_{r(2)}, \ldots, \tau_{r(k)}\}$ if these properties are satisfied for the family $\{\tau_{r(1)}, \ldots, \tau_{r(k-1)}\}$. (Note that due to (3.2.6) the property 3.2.A will hold automatically.)

2°. By the properties of AHAMF and definitions of $\alpha_t^s(\omega)$

$$\alpha_t^s \alpha_u^t = \alpha_u^s \qquad \text{a.s. } \overline{\mathbf{P}} \tag{3.2.7}$$

for any fixed $s<t<u$. Therefore, (3.2.7) is true for all rational $s, t$, and $u$ simultaneously. Since $\alpha_u^t$ is right-continuous in $u$ (3.2.7) is true a.s. $\overline{\mathbf{P}}$ for any rational $s$ and $t$ and all $u$. Put $n_1(\omega; \Gamma)=\alpha^{r(1)}(\Gamma|\omega)$ and put $\tau_{r(1)}(\tilde{\omega})=t_1$. The properties 3.2.B–3.2.D are satisfied trivially for the family consisting of a single element $\tau_{r(1)}$.

Suppose that the kernels $n_1, n_2, \ldots, n_{k-1}$ are constructed in such a way that the family of random variables $(\tau_{r(1)}=t_1,\ \tau_{r(2)}=t_2, \ldots, \tau_{r(k-1)}=t_{k-1})$ satisfies 3.2.B–3.2.D. Let $\Lambda_m=\{r_1, r_2, \ldots, r_m\}$ and $b=\min \Lambda_{k-1}\cap[r_k, +\infty[$, $a=\max \Lambda_{k-1}\cap]-\infty, r_k]$. Suppose that $a>-\infty$ and $b<+\infty$; consequently there exist $i\leqq n-1$ and $j\leqq n-1$ such that $a=r_i$, $b=r_j$. Put

$$\begin{aligned} n_k(\omega, t_1, t_2, \ldots, t_k; \Gamma) &= n_k(\omega, t_i, t_j; \Gamma) \\ &= \begin{cases} 1_{t_i}(\Gamma), & \text{if } t_i > r_k; \\ \alpha^{r(k)}(\Gamma'|\omega)+\alpha_b^{r(k)} 1_{t_j}(\Gamma''), & \text{if } t_i \leqq r_k. \end{cases} \end{aligned} \tag{3.2.8}$$

Here $\Gamma'=\Gamma\cap[r_k, b]$, $\Gamma''=\Gamma\cap]b, \infty[$.

3°. We have to check that the properties 3.2.B–3.2.D hold for the new family $\{\tau_{r(1)}, \ldots, \tau_{r(k-1)}, \tau_{r(k)}\}$. The property 3.2.C is trivially satisfied by the construction of the kernel (3.2.8).

To check 3.2.B we may consider only sets $\Gamma$ lying on the ray $[r_k, \infty[$. (In the following formulae $\mathbf{Q}_{\mathscr{F}}$ means conditioning with respect to $\mathscr{F}$; and all equations are satisfied a.s. $\mathbf{Q}$.)

$$\begin{aligned} \mathbf{Q}_{\mathscr{F}}\{t_k\in\Gamma\} &= \mathbf{Q}_{\mathscr{F}}\{t_k\in\Gamma,\ t_i>r_k\}+\mathbf{Q}_{\mathscr{F}}\{t_k\in\Gamma,\ t_i\leqq r_k\} \\ &= \mathbf{Q}_{\mathscr{F}}\{t_i\in\Gamma\}+\mathbf{Q}_{\mathscr{F}}\{t_k\in\Gamma|t_i<r_k\}\,\mathbf{Q}_{\mathscr{F}}\{t_i<r_k\} \\ &= \alpha^a(\Gamma)+(1-\alpha_{r(k)}^a)(\alpha^{r(k)}(\Gamma')+\alpha_b^{r(k)}\alpha^b(\Gamma'')). \end{aligned} \tag{3.2.9}$$

The last equality in (3.2.9) is due to the conditional independence of $\tau_a=t_i$ and $\tau_b=t_j$ given $\mathscr{F}$ on the set $\{t_i<b\}$ (property 3.2.D). By (3.2.7)

$$\alpha^a(\Gamma)=\alpha_{r(k)}^a\,\alpha^{r(k)}(\Gamma) \tag{3.2.10}$$

and

$$\alpha_b^{r(k)}\alpha^b(\Gamma'')=\alpha^{r(k)}(\Gamma''). \tag{3.2.11}$$

Comparing (3.2.10) and (3.2.11) with the right side of (3.2.9), we get that (3.2.9) is equal to $\alpha^{r(k)}(\Gamma)$.

