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Non Reversible Stationary Measures for Infinite Interacting Particle Systems

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Summary. Two results concerning the local conditional distributions of a stationary measure for a spin flip process with strictly positive and continuous rates are obtained: 1) The local conditional distributions and the rates of the reversed process determine each other. 2) Either all shift invariant stationary measures are Gibbs with the same potential or no shift invariant stationary Gibbs measure exist.

0. Introduction

Consider a countable set of particles located on the lattice $\mathbb{Z}^d$, each particle having a spin which is pointing either up or down. The configuration space of this particle system is $E=\{\pm 1, -1\}^{\mathbb{Z}^d}$. We let the system evolve in time according to the following rule: If at time t the configuration of the system is $\eta \in E$, then during the time interval $[t, t+dt]$ the particle at i changes its spin with probability $c_i(\eta)\,dt$ independently of the other particles. The interaction comes in only through the dependence of the flip rate $c_i(\eta)$ on the values η_j, $j \neq i$. The central question is the behaviour of such a process for large times. Infinite spin systems are only one possible application of these processes. For instance, if we interpret $\eta_i = +1$ (-1) as the site i being occupied (empty), they have a natural interpretation as birth-death processes and can be used for various biological models. However, we stick here to the spin-flip interpretation because our notations corresponds intuitively to this picture. Introductions to these processes are given in Liggett [15] and Durrett [3], for applications see also the papers in Dobrushin et al. [1].

In contrast to finite systems there is in general more than one stationary measure for these processes even if all rates c_i are bounded away from zero. In the known examples, this non-uniqueness is closely related with the phenomenon of phase transition for Gibbs measures (see e.g. Preston [17]). To a given

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interaction potential, one can construct rates c_i such that all Gibbs measures are stationary for the corresponding process. Since there are examples with more than one Gibbs measure, we immediately obtain examples where the stationary measure is not unique. A natural question is then if for these processes there are other stationary measures than the Gibbs measures. It has been proved that this is not the case if the dimension of the lattice is less than three (Holley-Stroock [10]) or if we restrict ourselves to shift invariant measures (Holley [8]). By the definition of Gibbs measures, this means that the local conditional distributions of a single spin given all other spins are the same for all stationary measures.

However, the processes which are constructed from a given interaction potential, are of a special type: the Gibbs measures are not only stationary, but even reversible. Clearly, there are many processes having no reversible measures at all, but we conjecture that a similar statement about the local conditional distributions of stationary measures holds for general processes with strictly positive and smooth flip rates c_i. In this paper we give some results which indicate why we believe in this conjecture. Our main result is the following: If there exists a shift invariant stationary measure which is Gibbs for some interaction potential, then any other shift invariant stationary measure is necessarily also Gibbs with the same interaction potential (Corollary 4.2). This result is obtained from a generalization of the free energy technique of Holley [8]. We first derive an equation which is satisfied by the local conditional distributions of a stationary measure (Theorem 3.1), and with this equation we can show that the derivative of the free energy is negative also in non reversible situations. In Sect. 2 we show that the local conditional distributions of a stationary measure and the flip rates of the reversed process determine each other. Our conjecture is therefore equivalent to the statement that all stationary measures have the same time reversal.

Compared with the reversible case, our results are weaker in three points: First, we do not know if in general all Gibbs measures with the same potential are stationary provided one of them is. For the proof of such a statement we need that the mixing coefficients decay quickly for extremal Gibbs measures (Theorem 3.3). Second, we could not prove in general the existence of a stationary Gibbs measure although this is very likely to hold. In view of Sect. 2, a stationary measure which is not Gibbs must have a peculiar time reversal, and in Sect. 5 we give at least a formal power series for the interaction potential of a stationary Gibbs measure. Finally, we have no result about those stationary measures which are not shift invariant, see the comment at the end of Sect. 4. Nevertheless, since it is much more difficult to get a description of the non reversible stationary measures, we think that our results are valuable for the understanding of the non-ergodicity of Markov processes on infinite product spaces.

1. Notations and Definitions

Elements of $E=\{+1,-1\}^{\mathbb{Z}^d}$ are denoted by η or ζ. η_i is the value of η at i, and X_i: $E\to\{-1,+1\}$ is the i-th projection $\eta\to\eta_i$. We introduce the σ-fields $\mathscr{E}$

$=\sigma(X_i,\ i\in\mathbb{Z}^d)$, $\mathscr{E}_V=\sigma(X_i,\ i\in V)$ and $\bar{\mathscr{E}}_V=\sigma(X_i,\ i\notin V)$. The symbol V always means a finite subset of $\mathbb{Z}^d$, including the empty set. We give $\{+1,-1\}$ the discrete topology and E the resulting product topology. $\mathscr{E}_b$ denotes the set of bounded measurable functions on E and $\mathscr{C}$ the set of continuous functions with the supremum norm $\|f\|=\sup_\eta|f(\eta)|$. The set $\mathscr{C}_0$ of functions depending only on finitely many η_i is dense in $\mathscr{C}$.

The processes described intuitively in the introduction are Markov processes on E whose generator $\mathscr{L}$ is given for functions $f\in\mathscr{C}_0$ by

$$\mathscr{L}f(\eta)=\sum_i c_i(\eta)\,\nabla_i f(\eta). \tag{1.1}$$

Here $\nabla_i f(\eta)=f({}_i\eta)-f(\eta)$ and ${}_i\eta$ is obtained from η by changing the spin at i. If the flip rates $c_i\in\mathscr{C}$, $c_i\geqq 0$ satisfy Liggett's condition

$$\sup_i\sum_k\|\nabla_k c_i\|<\infty,\qquad \sup_i\|c_i\|<\infty, \tag{1.2}$$

then the closure of $(\mathscr{L},\mathscr{C}_0)$ is the generator of a Feller semigroup P_t, see Liggett [15], I.1.2. Since E is compact, the set of stationary measures $\mathscr{I}=\{\mu,\ \mu_t=\mu$ for all $t\}$ where $\int f\,d\mu_t=\int P_t f\cdot d\mu$ is never empty. Moreover $\mu\in\mathscr{I}$ iff $\int\mathscr{L}f\cdot d\mu=0$ for all $f\in\mathscr{C}_0$.

A different way of constructing a process with given flip rates goes via the martingale problem, see Holley-Stroock [9]. In this approach the existence becomes easy for continuous rates, but the real problem is uniqueness which does not always hold. Gray-Griffeath [6] gave an example where uniqueness holds though $\overline{(\mathscr{L},\mathscr{C}_0)}$ is not the generator of a semigroup, but the general conditions needed for uniqueness of a solution of the martingale problem also assure that $\overline{(\mathscr{L},\mathscr{C}_0)}$ generates a Feller semigroup. For this reason and because (1.2) is sufficient for our purpose here, we will not use the martingale problem except at one place, in Proposition 2.4.

