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Most Stringent and Maximin Tests as Solutions of Linear Programming Problems

Willem Schaafsma

Summary. The problem to obtain the most stringent size-α test φ^* is formulated as a linear programming problem of type II (Section 2). If sample space and parameter space are finite, then we obtain a discrete linear programming problem (Section 4). The well-known results for this special case, and the results of Krafft and Witting for the maximin size-α test, point out how to formulate the dual problem of type I in the general case and how to develop the corresponding duality theory (Sections 5 and 6). It turns out that φ^* can be determined completely by the solution of the dual type I problem, which solution may be characterized by means of a least favorable pair $(\tilde{\lambda}, \tilde{\nu})$ of probability measures over Ω_H and Ω_K respectively (Section 7). Statistical interpretation shows further that φ^* can also be characterized by means of a least favorable distribution $\tilde{\nu}$ over Ω_K alone (Section 8).

1. Introduction

In [8], Witting characterizes the *most powerful* (M.P.) size-α test for testing problems with a simple alternative as the solution of a linear programming (L.P.) problem, the dual of which determines the least favorable a priori distribution over Ω_H. Krafft and Witting extended these results in [2] by considering the *maximin* size-α criterion for testing problems with a composite alternative. They proved an important *weak duality* theorem providing conditions under which Lehmann's sufficient conditions ([3] Section 8.1) for a maximin size-α test are also necessary.

In [6] we characterized the *most stringent* (M.S.) size-α test for certain testing problems with a composite alternative by means of a least favorable a priori distribution over the alternative. This result constituted the basis of numerical computations providing M.S. size-α tests for a special class of problems with an alternative restricted by a number of linear inequalities.

In this paper we generalize the results in [2] and [6] by applying the L.P. method of Krafft and Witting and by defining a criterion "*most stringent in a class C of tests with respect to a class D of tests*"([5] Chapter 2), which can be regarded as a generalization, both of the criterion maximin size-α and of the criterion most stringent size-α.

2. The Most Stringent (Φ_α, D)-Test φ^* as the Solution of a Linear Programming Problem

Let $(\mathfrak{X}, \mathfrak{A}, \mu)$ be the sample space $\mathfrak{X}$ with the σ-field $\mathfrak{A}$ and the σ-finite measure μ dominating the family $\{P_\theta;\ \theta\in\Omega\}$ of probability measures over the measurable space $(\mathfrak{X}, \mathfrak{A})$. Let $p_\theta = \mathrm{d}P_\theta/\mathrm{d}\mu$ be a measurable determination (p.d.f.) of the Radon-Nikodym derivative of P_θ with respect to μ.

The parameter space Ω is subdivided into two non-void mutually exclusive subsets $\Omega=\Omega_H\cup\Omega_K$ $(\Omega_H\cap\Omega_K=\emptyset)$ and we consider the testing problem (H,K) where H is true if and only if $\theta\in\Omega_H$ and K is true if and only if $\theta\in\Omega_K$. The hypothesis H will correspond with a certain standard situation whereas the alternative K corresponds with deviations from this standard situation. If it simplifies the notation, then Ω_H and Ω_K will be denoted by H and K respectively.

Let Φ denote the class of all measurable functions $\varphi\colon \mathfrak{X}\to[0,1]$ (test functions). $\varphi\in\Phi$ belongs to the class Φ_α of all size-α tests for Problem (H,K) if and only if φ satisfies the size-α restriction: $\mathrm{E}_\theta(\varphi)=\int\varphi\, p_\theta\,\mathrm{d}\mu\leqq\alpha$ for all $\theta\in\Omega_H$. The power function $\beta_\varphi\colon \Omega_K\to[0,1]$ of $\varphi\in\Phi$ is defined by $\beta_\varphi(\theta)=\mathrm{E}_\theta(\varphi)$. If D is an arbitrary class $D\subset\Phi\,(D\neq\emptyset)$ then we define the envelope power $\beta_D^*\colon \Omega_K\to[0,1]$ with respect to D by means of $\beta_D^*(\theta)=\sup_{\varphi\in D}\beta_\varphi(\theta)$, and the shortcoming $\gamma_{\varphi,D}\colon \Omega_K\to[-1,1]$ of $\varphi\in\Phi$ with respect to D as $\gamma_{\varphi,D}=\beta_D^*-\beta_\varphi$.

Let C denote an arbitrary class $C\subset\Phi$. $\varphi^*\in\Phi$ is said to be *most stringent in C with respect to D* (M.S. (C,D)) if (see [5] Chapter 2) (i) $\varphi^*\in C$ and (ii)

$$\sup_{\theta\in\Omega_K}\gamma_{\varphi^*,D}(\theta)=\inf_{\varphi\in C}\sup_{\theta\in\Omega_K}\gamma_{\varphi,D}(\theta). \tag{2.1}$$

Next we consider some special cases of the criterion M.S. (C,D).