4°. Now prove 3.2.D. Consider the case $t=r(k)$, $s=r(i)=a$ and $u<t<r$. Suppose $r<b$. Then

$$\begin{aligned}
&\mathbf{Q}_{\mathscr{F}}\{t_i<u, t_k>r\}\\
&\quad=\mathbf{Q}_{\mathscr{F}}\{t_i<u, r<t_k\leqq b\}+\mathbf{Q}_{\mathscr{F}}\{t_i<u, t_k>b\}\\
&\quad=\mathbf{Q}_{\mathscr{F}}\{t_i<u\}(\alpha_r^{r(k)}-\alpha_b^{r(k)})+\mathbf{Q}_{\mathscr{F}}\{t_i<u\}\,\alpha_b^{r(k)}\\
&\quad=\mathbf{Q}_{\mathscr{F}}\{t_i<u\}\,\alpha_r^{r(k)}=\mathbf{Q}_{\mathscr{F}}\{t_i<u\}\,\mathbf{Q}_{\mathscr{F}}\{t_k>r\}.
\end{aligned}$$

All the other cases are considered similarly.

5°. We have constructed a measure $\mathbf{Q}$ on $\tilde{\Omega}$ and a family $\tau_s(\tilde{\omega})$ ($s$-rational) which satisfy 3.2.A–3.2.D. Show that $\tau_s$ is a.s. $\mathbf{Q}$ right-continuous in $s$. Since $s$ takes on only a countable number of values it is sufficient to show that for any fixed $r$ $\tau_u\downarrow\tau_r$ a.s. $\mathbf{Q}$ when $u\downarrow r$. Owing to the fact that $\{\tau_r\neq\tau_u\}\subset\{\tau_r\leqq u\}$ a.s. $\mathbf{Q}$, we get

$$\mathbf{Q}\{\tau_u-\tau_r>u-r\}\leqq\mathbf{Q}\{\tau_u\neq\tau_r\}=\mathbf{Q}\{\tau_r\leqq u\}=\bar{\mathbf{P}}\{1-\alpha_u^r\}. \tag{3.2.12}$$

Since $\alpha_u^r(\omega)=\alpha_{u-r}(\theta_r\omega)$ then

$$\lim_{u\downarrow r}\alpha_u^r=\lim_{\varepsilon\downarrow 0}\alpha_\varepsilon(\theta_r\omega)=1 \qquad \text{a.s. } \bar{\mathbf{P}}, \tag{3.2.13}$$

(see 3.2.$\beta$). Therefore the limit in the right side of (3.2.12) is equal to zero whenever $u\downarrow r$. For irrational $u$ put

$$\tau_u(\tilde{\omega})=R-\limsup_{s\downarrow u}\tau_s(\tilde{\omega}). \tag{3.2.14}$$

Since a.s. $\mathbf{Q}$ $\tau_s$ is a nondecreasing right-continuous function of $s$ for rational $s$, then the right side of (3.2.14) has a finite limit for all $u$ a.s. $\mathbf{Q}$. Therefore, (all equalities below are true a.s. $\mathbf{Q}$.)

$$\begin{aligned}
\mathbf{Q}\{\tau_u\geqq t|\mathscr{F}\}&=R-\lim_{s\downarrow u}\mathbf{Q}\{\tau_s\geqq t|\mathscr{F}\}\\
&=R-\lim_{s\downarrow u}(\mathbf{Q}\{\tau_s>t|\mathscr{F}\}+\mathbf{Q}\{\tau_s=t|\mathscr{F}\})\\
&=R-\lim_{s\downarrow u}(\alpha_t^s+(\alpha_{t-}^s-\alpha_t^s)).
\end{aligned}$$