If μ is a probability measure on $(E,\mathscr{E})$, its local conditional distributions p_μ^i are defined as

$$p_\mu^i(\eta)=\mu[X_i=\eta_i\,|\,\bar{\mathscr{E}}_i](\eta). \tag{1.3}$$

Different μ's can have the same local conditional distributions; then we say that *phase transition* occurs. Usually not μ is given, but a family of conditional distributions p^i, and one looks for those μ whose local conditional distributions are equal to p^i. The set of these probability measures is denoted by $\mathscr{G}(p)$. In most cases, the p^i are of the *Gibbsian form*

$$p^i(\eta)/p^i({}_i\eta)=\exp\Big(-2\sum_{V\ni i}J_V\,\chi_V(\eta)\Big) \tag{1.4}$$

where $\chi_V(\eta)=\prod_{i\in V}\eta_i$, and the so-called *interaction potential* (J_V) satisfies $\sum_{V\ni i}|J_V|<\infty$. Since $p^i(\eta)+p^i({}_i\eta)=1$, (1.4) determines the p^i uniquely. If $\mathscr{G}(p)$ is not empty, the p^i must satisfy the following consistency condition

$$p^i(\eta)/p^i({}_i\eta)\cdot p^k({}_i\eta)/p^k({}_{ki}\eta)=p^k(\eta)/p^k({}_k\eta)\cdot p^i({}_k\eta)/p^i({}_{ik}\eta), \tag{1.5}$$

see e.g. Preston [17], Sect. 5. Gibbsian conditional distributions satisfy (1.5) automatically, and as a converse it can be shown that consistent conditional distributions are of the Gibbsian type if they are strictly positive and $\log(p^i(\eta)/p^i({}_i\eta))$ has an absolutely convergent Fourier series:

$$\sum_V |\int \log(p^i(\eta)/p^i({}_i\eta))\, \chi_V(\eta)\, d\lambda(\eta)| < \infty,$$

where λ is the Bernoulli measure with parameter 1/2. With a weaker summability condition of the potential, even the continuity of $p^i(\cdot)$ is sufficient, see Sullivan [20]. So the Gibbsian form of local conditional distributions is a rather weak condition, but it is difficult to check it. μ is called a *Gibbs measure* if its local conditional distributions are Gibbsian with some interaction potential. Finally, we say in analogy to (1.4) that flip rates are of the Gibbsian type if

$$c_i(\eta) = \exp(\sum_V A_{i,V}\, \chi_V(\eta)) \quad \text{with } \sum_V |A_{i,V}| < \infty. \tag{1.6}$$

This only means that the c_i are strictly positive and $\log c_i$ has an absolutely convergent Fourier series.

2. Local Conditional Distributions of Stationary Measures and Time Reversal

If P_t is a Markov semigroup with generator $(\overline{\mathscr{L}, \mathscr{C}_0})$, $\mathscr{L}$ as in (1.1), and if $\mu \in \mathscr{I}$, we can define a Markov process $X(t)$, $-\infty < t < \infty$, such that $X(t)$ is distributed according to μ for all t and $E(f(X(t))| X(s) = \eta) = P_{t-s} f(\eta)$ for $t \geqq s$. It is well known that the reversed process $Y(t) = X(-t)$ is again a time homogeneous Markov process, and with $\hat{P}_t f(\eta) = E(f(Y(t+s))| Y(s) = \eta)$ we have

$$\int P_t f(\eta)\, g(\eta)\, d\mu(\eta) = E(f(X(t+s))\, g(X(s))) = \int \hat{P}_t g(\eta)\, f(\eta)\, d\mu(\eta), \tag{2.1}$$

i.e. P_t and $\hat{P}_t$ are the duality with respect to μ. However it is not clear if one can choose $\hat{P}_t$ such that it becomes a semigroup because the Chapman-Kolmogorov condition holds only μ-a.s. Even if we could solve this, in general we don't know anything about the domain of the generator of $\hat{P}_t$. Here we calculate first a formal adjoint $\hat{\mathscr{L}}$ of $\mathscr{L}$ which is also defined on $\mathscr{C}_0$, and after this we discuss in which sense $\hat{\mathscr{L}}$ describes the reversed process.

Theorem 2.1. *Let $\mathscr{L}$ be an operator as in* (1.1) *with continuous and strictly positive rates c_i, and let μ be such that $\int \mathscr{L} f\, d\mu = 0$ $(f \in \mathscr{C}_0)$. Then there is an operator $\hat{\mathscr{L}}$ with the same properties as $\mathscr{L}$ such that $\int f\, \mathscr{L} g\, d\mu = \int g\, \hat{\mathscr{L}} f\, d\mu$ $(f, g \in \mathscr{C}_0)$ iff the local conditional distributions $p^i_\mu(\eta)$ are continuous and strictly positive. Moreover p^i_μ and $\hat{c}_i$ determine each other by*

$$p^i_\mu(\eta)/p^i_\mu({}_i\eta) = c_i({}_i\eta)/\hat{c}_i(\eta) = \hat{c}_i({}_i\eta)/c_i(\eta).$$

Proof. First we assume that the $p^i(.)$ are continuous and >0. Let i be fixed. By definition $E_\mu[g|\bar{\mathscr{E}}_i](\eta) = g(\eta)\, p^i_\mu(\eta) + g({}_i\eta)\, p^i_\mu({}_i\eta)$ $(g \in \mathscr{E}_b)$. Hence $E_\mu[g|\bar{\mathscr{E}}_i](\eta) = E_\mu[h|\bar{\mathscr{E}}_i](\eta)$ if $h(\eta) = g({}_i\eta)\, p^i_\mu({}_i\eta)/p^i_\mu(\eta)$. Choosing as $g(\eta) = f(\eta)\, c_i(\eta)$, we obtain

$$\int f(\eta)\, c_i(\eta)\, d\mu(\eta) = \int f({}_i\eta)\, \hat{c}_i(\eta)\, d\mu(\eta) \qquad (f \in \mathscr{E}_b) \tag{2.2}$$

where $\hat{c}_i(\eta) = c_i({}_i\eta)\, p_\mu^i({}_i\eta)/p_\mu^i(\eta)$.

Now let $f, g \in \mathscr{C}_0$ and take V such that f and g depend only on η_i, $i \in V$. Then

$$\begin{aligned} f(\eta)\,\mathscr{L} g(\eta) - g(\eta)\,\hat{\mathscr{L}} f(\eta) = &\sum_{i\in V} c_i(\eta)\, f(\eta)\, g({}_i\eta) - \sum_{i\in V} \hat{c}_i(\eta)\, f({}_i\eta)\, g(\eta) \\ &+ \sum_{i\in V} \hat{c}_i(\eta)\, f(\eta)\, g(\eta) - \sum_{i\in V} c_i(\eta)\, f(\eta)\, g(\eta). \end{aligned}$$

Integrating with respect to $d\mu$ and using (2.2) for the second and third sum, we get

$$\begin{aligned} &\int f(\eta)\,\mathscr{L} g(\eta)\, d\mu(\eta) - \int g(\eta)\,\hat{\mathscr{L}} f(\eta)\, d\mu(\eta) \\ &\quad = \int \sum_{i\in V} c_i(\eta)\,(f({}_i\eta)\, g({}_i\eta) - f(\eta)\, g(\eta))\, d\mu(\eta) = \int \mathscr{L}(f g)(\eta)\, d\mu(\eta) = 0. \end{aligned}$$

For the converse we assume that $\int f \mathscr{L} g\, d\mu = \int g \hat{\mathscr{L}} f\, d\mu$. First we show that then (2.2) holds. The argument is the same as in Liggett [15], Theorem 4.1.3. We choose $f(\eta) = h_V(\eta) = \prod_{j\in V} (\eta_j + 1)$ and $g(\eta) = h_V({}_i\eta)$ for some $i \in V$. Then $f(\eta)\,\mathscr{L} g(\eta) = f(\eta)\, c_i(\eta)\, 2^{|V|}$ and $g(\eta)\,\hat{\mathscr{L}} f(\eta) = g(\eta)\, \hat{c}_i(\eta)\, 2^{|V|}$, so by assumption

$$\int h_V(\eta)\, c_i(\eta)\, d\mu(\eta) = \int h_V({}_i\eta)\, \hat{c}_i(\eta)\, d\mu(\eta) \qquad (i \in V). \tag{2.3}$$

Obviously (2.3) also holds with c_i and $\hat{c}_i$ exchanged. On the other hand $h_V(\eta) + h_V({}_i\eta) = 2 h_{V\setminus i}(\eta)$ which shows that (2.3) is also true for $i \notin V$. Finally we note that any $f \in \mathscr{E}_b$ can be approximated by linear combinations of functions h_V, so (2.2) follows.