Case (i). Let C be the class of all tests which are somewhere most powerful with respect to D: $\varphi\in C$ if and only if $\varphi\in D$ and $\gamma_{\varphi,D}(\theta)=0$ for some $\theta\in\Omega_K$. In this case φ^* is said to be M.S.S.M.P. (D) *(most stringent somewhere most powerful with respect to the class D)* if and only if φ^* is M.S. (C,D). In [5] we constructed M.S.S.M.P. (D) tests for many testing problems from actual practice with Ω_K restricted by a number of inequalities. By choosing D appropriately as the class Φ_α of all size-α tests or as the class of all similar (or unbiased) size-α tests, we obtained easily applicable tests for these problems.

Case (ii). Take $C=D=\Phi_\alpha$. In this case φ^* is said to be *most stringent size-α*. It is well-known that for many classical hypothesis testing problems the likelihood-ratio tests are (approximately) uniformly M.P. invariant size-α (see [3]) and most stringent size-α. This does not hold for hypothesis testing problems with a restricted alternative. For a very special class of such problems we arrived at the most stringent size-α test by characterizing it with a least favorable distribution over the alternative. Thus in [6] we could compare the M.S.S.M.P. and the M.S. size-α test for these special problems.

Case (iii). Take $C=\Phi_\alpha$ and $D=\Phi$. In this case $\beta_D^*(\theta)=1$ for all $\theta\in\Omega_K$ and φ^* is M.S. (Φ_α,Φ) if and only if (i) $\varphi^*\in\Phi_\alpha$ and (ii)

$$\inf_K\beta_{\varphi^*}(\theta)=\sup_{\Phi_\alpha}\inf_K\beta_\varphi(\theta) \tag{2.2}$$

or in other words if and only if φ^* is a *maximin size-α* test. Such tests have been considered by Lehmann ([3] Chapter 8) a.o. In a beautiful paper [2], Krafft and Witting characterized the maximin size-α test as the solution of a L.P. problem, the dual of which characterizes the least favorable pair of distributions over Ω_H and Ω_K respectively.

In this paper we deal with the cases (ii) and (iii) at the same time by considering M.S. (Φ_α, D) tests where D is an arbitrary class of tests. For $D=\Phi_\alpha$ we obtain a theory for the most stringent size-α test and for $D=\Phi$ we arrive at the results of Krafft and Witting for the maximin size-α test.

Lemma 2.1. *If $\Phi_\alpha \subset D \subset \Phi$ then φ^* is* M.S. (Φ_α, D) *if and only if there exists a number $\gamma^* \in [0, 1-\alpha]$ such that (φ^*, γ^*) is a solution to the* L.P. *problem of type* II *where the linear form b, which is defined by*

$$b(\varphi, \gamma)=\gamma, \tag{2.3}$$

has to be minimized among the elements (φ, γ) of the class Y that is defined by the following restrictions

$$\varphi(x) \geqq 0 \qquad \forall x \in \mathfrak{X}, \tag{2.4}$$

$$\gamma \geqq 0 \tag{2.5}$$

$$-\varphi(x) \geqq -1 \qquad \forall x \in \mathfrak{X}, \tag{2.6}$$

$$-\int \varphi(x)\, p_\theta(x)\, d\mu \geqq -\alpha \qquad \forall \theta \in \Omega_H, \tag{2.7}$$

$$\gamma + \int \varphi(x)\, p_\theta(x)\, d\mu \geqq \beta_D^*(\theta) \qquad \forall \theta \in \Omega_K. \tag{2.8}$$

Proof. If φ^* is M.S. (Φ_α, D) and $\gamma^* = \sup_K \gamma_{\varphi^*, D}(\theta)$, then $\gamma^* \geqq 0$; for $\varphi^* \in \Phi_\alpha \subset D$ implies that $\gamma_{\varphi^*, D}(\theta) \geqq 0$ for all $\theta \in \Omega_K$. Moreover the trivial size-α test φ_α with $\varphi_\alpha(x)=\alpha$ for all $x \in \mathfrak{X}$ satisfies $\gamma_{\varphi_\alpha, D}(\theta) \leqq 1-\alpha$ for all $\theta \in \Omega_K$. Hence $\gamma^* \in [0, 1-\alpha]$ and (φ^*, γ^*) satisfies the restrictions (2.4), ..., (2.8). Moreover the pair (φ, γ) cannot satisfy these restrictions unless $\gamma \geqq \gamma^*$.

If (φ, γ) is a solution to Problem II then $\varphi \in \Phi_\alpha$ on account of (2.4), (2.6) and (2.7). Moreover $\sup_K \gamma_{\varphi, D}(\theta) \leqq \gamma$ on account of (2.8). If φ is not M.S. (Φ_α, D) with $\sup_K \gamma_{\varphi, D}(\theta)=\gamma$ then there exists $\varphi' \in \Phi_\alpha$ with $0 \leqq \sup_K \gamma_{\varphi', D}(\theta)=\gamma' < \gamma$ and hence a feasible element $(\varphi', \gamma') \in Y$ with $\gamma' < \gamma$.