By virtue of 2.1.A $\alpha_t=\alpha_{t-}$ a.s. $\bar{\mathbf{P}}$ for each fixed $t$. Consequently, $\alpha_{t-}^s=\alpha_t^s$ a.s. $\bar{\mathbf{P}}$ and

$$\mathbf{Q}\{\tau_u\geqq t|\mathscr{F}\}=R-\lim_{s\downarrow u}\alpha_t^s=R-\lim_{s\downarrow u}(\alpha_t^u/\alpha_s^u)=\alpha_t^u. \tag{3.2.15}$$

The last equality in (3.2.15) is due to (3.2.13). Applying right-continuity of $\alpha_t^u$ in $t$, we get

$$\mathbf{Q}\{\tau_u>t|\mathscr{F}\}=\alpha_{t+}^u=\alpha_t^u.$$

Therefore, 3.2.B holds for all real $s$. To verify 3.2.C and 3.2.D one has to pass to the limit in the corresponding relations for rational $s$ and $t$.

**Lemma 3.2.3.** *Let $\mathscr{C}_t$ be the σ-field in $\tilde{\Omega}$ generated by all random variables of the form $g(\tau_s)$, where $s<t$ and $g$ is a measurable function with support on $[-\infty, t]$. Then for each $t$ the random variable $\tau_t$ and the σ-field $\mathscr{C}_t \vee \mathscr{F}_t$ are conditionally independent given $x_t$.*

*Proof.* Let $\xi$ be $\mathscr{F}_t$-measurable and $s_1<s_2<\ldots<s_k=t$. Let $g_i(x)=1_{]u_i, v_i]}$, where $s_i \leqq u_i < v_i \leqq s_{i+1}$, $i=1, 2, \ldots, k$ (we assume $s_{k+1}=\infty$). By 3.2.B and 3.2.D

$$\begin{aligned}
&\mathbf{Q}\{\xi g_1(\tau_{s_1}) g_2(\tau_{s_2}) \ldots g_k(\tau_t)|x_t\} \\
&=\mathbf{Q}\{\mathbf{Q}\{\xi g_1(\tau_{s_1}) \ldots g_k(\tau_t)|\mathscr{F}\}|x_t\} \\
&=\mathbf{Q}\{\xi(\alpha_{u_1}^{s_1}-\alpha_{v_1}^{s_1}) \ldots (\alpha_{u_k}^t-\alpha_{v_k}^t)|x_t\}.
\end{aligned} \tag{3.2.16}$$

Because $\alpha_{u_i}^{s_i}-\alpha_{v_i}^{s_i}$ is $\mathscr{F}_{v_i}$-measurable and $\alpha_{u_k}^t-\alpha_{v_k}^t$ is $\mathscr{F}^t$-measurable, then by the Markov property for $(x_t, \bar{\mathbf{P}})$ the right side of (3.2.16) is equal to

$$\begin{aligned}
&\mathbf{Q}\left\{\xi \prod_{i=1}^{k-1} \alpha_{u_i}^{s_i}-\alpha_{v_i}^{s_i}|x_t\right\} \mathbf{Q}\{\alpha_{u_k}^t-\alpha_{v_k}^t|x_t\} \\
&=\mathbf{Q}\left\{\xi \prod_{i=1}^{k-1} g_i(\tau_{s_i})|x_t\right\} \mathbf{Q}\{g_k(\tau_t)|x_t\}.
\end{aligned} \tag{3.2.17}$$

Standard arguments show that (3.2.17) holds for all functions $g_1, g_2, \ldots, g_{k-1}$ with support on $]-\infty, t[$ and all functions $g_k$ with support on $[t, \infty[$.