The rest of the proof is now an application of (2.2). We put $p^i(\eta) = c_i({}_i\eta)/(c_i({}_i\eta) + \hat{c}_i(\eta))$. Then we have by (2.2)

$$\int f({}_i\eta)\, p^i({}_i\eta)\, d\mu(\eta) = \int \hat{c}_i(\eta)\, f(\eta)/(c_i({}_i\eta) + \hat{c}_i(\eta))\, d\mu(\eta) = \int f(\eta)\,(1 - p^i(\eta))\, d\mu(\eta).$$

Hence $\int (f(\eta)\, p^i(\eta) + f({}_i\eta)\, p^i({}_i\eta))\, d\mu(\eta) = \int f(\eta)\, d\mu(\eta)$, i.e. the p^i are the local conditional distributions of μ.

But then we must have $p^i(\eta) + p^i({}_i\eta) = 1$ and therefore $p^i(\eta)/p^i({}_i\eta) = c_i({}_i\eta)/\hat{c}_i(\eta)$. Finally it is clear that the results are the same if we exchange the role of c_i and $\hat{c}_i$, at least μ-a.s. But since c_i and $\hat{c}_i$ are assumed to be continuous and μ is everywhere dense (see Holley-Stroock [10], Lemma 1.16), we even have equality everywhere. $\square$

It should be noted that for Theorem 2.1 we did not need that $\overline{(\mathscr{L}, \mathscr{C}_0)}$ and $\overline{(\hat{\mathscr{L}}, \mathscr{C}_0)}$ generate Feller semigroups. If we assume this, it is straightforward that the two semigroups are in duality with respect to μ. Concerning the principal conjecture of this paper, we can therefore formulate the following consequence of Theorem 2.1.

Corollary 2.2. *Let flip rates of the Gibbsian type* (1.6) *be given with* $\sup_i \sum_V |A_{i,V}|\,|V| < \infty$, *and assume that μ is stationary for the semigroup generated* *by $\overline{(\mathscr{L},\mathscr{C}_0)}$. If μ is Gibbs with potential (J_V) and $\sup_i \sum_{V\ni i} |J_V|\,|V|<\infty$, then the reversed process for μ is also a spin flip process with rates*

$$\hat{c}_i(\eta)=\exp\Big(\sum_{V\ni i}(2J_V-A_{i,V})\,\chi_V(\eta)+\sum_{V\not\ni i}A_{i,V}\,\chi_V(\eta)\Big).$$

Moreover any other stationary $\tilde{\mu}$ is also Gibbs with the same potential iff the reversed provess for $\tilde{\mu}$ is the same as for μ.

Proof. It is straightforward to show that under the above assumptions Liggetts condition (1.2) is satisfied for c_i and $\hat{c}_i$. The corollary follows then immediately from Theorem 2.1. □

However, because in general the local conditional distributions of a stationary measure are unknown, we would like to avoid additional assumptions for the $\hat{c}_i$. We are going to show now that in any case the reversed process defines μ-a.s. a solution of the martingale problem for the operator $\hat{\mathscr{L}}$ of Theorem 2.1. This means that in a weak sense the $\hat{c}_i$ are the flip rates of the reversed processes, compare the introduction of Holley-Stroock [9].

First we prepare a lemma.

Lemma 2.3. *Let $\mathscr{L}$ be an operator as in (1.1) with $c_i\in\mathscr{C}$. If $\overline{(\mathscr{L},\mathscr{C}_0)}$ generates a semigroup, then $fg\in\mathscr{D}$ for any $f\in\mathscr{D}$, $g\in\mathscr{C}_0$ where $\mathscr{D}$ denotes the domain of the generator.*

Proof. We choose a sequence of functions $f_n\in\mathscr{C}_0$ such that $\|f_n-f\|\to 0$ and $\|\mathscr{L}f_n-\mathscr{L}f\|\to 0$. Then $f_n g\in\mathscr{C}_0$ and $\|f_n g-fg\|\to 0$. Moreover, if we take a V such that g is $\mathscr{E}_V$-measurable, $\mathscr{L}(f_n g)=(\mathscr{L}f_n)\,g+f_n\mathscr{L}g+\sum_{i\in V}c_i\,\nabla_i f_n\,\nabla_i g$. Hence $\mathscr{L}(f_n g)$ converges in the supremum norm to $(\mathscr{L}f)\,g+f\mathscr{L}g+\sum_{i\in V}c_i\,\nabla_i f\,\nabla_i g$ which means that $fg\in\mathscr{D}$. □

In order to state our result, we choose a right continuous version of the process $Y(t)$ defined at the beginning of this section, and we choose a regular conditional probability distribution Q_η of $Y(t)$, $t\geqq 0$, with respect to $\sigma(Y(0))$.

Proposition 2.4. *Assume that $\overline{(\mathscr{L},\mathscr{C}_0)}$ generates a semigroup and $\int f\,\mathscr{L}g\,d\mu=\int g\,\hat{\mathscr{L}}f\,d\mu$ $(f,g\in\mathscr{C}_0)$ for some operator $\hat{\mathscr{L}}\colon\mathscr{C}_0\to\mathscr{C}$. Then there is a set $N\in\mathscr{E}$ with $\mu(N)=0$ such that for all $\eta\notin N$ Q_η is a solution of the martingale problem for $\hat{\mathscr{L}}$.*

Proof. Let $X(t)$, $-\infty<t<\infty$, be the process defined at the beginning of this section. We are going to show that for all n, $0=s_0<s_1<\ldots<s_n<t$, $f_0,\ldots,f\in\mathscr{C}_0$

$$E\left[\prod_{i=0}^{n} f_i(X(-s_i))\,\Big(f(X(-t))-f(X(-s_n))-\int_{s_n}^{t}\hat{\mathscr{L}}f(X(-u))\,du\Big)\right]=0. \tag{2.4}$$

By the Markov property of $X(t)$ and Fubini's theorem, the left hand side of (2.4) is equal to

$$\int f(\eta)\,P_{t-s_n}h(\eta)\,d\mu(\eta)-\int f(\eta)\,h(\eta)\,d\mu(\eta)-\iint_{s_n}^{t}\mathscr{L}f(\eta)\,P_{u-s_n}h(\eta)\,d\mu(\eta)\,du$$

where $h=f_n P_{s_n-s_{n-1}}(f_{n-1} P_{s_{n-1}-s_{n-2}}(\dots P_{s_1} f_0))\dots)$. Since $f\in\mathscr{D}$ implies that $P_t f\in\mathscr{D}$ for all t, we conclude by a repeated application of Lemma 2.3 that $h\in\mathscr{D}$. But it is obvious that $\int f\mathscr{L} g\,d\mu=\int g\hat{\mathscr{L}} f\,d\mu$ for all $f\in\mathscr{C}_0$, $g\in\mathscr{D}$, so the last integral above is equal to

$$\int\int_{s_n}^{t} f(\eta)\,\mathscr{L} P_{u-s_n} h(\eta)\,d\mu(\eta)\,du=\int f(\eta)\int_{s_n}^{t}\frac{d}{du}P_{u-s_n}h(\eta)\,du\,d\mu(\eta)$$
$$=\int f(\eta)\,(P_{t-s_n}h(\eta)-h(\eta))\,d\mu(\eta),$$

and (2.4) is proved.

Therefore there is a set $N\in\mathscr{E}$ with $\mu(N)=0$ such that for $\eta\notin N$

$$E_{Q_\eta}\left[\prod_{i=1}^{n} f_i(Y(s_i))\,(f(Y(t))-f(Y(s_n))-\int_{s_n}^{t}\hat{\mathscr{L}} f(Y(u))\,du)\right]=0$$

for all s_i rational and $f_i\in\mathscr{C}_0$ (note that $\mathscr{C}_0$ is countable). It is then a straightforward approximation argument to show that $f(Y(t))-\int_0^t\hat{\mathscr{L}} f(Y(u))\,du$ is a Q_η-martingale for $\eta\notin N$. □

From Theorem 2.1 and the consistency condition (1.5) for the local conditional distributions we obtain the following relation between the c_i's and the $\hat{c}_i$'s:

$$\hat{c}_i({}_i\eta)/c_i(\eta)\cdot\hat{c}_k({}_{ik}\eta)/c_k({}_i\eta)=\hat{c}_k({}_k\eta)/c_k(\eta)\cdot\hat{c}_i({}_{ik}\eta)/c_i({}_k\eta). \tag{2.5}$$

(2.5) is most interesting in the reversible case $c_i=\hat{c}_i$ because it gives a condition for the existence of reversible measures. In fact, (2.5) with $c_i=\hat{c}_i$ is nothing else than Kolmogorov's [11] condition for the reversibility of a Markov chain: it means that the closed path $\eta\to{}_i\eta\to{}_{ik}\eta\to{}_k\eta\to\eta$ has the same probability as the reversed path $\eta\to{}_k\eta\to{}_{ik}\eta\to{}_i\eta\to\eta$. For reversible rates of the Gibbsian type, (2.5) takes the simple form $A_{i,V}=A_{k,V}$ for $V\supset\{i,k\}$.

An analogous result to Theorem 2.1 also holds for discrete time Markov chains on E (Vasiliyev [22], Künsch [14]) and for finite dimensional diffusions (Kolmogorov [12], Nagasawa [16]). But in the latter case we can do better than simply express the density of the stationary measure with the original and the reversed drift because the Fokker-Planck equation gives us additional information on this density. In the next section we obtain an equation which can be viewed as an analogue to the Fokker-Planck equation (Formula (3.2) below).

3. An Equation for the Local Conditional Distributions of a Stationary Measure

We consider now the problem of how to find the local conditional distributions of a stationary measure for given flip rates c_i. Because of Sect. 2, this is equivalent to finding the possible time reversals of a given process. In this section we show that the local conditional distributions p^i of a stationary measure are determined by the equation

$$\sum_i \nabla_k (c_i - \hat{c}_i)(\eta) = 0 \quad (k \in \mathbb{Z}^d) \text{ with } \hat{c}_i(\eta) = c_i({}_i\eta)\, p^i({}_i\eta)/p^i(\eta). \tag{3.1}$$

Under a weak condition guaranteeing that (3.1) is well defined, we first show that (3.1) is necessary for the existence of a stationary $\mu \in \mathscr{G}(p)$.

Theorem 3.1. *Let continuous flip rates c_i be given such that $\inf_i \inf_\eta c_i(\eta) > 0$, $\sup_i \sum_k \|\nabla_k c_i\| < \infty$ and $\sum_i \|\nabla_k c_i\| < \infty$ $(k \in \mathbb{Z}^d)$. If μ is stationary for the associated semigroup and its local conditional distributions p^i_μ are continuous with $\inf_i \inf_\eta p^i_\mu(\eta) > 0$, $\sup_i \sum_k \|\nabla_k p^i_\mu\| < \infty$ and $\sum_i \|\nabla_k p^i_\mu\| < \infty$ $(k \in \mathbb{Z}^d)$, then p^i_μ satisfies (3.1).*

Proof. We fix $k \in \mathbb{Z}^d$ and put $h_i(\eta) = \nabla_k (c_i - \hat{c}_i)(\eta)$. By (2.2)

$$\int h_i(\eta)(\hat{c}_j(\eta) - c_j(\eta))\, d\mu(\eta) = \int c_j(\eta)\, \nabla_j h_i(\eta)\, d\mu(\eta).$$

Moreover, applying (2.2) twice, we have

$$\begin{aligned}
&\int h_i(\eta)(c_j({}_k\eta) - \hat{c}_j({}_k\eta))\, d\mu(\eta)\\
&\quad = \int h_i(\eta)(c_j({}_k\eta) - \hat{c}_j({}_k\eta))/c_k(\eta) \cdot c_k(\eta)\, d\mu(\eta)\\
&\quad = \int (h_i({}_k\eta)\, \hat{c}_k(\eta)/c_k({}_k\eta)\, c_j(\eta) - h_i({}_k\eta)\, \hat{c}_k(\eta)/c_k({}_k\eta)\, \hat{c}_j(\eta))\, d\mu(\eta)\\
&\quad = -\int c_j(\eta)(h_i({}_{jk}\eta)\, \hat{c}_k({}_j\eta)/c_k({}_{jk}\eta) - h_i({}_k\eta)\, \hat{c}_k(\eta)/c_k({}_k\eta))\, d\mu(\eta).
\end{aligned}$$

Therefore with $g_i(\eta) = h_i({}_k\eta)\, \hat{c}_k(\eta)/c_k({}_k\eta) = h_i({}_k\eta)\, p^k_\mu({}_k\eta)/p^k_\mu(\eta)$

$$\int h_i(\eta)\, h_j(\eta)\, d\mu(\eta) = \int c_j(\eta)\, \nabla_j(h_i - g_i)(\eta)\, d\mu(\eta).$$

Now because of the conditions on c_i and p^i_μ, $\sum_j \|\nabla_j h_i\| < \infty$, $\sum_j \|\nabla_j g_i\| < \infty$, so $\sum_j c_j(\eta)\, |\nabla_j(h_i - g_i)(\eta)|$ converges uniformly and $h_i - g_i$ is in the domain of the generator of the process. Hence by the stationarity of μ

$$\int h_i(\eta)\Big(\sum_j h_j(\eta)\Big)\, d\mu(\eta) = \int \mathscr{L}(h_i - g_i)(\eta)\, d\mu(\eta) = 0.$$

But then $\int (\sum_i h_i(\eta))^2\, d\mu(\eta) = 0$, and because μ is everywhere dense, we conclude that $\sum_i h_i(\eta) = \sum_i \nabla_k (c_i - \hat{c}_i)(\eta) = 0$. □

Remarks 3.2. i) Liggett's condition (1.2) says that the total influence from all particles on any fixed particle is finite. In Theorem 3.1 we needed also the dual condition that the influence from any particle on all other particles is finite. In the shift invariant case the two conditions are equivalent.

ii) For a finite system with $E = \{+1, -1\}^V$, $|V| < \infty$, a probability measure μ is stationary iff $\sum_{i \in V} c_i(\eta)\, \mu(\eta) = \sum_{i \in V} c_i({}_i\eta)\, \mu({}_i\eta)$, or $\sum_{i \in V} c_i(\eta) = \sum_{i \in V} c_i({}_i\eta)\, \mu({}_i\eta)/\mu(\eta)$

$=\sum_{i\in V} c_i({}_i\eta)\, p^i_\mu({}_i\eta)/p^i_\mu(\eta)=\sum_{i\in V}\hat{c}_i(\eta)$. For infinite systems this condition does not make sense, but (3.1) which follows from it does in many cases. Moreover, for finite systems (3.1) is equivalent to the stationarity of μ, see Theorem 3.3 below.

iii) From approximations with finite systems one finds the following heuristic equations for the evolution of the local conditional distributions under P_t:

$$\frac{d}{dt}\log(p^k_{\mu_t}(\eta)/p^k_{\mu_t}({}_k\eta))$$
$$=\sum_i (c_i({}_k\eta)-c_i(\eta)-c_i({}_{ki}\eta)\, p^i_{\mu_t}({}_{ki}\eta)/p^i_{\mu_t}({}_k\eta)+c_i({}_i\eta)\, p^i_{\mu_t}({}_i\eta)/p^i_{\mu_t}(\eta)). \quad (3.2)$$

Clearly, (3.2) would imply Theorem 3.1, but we have no idea how to prove (3.2). We even don't know if the class of local conditional distributions for which the right hand side of (3.2) makes sense is stationary for P_t.