3.3. Let $M'(\omega)=\{u: u=\tau_s(\tilde{\omega})$ for some $s\}$ and $M(\omega)$ be a closure of $M'(\tilde{\omega})$. Note that $M'$ is closed from the right and $M \setminus M'$ consists of no more than a countable number of points a.s. $\mathbf{Q}$. It is obvious that

$$\tau_s(\tilde{\omega})=\inf\{t: t>s, t \in M(\tilde{\omega})\}, \quad \text{a.s. } \mathbf{Q}. \tag{3.3.1}$$

Let $V$ be a replica of $D$ and $x'$ be the image in $V$ of the point $x$ in $D$. Put

$$x_t^*(\tilde{\omega})=\begin{cases} x_t(\omega) & \text{if } t \bar{\in} M(\tilde{\omega}) \\ x_t'(\omega) & \text{if } t \in M(\tilde{\omega}). \end{cases}$$

**Lemma 3.3.1.** Let $G^*$ be the universal completion of $\tilde{G}$. For each fixed $t$ $x_t^*$ is $G^*$-measurable,

$$\mathbf{Q}\{x_t \neq x_t^*\}=0$$

and the finite dimensional distributions of $(x_t^*, \mathbf{Q})$ are equal to those of $(x_t, \bar{\mathbf{P}})$.

*Proof.* 1°. It is easy to see that $\tau_s$ is a stopping time with respect to the filtration $\mathscr{C}_{t+}$. The set $M=\{t, \tilde{\omega}: t \in M(\tilde{\omega})\}$ is the closure of $\bigcup_k [[\tau_{r_k}]]$, where $[[\tau_s]]$ stands for the graph of $\tau_s$ in $T \times \tilde{\Omega}$. By T4, Chap. 6 of [2] $M$ is a measurable set in $T \times \tilde{\Omega}$. Therefore, its $t$-section is a measurable set and its projection on $\tilde{\Omega}$ is universal measurable (see T32, Chap. 1 of [2]).

The third statement of the lemma is a trivial consequence of the second one. To prove the second one it is enough to show that for any fixed $t$

$$\mathbf{Q}\{t \in M\}=0.$$

Fix $t \in T$. If $t \in M'$ then either $\tau_t = t$ or $\tau_s = t$ for some rational $s < t$. Due to the condition 2.1.A for each fixed $t$

$$\begin{aligned} 0 &= \mathbf{T}_{t-} 1(x) - \mathbf{T}_t 1(x) \\ &= \lim_{s \uparrow t} \bar{\mathbf{P}}_x\{x_s \in D\} - \bar{\mathbf{P}}_x\{x_t \in D\} \\ &= \bar{\mathbf{P}}_x\{\alpha_{t-} - \alpha_t\}. \end{aligned}$$

Consequently, $\mathbf{P}_x\{\alpha_{t-} - \alpha_t\} = 0$ and

$$\mathbf{Q}\{\tau_s = s + t\} = \mathbf{Q}\{\mathbf{Q}\{1_{t+s}(\tau_s) | \mathscr{F}\}\} = \bar{\mathbf{P}}\{\bar{\mathbf{P}}_{x_s}\{\alpha_{t-} - \alpha_t\}\}.$$

Similarly, applying 3.2.$\beta$, we get

$$\mathbf{Q}\{\tau_t = t\} = \bar{\mathbf{P}}\{\lim_{u \downarrow t} \alpha_u^t \neq 1\} = 0.$$

Therefore, $\mathbf{Q}\{t \in M'\} = 0$.

2°. If $t \in M \setminus M'$ then $\tau_s < t$ a.s. $\mathbf{Q}$ for each rational $s < t$ and vice versa. By 3.2.B

$$\mathbf{Q}\{\tau_s \leqq t\} = \bar{\mathbf{P}}\{1 - \alpha_t^s\}.$$

Therefore,

$$\mathbf{Q}\{\tau_s < t \text{ for each rational } s < t\} \leqq \bar{\mathbf{P}}\{R - \lim_{s \uparrow t} \alpha_t^s \neq 1\}. \tag{3.3.2}$$

Since $\bar{\mathbf{P}}$ is a stationary measure and $\alpha_{v+r}^{u+r}(\omega) = \alpha_v^u(\theta_r \omega)$, then the right side of (3.3.2) does not depend on $t$. Applying the Fubini theorem, we get

$$\mathbf{Q}\{t \in M \setminus M'\} = \int_0^1 \mathbf{Q}\{s \in M \setminus M'\}\, ds = \mathbf{Q}\left\{\int_0^1 1_s(M \setminus M')\, ds\right\} = 0.$$