It is much more difficult to show that (3.1) is also sufficient to conclude that $\mathscr{G}(p)\cap\mathscr{I}\neq\emptyset$ or even $\mathscr{G}(p)\subset\mathscr{I}$. For a partial result to this question we need the mixing coefficients which are defined as usual by

$$\alpha_\mu(V,n)=\sup\{|\mu(A\cap B)-\mu(A)\,\mu(B)|,\ A\in\mathscr{E}_V,\ B\in\bar{\mathscr{E}}_{V_n}\} \quad (3.3)$$

where $V_n=[-n,n]^d$. Furthermore we need that the speed of convergence in $\sum_i \nabla_k c_i$ and $\sum_i \nabla_k \hat{c}_i$ is uniform in k. For this we introduce

$$\beta(m)=\sup_k \max(\sum_{i,|i-k|>m}\|\nabla_k c_i\|,\ \sum_{i,|i-k|>m}\|\nabla_k p^i\|). \quad (3.4)$$

where $|i|=\max(|i_1|,\ldots,|i_d|)$. Then we have

Theorem 3.3. *Assume that we have flip rates c_i and local conditional distributions p^i such that the conditions of Theorem* 3.1 *are satisfied, $\beta(0)<\infty$, $\beta(m)\to 0$ as $m\to\infty$, and* (3.1) *holds. Then all $\mu\in\mathscr{G}(p)$ are stationary for the associated semigroup if for all extremal $\mu\in\mathscr{G}(p)$ the mixing coefficients satisfy $\sum_n \alpha_\mu(V,n)\,n^{d-1}<\infty$ for any finite V. If $\sum_m\beta(m)<\infty$, then $\alpha_\mu(V,n)=o(n^{-d+1})$ for any finite V is sufficient.*

Proof. Because $\mathscr{G}(p)$ is a Choquet simplex (see Preston [17], Sect. 2), it is sufficient to show that $\int\mathscr{L}f(\eta)\,d\mu(\eta)=0$ for any extremal $\mu\in\mathscr{G}(p)$. We fix $f\in\mathscr{C}_0$ and take V such that f is $\mathscr{E}_V$-measurable. Then we have by (2.2) that

$$\int\mathscr{L}f(\eta)\,d\mu(\eta)=-\int f(\eta)\sum_{i\in V_n}(c_i(\eta)-\hat{c}_i(\eta))\,d\mu(\eta) \quad (3.5)$$

for any n with $V_n\supset V$. In order to make the notation shorter, we write $\varphi_n(\eta)$ for $\sum_{i\in V_n}(c_i(\eta)-\hat{c}_i(\eta))$. We note that by (2.2) $\int\varphi_n\,d\mu=0$, so the right-hand side of (3.5) is the covariance between f and φ_n. The idea of the proof is then as follows. Because of (3.1), $\varphi_n(\eta)$ depends essentially only on those η_i with $||i|-n|$ small. If this were exactly true, we could estimate the covariance between f and φ_n by const. $\alpha_\mu(V,n-m)\,n^{d-1}m$ with some fixed m and let n go to infinity. For our proof we therefore have to estimate how much $\varphi_n(\eta)$ differs from a function depending only on η_i, $||i|-n|\leqq m$.

In order to do this, we introduce r_n: $E\to E$ defined by $(r_n\eta)_i=\eta_i$ if $|i|\geqq n$, $(r_n\eta)_i=1$ if $|i|<n$. In particular, $r_0\eta=\eta$. With this we can write $\varphi_n(\eta)=\sum_{m=0}^{\infty}(\varphi_n(r_m\eta)-\varphi_n(r_{m+1}\eta))+\varphi_n(1)$. The m-th summand is $\bar{\mathscr{E}}_{V_{m-1}}$-measurable, and it is well known that

$$|\int f_1 f_2\, d\mu-\int f_1\, d\mu \int f_2\, d\mu|\leqq 4\alpha_\mu(V,m)\,\|f_1\|\,\|f_2\|$$

for f_1 $\mathscr{E}_V$-measurable and f_2 $\bar{\mathscr{E}}_{V_m}$-measurable. Therefore we conclude that

$$|\int f(\eta)\,\varphi_n(\eta)\,d\mu(\eta)|\leqq \text{const.}\left(\sum_{m=0}^{\infty}\alpha_\mu(V,m-1)\sum_{|k|=m}\|V_k\varphi_n\|\right)$$

(we set $\alpha_\mu(V,-1)=1$). But for $|k|=m\leqq n$ we have by (3.1)

$$\|V_k\varphi_n\|\leqq\sum_{i\notin V_n}(\|V_k c_i\|+\|V_k\hat{c}_i\|)\leqq \text{const.}\,\beta(n-m),$$

while for $|k|=m>n$

$$\|V_k\varphi_n\|\leqq\sum_{i\in V_n}(\|V_k c_i\|+\|V_k\hat{c}_i\|)\leqq \text{const.}\,\beta(m-n-1).$$

With our conditions on $\alpha_\mu(V,m)$ and $\beta(m)$ it is then straightforward to show that $\int f\varphi_n\,d\mu=o(1)$ as n tends to infinity. Since n was arbitrary, the theorem is proved. □

Let us make some comments on the conditions of Theorem 3.3. For any extremal μ, $\alpha_\mu(V,n)$ tends to zero as n goes to infinity because $\bar{\mathscr{E}}_\infty=\bigcap_n\bar{\mathscr{E}}_{V_n}$ is trivial for an extremal μ. However, the convergence may be extremely slow as the example of the Ising model in two dimensions at the critical temperature shows. So for $d>1$ neither $\sum_n\alpha_\mu(V,n)\,n^{d-1}<\infty$ nor $\alpha_\mu(V,n)=o(n^{-d+1})$ is satisfied in general. The decay of the mixing coefficients has been studied by several authors, mainly because of its connection with the central limit theorem, see e.g. Dobrushin-Tirozzi [2], Sect. 1.3. We mention here two situations where the conditions of Theorem 3.3 have been established.

The first one is Dobrushin's uniqueness condition $\sup_i\sum_k\|V_k p^i\|<1$. In this case the mixing coefficients can be estimated, but it is more convenient to estimate directly the right hand side of (3.5) with the covariance estimates of Künsch [13] and Föllmer [4]. It then turns out that the Dobrushin condition is sufficient for the statement of Theorem 3.3.

The second one is the case of Gibbs measures with attractive pair potentials: $J_V=0$ if $|V|\neq 2$, $J_{\{i,k\}}=J_{i-k}<0$. In this case the mixing coefficients for extremal Gibbs measures decay exponentially fast if $\sum_k|J_k|$ is large enough. This is interesting because for $\sum_k|J_k|$ large phase transition can occur. So the conditions of Theorem 3.3 include also cases where there is more than one stationary measure.

It seems somehow that the conditions of Theorem 3.3 are too strong because we do not need covariance estimates for arbitrary functions, we only want to estimate the right-hand side of (3.5) using (3.1). In the next section we will show by completely different methods that in the shift invariant case $\mathcal{G}(p)\cap\mathcal{I}$ is never empty if (3.1) holds. It will also follow that a solution to (3.1) is unique among the shift invariant Gibbsian p^i.

4. Free Energy

We are going to generalize here the well known results of Holley [8] on the decrease of free energy to the non reversible case. Our arguments follow the same lines as in Holley's paper, but at one place an additional term occurs which is zero only if $c_i=\hat{c}_i$. With the help of (3.1) we can estimate this term and show that it does not contribute in the infinite volume limit. Even if the c_i have finite range, this is in general not true for the $\hat{c}_i$; for this reason we have to follow the more refined arguments of Higuchi-Shiga [7].