**Lemma 3.3.2.** *The subprocess in D of the process* $(x_t^*, \mathbf{Q})$ *is equal to* $(w(s), \mathbf{P})$.

*Proof.* It is enough to consider the finite-dimensional distribution of the subprocess, that is the expressions

$$\begin{gathered} \mathbf{Q}\{x_{t_1}^* \in \Gamma_1, x_{t_2}^* \in \Gamma_2, \ldots, x_{t_n}^* \in \Gamma_n, [t_1, t_n] \cap M = \emptyset\}, \\ t_1 < t_2 < \ldots < t_n; \quad \Gamma_1, \Gamma_2, \ldots, \Gamma_n \subset \mathrm{D}. \end{gathered} \tag{3.3.3}$$

Put $A_i = \{x_{t_i} \in \Gamma_i\}$, $i = 1, 2, \ldots, n$. By (3.3.1) the expression (3.3.3) equals

$$\mathbf{Q}\{A_1 A_2 \ldots A_n, \tau_{t_1} > t_n\} = \mathbf{Q}\{\mathbf{Q}\{A_1 \ldots A_n, \tau_{t_1} > t_n | \mathscr{F}\}\}. \tag{3.3.4}$$

By 3.2.B the expression (3.3.4) equals

$$\mathbf{Q}\{1_{A_1 \ldots A_n} \alpha_{t_n}^{t_1}\} = \bar{\mathbf{P}}\{1_{A_1} \alpha_{t_2}^{t_1} 1_{A_2} \alpha_{t_3}^{t_2} \ldots 1_{A_{n-1}} \alpha_{t_n}^{t_{n-1}} 1_{A_n}\}. \tag{3.3.5}$$

We know that $\alpha_b^a$ is $\mathscr{F}^a \wedge \mathscr{F}_b$-measurable. Therefore, we may apply the Markov property to the right side of (3.3.5). Doing so, we get

$$\mathbf{Q}\{A_1, \ldots, A_n, \tau_{t_1} > t_n\} = \{\bar{\mathbf{P}}\{1_{A_1} \alpha_{t_2}^{t_1} \ldots 1_{A_{n-1}} \varphi(x_{t_{n-1}})\},$$

where

$$\varphi(x)=\bar{\mathbf{P}}_x\{1_{\Gamma_n}(x_{t_n})\alpha_{t_n-t_{n-1}}\}=p(t_n-t_{n-1},x;\Gamma_n),$$

Repeating this argument $(n-1)$ times, we get

$$\begin{aligned}&\mathbf{Q}\{A_1,A_2,\ldots,A_n,\tau_{t_1}>t_n\}\\&\quad=\int_{\Gamma_1}\nu(dx_1)p(t_2-t_1,x_1;dx_2)\int_{\Gamma_2}p(t_3-t_2,x_2;dx_3)\ldots\int_{\Gamma_{n-1}}p(t_n-t_{n-1},x_{n-1};\Gamma_n),\end{aligned}$$

and that is the finite dimensional distribution of $(w(s),\mathbf{P})$.

Lemma 3.3.2 completes the proof of Theorem 3.3.1.

3.4. Suppose that $\tilde{\mathbf{T}}_t$ and $\hat{\mathbf{T}}_t$ are two semigroups which are enhancing of $\mathbf{T}_t$. Let $\tilde{p}$ and $\hat{p}$ be the transition functions of these semigroups. By virtue of Theorem 3.3.1 it is possible to construct Markov processes $(\tilde{y}_t,\tilde{\mathbf{Q}})$ and $(\hat{y}_t,\hat{\mathbf{Q}})$ in $D+V$ in such a way that the one-dimensional distribution of $(\tilde{y}_t,\tilde{\mathbf{Q}})$ (of $(\hat{y}_t,\hat{\mathbf{Q}})$) is concentrated on $D$ and is equal to $\nu$ and the transition function of $(\tilde{y}_t,\tilde{\mathbf{Q}})$ (of $(\hat{y}_t,\hat{\mathbf{Q}})$) is equal to $\tilde{p}$ (to $\hat{p}$); and both $(\tilde{y}_t,\tilde{\mathbf{Q}})$ and $(\hat{y}_t,\hat{\mathbf{Q}})$ are covering processes for $(w(s),\mathbf{P})$.