We start with some definitions. Let τ_i: $E\to E$ be the shift operator defined by $(\tau_i\eta)_k=\eta_{k-i}$. With this we call flip rates *shift invariant* if $c_i(\eta)=c_0(\tau_{-i}\eta)$. A probability measure μ on E is called shift invariant if $\mu(\tau_i A)=\mu(A)$ $(A\in\mathcal{E}$, $i\in\mathbb{Z}^d)$, and $\mathcal{S}$ denotes the set of all shift invariant μ's. Finally a potential (J_V) is shift invariant if $J_V=J_{V-i}$ for all i and V.

For a shift invariant potential (J_V) we put

$$p(n,\eta)=\exp\Big(-\sum_{W\subseteq V_n} J_W\chi_W(\eta)\Big)\qquad \text{where } V_n=[-2^n+1,2^n-1]^d. \tag{4.1}$$

The free energy in V_n of a probability measure ν is

$$A_n(\nu)=\sum_{\zeta\in E_n}\nu(n,\zeta)\log(\nu(n,\zeta)/p(n,\zeta))$$

where

$$\nu(n,\zeta)=\nu(S(n,\zeta)),\qquad S(n,\zeta)=\{\eta,\eta=\zeta \text{ on } V_n\} \tag{4.2}$$

and

$$E_n=\{+1,-1\}^{V_n}.$$

The free energy per particle is then

$$A(\nu)=\limsup|V_n|^{-1}A_n(\nu). \tag{4.3}$$

It is well known that the free energy is minimal for $\nu\in\mathcal{G}(J)\cap\mathcal{S}$ (Gibbs variational principle). The following theorem is a dynamical version of this.

Theorem 4.1. *Assume that we have shift invariant rates $c_i\in\mathcal{C}$, $c_i>0$, and a shift invariant potential (J_V) such that $\sum_j\|\nabla_j c_0\|<\infty$, $\sum_j\|\nabla_j\hat{c}_0\|<\infty$ and* (3.1) *holds where $\hat{c}_i(\eta)=c_i({}_i\eta)\exp(2\sum_{V\ni i}J_V\chi_V(\eta))$. Then*

i) *For any initial distribution* ν, $A(\nu_t)$ *is non increasing in* t.

ii) *For any* $\nu \in \mathscr{S}$, $\nu \notin \mathscr{G}(p)$ *there exists a weakly open set* G_ν *containing* ν *and* $\varepsilon, \delta > 0$ *such that if* $\nu' \in G_\nu$ *and* $0 \leqq s \leqq \varepsilon$, *then* $A(\nu'_s) - A(\nu') \leqq -\delta s$.

Proof. We only prove those points which differ from Holley [8] and Higuchi-Shiga [7]. Our notation is also similar to what they used. As in Lemmas (2.3) −(2.5) of Holley and Lemmas 3.2 and 3.3 of Higuchi-Shiga respectively we have

$$\begin{aligned}\frac{d}{dt}(A_n(\nu_t))|_{t=0} &= \sum_{\zeta \in E_n} \sum_{i \in V_n} \Big(\int_{S(n,{}_i\zeta)} c_i(\eta)\, d\nu(\eta) - \int_{S(n,\zeta)} c_i(\eta)\, d\nu(\eta)\Big) \log \frac{\nu(n,\zeta)}{p(n,\zeta)} \\ &= \sum_{\zeta \in E_n} \sum_{i \in \bar{V}_n} \Big(\int_{S(n,{}_i\zeta)} c_i(\eta)\, d\nu(\eta) - \int_{S(n,\zeta)} c_i(\eta)\, d\nu(\eta)\Big) \log \frac{\nu(n,\zeta)}{p(n,\zeta)} \\ &\quad + o(|V_n|) \\ &\quad \text{with } \bar{V}_n = [-2^n + n + 1, 2^n - n - 1]^d.\end{aligned} \tag{4.4}$$

Then we define

$$F(\nu, n, \zeta, i) = \begin{cases} F_0(\nu(n,{}_i\zeta)/\nu(n,\zeta)\, p(n,\zeta)/p(n,{}_i\zeta))\, \nu(n,\zeta) & \text{if } \nu(n,\zeta) > 0 \\ -\infty \quad \text{if } \nu(n,\zeta) = 0,\ \nu(n,{}_i\zeta) > 0 \\ 0 \quad \text{if } \nu(n,\zeta) = \nu(n,{}_i\zeta) = 0 \end{cases} \tag{4.5}$$

where $F_0(u) = u - u \log u - 1$ if $u > 0$, $F_0(0) = -1$.

We are going to show that

$$\frac{d}{dt} A_n(\nu_t)|_{t=0} = \sum_{\zeta \in E_n} \sum_{i \in \bar{V}_n} F(\nu, n, \zeta, i)\, c_i^{(n)}({}_i\zeta)\, p(n,{}_i\zeta)/p(n,\zeta) + o(|V_n|) \tag{4.6}$$

where

$$c_i^{(n)}(\zeta) = \begin{cases} \nu(n,\zeta)^{-1} \int_{S(n,\zeta)} c_i(\eta)\, d\nu(\eta) & \text{if } \nu(n,\zeta) > 0 \\ c_i(r_n \zeta) & \text{if } \nu(n,\zeta) = 0 \end{cases} \tag{4.7}$$

and

$$(r_n \zeta)_i = \zeta_i \text{ if } i \in V_n, \quad (r_n \zeta)_i = 1 \text{ if } i \notin V_n. \tag{4.8}$$

The assertion i) of the theorem will follow immediately from (4.6) because F is negative.

From (4.4) it follows by a simple calculation that (4.6) is equivalent to

$$\begin{aligned}&\sum_{\zeta \in E_n} \sum_{i \in \bar{V}_n} \Big(\int_{S(n,{}_i\zeta)} c_i(\eta)\, d\nu(\eta) - \nu(n,\zeta)/\nu(n,{}_i\zeta) \int_{S(n,{}_i\zeta)} c_i(\eta)\, p(n,{}_i\eta)/p(n,\eta)\, d\nu(\eta)\Big) \\ &\quad = o(|V_n|).\end{aligned} \tag{4.9}$$

With r_n defined in (4.8), the left hand side of (4.9) can be decomposed as $I_1 + I_2 + I_3$ where

$$I_1 = \sum_{\zeta \in E_n} \nu(n, \zeta) \sum_{i \in \bar{V}_n} (c_i(r_n \zeta) - \hat{c}_i(r_n \zeta))$$

$$I_2 = \sum_{\zeta \in E_n} \sum_{i \in \bar{V}_n} \int_{S(n,\zeta)} (c_i(\eta) - c_i(r_n \eta))\, d\nu(\eta)$$

$$I_3 = \sum_{\zeta \in E_n} \nu(n, \zeta) \sum_{i \in \bar{V}_n} \nu(n, {}_i\zeta)^{-1} \int_{S(n, {}_i\zeta)} (c_i(\eta)\, p(n, {}_i\eta)/p(n, \eta) - \hat{c}_i(r_n({}_i\eta)))\, d\nu(\eta).$$

I_2 and I_3 are of the order $o(|V_n|)$ because each integrand is of the order $o(1)$, compare formula (3.11) of Higuchi-Shiga. The term I_1 does not appear in the reversible case. It can be estimated as follows. First we observe that for $i \in V_n$

$$\sum_{\zeta \in E_n} \exp\Big(- \sum_{W \cap V_n \neq \emptyset} J_W \chi_W(r_n \zeta)\Big)(c_i(r_n \zeta) - \hat{c}_i(r_n \zeta)) = 0.$$