By Theorem 2 of [11] the finite-dimensional distributions of $(\tilde{y}_t,\tilde{\mathbf{Q}})$ coincide with those of $(\hat{y}_t,\hat{\mathbf{Q}})$. Therefore, for $\Gamma_1,\Gamma_2\subset D$ and $t\in T$

$$\int_{\Gamma_1}\nu(dx)\hat{p}(t,x;\Gamma_2)=\int_{\Gamma_1}\nu(dx)\tilde{p}(t,x;\Gamma_2). \tag{3.4.1}$$

Consequently

$$\hat{p}(t,x;\Gamma_2)=\tilde{p}(t,x;\Gamma_2)\qquad\text{for }\nu\text{-a.e. }x. \tag{3.4.2}$$

Standard arguments show that (3.4.2) is true for all $\Gamma\subset D$ and for all $t\in T$ a.s. $\nu$.

It is necessary to mention that in our situation not all the conditions of Theorem 2 of [11] hold. The set $M$ does not satisfy 5.1.A of [11], i.e., it is not progressive measurable with respect to the filtrations generated by the process $(x_t^*,\mathbf{Q})$. Nevertheless, if we replace 5.1.A of [11] by the condition 3.4.A below all the proofs remain the same without any changes.

3.4.A. For any $\mathscr{F}_t$-measurable random variable $\xi$, any $k$, and any functions $f_1,f_2,\ldots,f_k$ on $T$ with support on $]-\infty,t[$ and any $s_1<s_1<\ldots<s_k$ the random variable $\xi f_1(\tau_{s_1})f_2(\tau_{s_2})\ldots f_k(\tau_{s_k})$ and the random variable $\tau_t$ are conditionally independent, given $x_t^*$.

In our situation 3.4.A is an immediate consequence of Lemma 3.2.3 and Lemma 3.3.1.

The following example shows that we can not prove that $\tilde{\mathbf{T}}_t$ in Theorem 2 is unique everywhere (not up to the measure $\nu$).

Let $D_1$ be a Borel space and $\mathbf{T}_t^{(1)}$ be a semigroup such that there exist at least two semigroups $R_t^1$ and $R_t^2$, $R_\cdot^1\neq R_\cdot^2$ and both semigroups are enhancing of $\mathbf{T}_t^{(1)}$. (The example of such semigroup $\mathbf{T}_t^{(1)}$ was given in Sect. 1.3 of paper [11].) Let $D_2$ be another Borel space and $\mathbf{T}_t^{(2)}$ be a semigroup on $D_2$. Put $D=D_1\cup D_2$ and consider a semigroup $\mathbf{T}_t$ on $D$ defined

$$\mathbf{T}_tf(x)=1_{D_1}(x)\mathbf{T}_t^{(1)}(f1_{D_1})(x)+1_{D_2}(x)\mathbf{T}_t^{(2)}(f1_{D_2})(x).$$

We write $\mathbf{T}_t = \mathbf{T}_t^{(1)} + \mathbf{T}_t^{(2)}$. It is easy to see that if $\bar{v}$ is a measure on $D_2$ which is excessive with respect to $\mathbf{T}_t^{(2)}$ then the measure $v$ on $D$ defined

$$v(\Gamma) = \bar{v}(\Gamma \cap D_2)$$

is excessive with respect to $\mathbf{T}_t$. Moreover, if $\bar{v}$ is extreme, then so is $v$. Let $R_t$ be a semigroup on $D_2$ which enhances $\mathbf{T}_t^{(2)}$ and preserves $\bar{v}$. Put $\tilde{\mathbf{T}}_t' = R_t^1 + R_t$ and $\tilde{\mathbf{T}}_t'' = R_t^2 + R_t$. Then both $\tilde{\mathbf{T}}_t'$ and $\tilde{\mathbf{T}}_t''$ are enhancing of $\mathbf{T}_t$ and $v$ is an invariant measure with respect to both $\tilde{\mathbf{T}}_t'$ and $\tilde{\mathbf{T}}_t''$.

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