This implies that for $\varphi_n(\zeta) = \sum_{i \in \bar{V}_n} (c_i(r_n \zeta) - \hat{c}_i(r_n \zeta))$ $\sup \varphi_n(\zeta) \geqq 0 \geqq \inf \varphi_n(\zeta)$. Therefore by (3.1)

$$\|\varphi_n\| \leqq \sum_{k \in V_n} \|\nabla_k \varphi_n\| \leqq \sum_{k \in V_n} \sum_{i \notin \bar{V}_n} (\|\nabla_k c_i\| + \|\nabla_k \hat{c}_i\|).$$

With $V_{n,m} = [-2^n + n + m + 1, 2^n - n - m - 1]^d$ we therefore conclude by the shift invariance of c_i and $\hat{c}_i$ that

$$|I_1| \leqq |V_{n,m}| \sum_{|i| \geqq m} (\|\nabla_0 c_i\| + \|\nabla_0 \hat{c}_i\|)) + |V_n \setminus V_{n,m}| \sum_i (\|\nabla_0 c_i\| + \|\nabla_0 \hat{c}_i\|).$$

Hence for any m $\limsup |V_n|^{-1} |I_1| \leqq \sum_{|i| \geqq m} (\|\nabla_0 c_i\| + \|\nabla_0 \hat{c}_i\|)$, and therefore, by letting m tend to infinity, $I_1 = o(|V_n|)$.

The assertion ii) is proved by showing that for $\nu \in \mathscr{S}$

$$\lim |V_n|^{-1} \sum_{\zeta \in E_n} \sum_{i \in \bar{V}_n} F(\nu, n, \zeta, i)\, c_i^{(n)}({}_i\zeta)\, p(n, {}_i\zeta)/p(n, \zeta)$$

exists, is upper semicontinuous in ν, $\leqq 0$ and $= 0$ iff $\nu \in \mathscr{G}(p)$. The arguments are the same as in Higuchi-Shiga; all estimates in their paper which use the Gibbsian form of c_i can be replaced by estimates using $\|\nabla_j c_0\|$ and $\|\nabla_j \hat{c}_0\|$. $\square$

In the next corollaries we use the notation $\mathscr{P}_2$ for the class of shift invariant potentials (J_V) with $\sum_{V \ni 0} |V|\, |J_V| < \infty$. If the p^i are Gibbs with potential (J_V), we write $\mathscr{G}(J)$ instead of $\mathscr{G}(p)$, and we assume that rates c_i satisfying the conditions of Theorem 4.1 are given.

Corollary 4.2. *Either $\mathscr{G}(J) \cap \mathscr{I} = \emptyset$ for all $(J_V) \in \mathscr{P}_2$ or $\mathscr{I} \cap \mathscr{S} \subseteq \mathscr{G}(J^0) \cap \mathscr{S}$ for some $(J_V^0) \in \mathscr{P}_2$.*

Corollary 4.3. *If $\mathscr{G}(J) \cap \mathscr{I} \neq \emptyset$ for some $(J_V) \in \mathscr{P}_2$, then ν_t converges weakly to $\mathscr{G}(J) \cap \mathscr{S}$ for all initial distributions $\nu \in \mathscr{S}$. In particular, if the Gibbs measure to this potential is unique, the process is ergodic for all shift invariant initial distributions.*

Corollary 4.4. *If* (3.1) *has a Gibbsian solution* p^i *with potential* $(J_V)\in\mathscr{P}_2$, *then* $\mathscr{I}\cap\mathscr{S}\subset\mathscr{G}(J)\cap\mathscr{S}$. *In particular,* $\mathscr{G}(J)\cap\mathscr{I}\neq\emptyset$.

From Theorem 4.1, i) and Gibbs variational principle it is clear that $\nu_t\in\mathscr{G}(J)\cap\mathscr{S}$ for all t if $\nu\in\mathscr{G}(J)\cap\mathscr{S}$ and (3.1) holds. With a little additional argument we can also show that ν_t is an extremal point in $\mathscr{G}(J)\cap\mathscr{S}$ for all t if ν is so. Then we obtain

Proposition 4.5. *Under the conditions of Theorem* 4.1 *we have* $\mathscr{G}(J)\cap\mathscr{S}=\mathscr{I}\cap\mathscr{S}$ *provided the set of extremal points in* $\mathscr{G}(J)\cap\mathscr{S}$ *is totally disconnected in the weak topology.*

Proof. It is well known that $\nu\in\mathscr{G}(J)\cap\mathscr{S}$ is an extremal point iff ν is ergodic. By Corollary 2 of Sullivan [21] ν_t is ergodic for all t if ν is so. Therefore ν_t stays in the set of extremal points, and on the other hand ν_t varies continuously in t, so $\nu_t=\nu$ for all t. □

I have the feeling that the above condition on the extremal points of $\mathscr{G}(J)\cap\mathscr{S}$ has more chances to be satisfied than the mixing conditions of Theorem 3.3. The extremal points of $\mathscr{G}(J)\cap\mathscr{S}$ have been studied by Slawny [18, 19] for ferromagnetic potentials $-\beta J_V$, $J_V\geqq 0$ for all V, $\beta>0$. He has shown that for large β there are at most two extremal points if we have a connected two-body potential or if $J_{\{0,e_i\}}>0$ for $i=1,\dots,d$ where e_i is the i-th basic vector in $\mathbb{Z}^d$. However he has an example of a four-body potential where there are continuously many extremal points, so Proposition 4.5 does not cover all cases.

For $d=1$ the Gibbs measure is unique for all shift invariant potentials with $\sum_{V\ni 0}|J_V|\operatorname{diam}(V)<\infty$. So if we could show the existence of a stationary Gibbs measure with such a potential, then by Corollary 4.4 $\mathscr{I}\cap\mathscr{S}$ would have only one element. This would then give a proof of the positive rates conjecture (see Gray [5]) for attractive interactions.

Holley-Stroock [10] have used the free energy in order to show that in the time reversible case $\mathscr{I}\subset\mathscr{G}(J)$ at least in dimensions less than three. However, this application cannot be extended to the non reversible case because the basic expression for the derivative of the free energy (Lemma 1.10 in [10]) is specific for the situation $c_i=\hat{c}_i$. Even for finite systems it is easy to see that the formula

$$2\frac{d}{dt}A(\mu_t)|_{t=0}=-\sum_{k\in V}\sum_{\eta}(c_k(\eta)\,\mu(\eta)-\hat{c}_k({}_k\eta)\,\mu({}_k\eta))\log(c_k(\eta)\,\mu(\eta)/\hat{c}_k({}_k\eta)\,\mu({}_k\eta)),$$

which would be the obvious generalization of formula (1.11) of [10], is not true in general.

5. Stationary Gibbs Measures

Clearly, the stationary measure is Gibbs for finite systems. But if we change the flip rate at some point i_0, we don't have an efficient way to estimate how much the local conditional distribution p^i of the stationary measure changes for points i far away from i_0. For this reason an approximation by finite systems doesn't give us a proof for the existence of a stationary Gibbs measure. A different way is to look for a Gibbsian solution of (3.1). We give here at least a formal solution for shift invariant rates c_i. If we can represent c_0 and $\hat{c}_0$ as a Fourier series

$$c_0(\eta)=\sum_V F_V\,\chi_V(\eta), \qquad \hat{c}_0(\eta)=\sum_V \hat{F}_V\,\chi_V(\eta),$$

then the Eq. (3.1) is easy to solve. Namely, by the shift invariance of c_i and $\hat{c}_i$, it is equivalent to

$$\sum_i F_{V-i}=\sum_i \hat{F}_{V-i} \qquad \text{for all } V\neq\emptyset \tag{5.1}$$

where $V-i=\{j\in Z^d,\ j+i\in V\}$. Of course, there are many $\hat{F}_V$ satisfying (5.1), but the $\hat{c}_0$ we are looking for must be of the form

$$\hat{c}_0(\eta)=c_0({}_0\eta)\exp\Big(2\sum_{V\ni 0} J_V\,\chi_V(\eta)\Big) \tag{5.2}$$

with some shift invariant potential (J_V). We therefore have to express $\hat{F}_V$ in terms of the unknown potential (J_V) and then to determine (J_V) from (5.1). In order to do this, it is convenient to assume that c_0 is of the Gibbsian type (1.6) with an additional parameter of interaction β

$$c_0(\eta)=\exp\Big(\beta\sum_V A_V\,\chi_V(\eta)\Big). \tag{5.3}$$

We then look for a potential $J_V=J_V(\beta)$ which is analytic in β:

$$J_V(\beta)=\sum_{n=1}^{\infty} J_{V,n}\,\beta^n \qquad (J_{V,0}=0 \text{ is obvious}).$$

From (5.3)

$$\frac{\partial^n}{\partial\beta^n}\,c_0(\eta)|_{\beta=0}=\alpha(\eta)^n \qquad \text{with } \alpha(\eta)=\sum_V A_V\,\chi_V(\eta),$$

while from (5.2) we obtain

$$\frac{\partial^n}{\partial\beta^n}\,\hat{c}_0(\eta)|_{\beta=0}=n!\sum_{k=1}^{n}\Big(\sum_{\substack{i_1+\dots i_k=n\\ i_j\geqq 1}} \hat{\alpha}_{i_1}(\eta)\dots\hat{\alpha}_{i_k}(\eta)\Big)\Big/k!$$

with $\hat{\alpha}_1(\eta)=2\sum_{V\ni 0}(J_{V,1}-A_V)\,\chi_V(\eta)+\alpha(\eta)$, $\hat{\alpha}_n(\eta)=2\sum_{V\ni 0} J_{V,n}\,\chi_V(\eta)$ $(n>1)$. Using this, we can express the derivatives of the Fourier coefficients $F_V(\beta)$ and $\hat{F}_V(\beta)$ at $\beta=0$. Moreover, by the shift invariance of (J_V) and (5.1), we obtain the following recursive equations for $J_{V,n}$

$$J_{V,1}=|V|^{-1}\sum_{i\in V} A_{V-i},$$

$$\begin{aligned} J_{V,n}=(2\,|V|)^{-1}\sum_i \Big(&(A*\dots*A)_{V-i}/n!\\ &-\sum_{k=2}^{n}\Big(\sum_{\substack{i_1+\dots+i_k=n\\ i_j\geqq 1}} (\hat{A}_{i_1}*\dots*\hat{A}_{i_k})_{V-i}\Big)\Big/k!\Big) \qquad (n\geqq 1) \end{aligned} \tag{5.4}$$

where $\hat{A}_{V,1}=2J_{V,1}-A_V$ ($V\ni 0$), $\hat{A}_{V,1}=A_V$ ($V\not\ni 0$), $\hat{A}_{V,n}=2J_{V,n}$ ($V\ni 0$), $\hat{A}_{V,n}=0$ ($V\not\ni 0$), and the convolution of two sets of Fourier coefficients (B_V) and (C_V) is defined as

$$(B_* C)_V=\sum_W B_W C_{V\triangle W} \quad \text{with } V\triangle W=(V\cap W^c)\cup(V^c\cap W).$$

So for shift invariant flip rates as in (5.3) we have a formal expression for the potential of a stationary Gibbs measure. Unfortunately, because of the complexity of the recursion formula (5.4), we were unable to prove a result on the convergence of $\sum_{n=1}^{\infty}\beta^n\sum_{V\ni 0}J_{V,n}\chi_V(\eta)$. The rough estimate

$$\|\hat{A}_n\|\leqq\|A\|^n/n!+\sum_{k=2}^{n}\Big(\sum_{\substack{i_1+\dots+i_k=n\\ i_j\geqq 1}}\|\hat{A}_{i_1}\|\dots\|\hat{A}_{i_k}\|\Big)/k!$$

with $\|B\|=\sum_i|B_V|$ is useless. On the other hand we know that for finite systems $J_V(\beta)$ is analytic in a neighborhood of $\{\mathrm{Im}(\beta)=0\}$, so $\sum_n J_{V,n}\beta^n$ has a non zero convergence radius. In the following two examples we even have $J_{V,n}=0$ for all $n\geqq 1$.

Example 5.1. $d=1$, $c_0(\eta)=\exp(\beta\alpha(\eta))$ where

$$\alpha(\eta)=\eta_0+a_1\eta_1\eta_0+a_2\eta_{-1}\eta_0+a_3(\eta_{-1}+\eta_1)+a_4\eta_{-1}\eta_1.$$

Then by (5.4) $\hat{\alpha}_1(\eta)=\alpha(\theta\eta)$ where $(\theta\eta)_i=\eta_{-i}$. Moreover $(A*\dots*A)_V=0$ except for $V\subset\{-1,0,1\}$, and for such V we have $-V=V-i$ for some i. Therefore

$$\sum_i((A*\dots*A)_{V-i}-(\hat{A}_1*\dots*\hat{A}_1)_{V-i}))=0$$

for all $n>1$ whence by (5.4) $J_{V,n}=0$ for all $n>1$. This means that

$$J_{\{i\}}(\beta)=\beta,\qquad J_{\{i,i+1\}}(\beta)=\beta(a_1+a_2)/2,$$

$J_V(\beta)=0$ otherwise, and the unique shift invariant stationary measure is a Markov chain on $\{-1,1\}$. Note that even in this simple case the ergodicity of the process has not yet been proved for all values of a_1,a_2,a_3,a_4, see Gray [5] and the references there. In order to treat the general case of flip rates with range one, we would have to replace the term $a_3(\eta_1+\eta_{-1})$ by $a_3\eta_1+a_5\eta_{-1}$ and to include an additional term $a_6\eta_{-1}\eta_0\eta_1$. But then the simplicity is lost completely, in fact it can be shown that a solution to (3.1) must have infinite range in this case. From the recursion formula (5.4) we also see that in general (J_n) has range $2^n r$ if c_0 has range r.

Example 5.2. $d=2$, $c_0(\eta)=\exp(\beta\alpha(\eta))$ with $\alpha(\eta)=a_1\eta_0\eta_{e_1}+a_2\eta_0\eta_{e_2}$ where $e_1=(1,0)$, $e_2=(0,1)$. With the same argument as in Example 5.1 we can show that the Gibbs measures with potential $J_{\{i,i+e_1\}}=\beta a_1$, $J_{\{i,i+e_2\}}=\beta a_2$, $J_V=0$ otherwise, are stationary. It is well known that for this potential there is phase transition if $|\beta|$ is large and $a_1a_2\neq 0$. Moreover, it has been proved at least for

$a_1 = a_2$ that $\mathscr{G}(J) \cap \mathscr{S}$ has one or two extremal points and $\mathscr{G}(J) \cap \mathscr{S} = \mathscr{G}(J)$. Therefore our results imply then that $\mathscr{I} \cap \mathscr{S} = \mathscr{G}(J)$; in particular we have an example with infinitely many stationary, but no reversible measure.

Finally let us close with the following trivial example of a stationary measure which is not Gibbs. Let $d=1$ and $c_i(\eta) = 1 - \beta \eta_0 \eta_i$. It is easy to show that this process has the following stationary measure: Given η_0, the other spins are Bernoulli with parameter $(1+\beta\eta_0)/2$, and η_0 is equal to one with probability 1/2. This measure is not Gibbs because $p^0(\eta)$ depends on the tail field and is therefore not continuous. But in this example we have $\sum_i \|\nabla_0 c_i\| = \infty$, so the condition of Theorem 3.1 is violated and I have the feeling that this is the reason why there is a stationary measure which is not Gibbs.

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