Remark 1. The condition $\Phi_\alpha \subset D \subset \Phi$ is not necessary in Lemma 2.1 but is a simple sufficient condition in order that $\sup_K \gamma_{\varphi^*, D}(\theta) \geqq 0$ for the M.S. (Φ_α, D) test φ^*, which latter condition is necessary and sufficient.

Up to now β_D^* in the formulation of Problem II denotes a certain envelope power function. In the following theory we regard β_D^* as an arbitrary function $\beta_D^*: \Omega_K \to [0, 1]$. Lemma 2.1 shows that for the important simple case of an envelope power function β_D^* with $\Phi_\alpha \subset D \subset \Phi$, the solution (φ^*, γ^*) to Problem II characterizes the M.S. (Φ_α, D) test for our testing problem.

Remark 2. In this paper we restrict the attention to the class Φ_α of all size-α tests. A generalization is obtained when we consider a certain function $\alpha: \Omega_H \to [0, 1]$ (instead of the constant function $\alpha(\theta)=\alpha$) and restrict the attention to the test functions φ satisfying

$$\int_{\mathfrak{X}} \varphi(x)\, p_\theta(x)\, d\mu \leqq \alpha(\theta) \qquad \forall \theta \in \Omega_H.$$

It will be possible to generalize the following theory to this case, thus implying a generalization of the Neyman-Pearson fundamental lemma ([3] Section 3.6).

Remark 3. The L.P. problem in Lemma 2.1 is called of type II, or briefly Problem II, because the linear form b has to be *minimized* (this is in agreement with the notation in the theory for discrete L.P. problems as developed in Karlin [1]). In Section 5 we shall formulate the dual to Problem II where a certain linear form c has to be maximized. Of course this dual problem will be called of type I or briefly Problem I.

3. A Weak Compactness Theorem Establishing the Existence of a Solution to Problem II

Lehmann [3] p. 354, Nölle and Plachky [4] and Witting [8] showed that $\{\Phi\}$ is weakly *sequentially* compact and used this result for proving the existence of maximin size-α tests, most stringent size-α tests and so on. Krafft and Witting [2] remarked that also weak *covering* compactness holds; they used (and we shall use) this result for proving a fundamental weak duality theorem.

Let Ψ denote the class of all measurable $f\colon \mathfrak{X}\to R$; Ψ is partitioned into equivalence classes by means of $\sim$: $f\sim g$ if and only if $f=g$ a.e. (μ). Let $\{f\}$ denote the equivalence class generated by f, $\{\Psi\}$ the class of all $\{f\}$ and $\{\Phi\}$ the class of all $\{f\}$ with $f\in\Phi$. Then $L_1(\mathfrak{X},\mathfrak{A},\mu)\subset\{\Psi\}$ denotes the class of all $\{f\}$ such that

$$\|\{f\}\|_1=\int_{\mathfrak{X}}|f(x)|\,d\mu<\infty$$

and $L_\infty(\mathfrak{X},\mathfrak{A},\mu)\subset\{\Psi\}$ denotes the class of all $\{g\}\in\{\Psi\}$ with

$$\|\{g\}\|_\infty=\operatorname{ess\,sup}|g(x)|<\infty.$$

It is well-known that L_1 and L_∞ are Banach spaces. Moreover μ was assumed to be σ-finite and hence L_∞ can be regarded as the continuous dual L_1' consisting of all continuous linear functionals over L_1, when $\{g\}\in L_\infty$ is regarded as the functional which is defined by

$$\{g\}(\{f\})=\int_{\mathfrak{X}} f(x)\,g(x)\,d\mu \qquad \{f\}\in L_1. \tag{3.1}$$

Thus (L_1, L_∞) constitute a dual pair of vector spaces with respect to the bilinear form $\langle\ \rangle\colon L_1\times L_\infty\to R$ where $\langle\{f\},\{g\}\rangle=\int fg\,d\mu$. Accordingly various topologies can be defined over L_1 and L_∞ respectively. We consider the weak $\sigma(L_\infty, L_1)$-topology over L_∞ which is sometimes called the weak-star topology over the dual $L_1'=L_\infty$ of the Banach space L_1 and which is the coarsest topology over L_∞ such that all functionals $\{f\}\in L_1$, which are defined by $\{f\}(\{g\})=\langle\{f\},\{g\}\rangle$ over L_∞, are continuous. A basis of (closed) neighbourhoods of the origin is obtained when we consider all sets of the form

$$U_{\{f_1,\dots,f_n\}}=\{\{g\};\ \{g\}\in L_\infty,\ |\langle\{f_i\},\{g\}\rangle|\leqq 1\ \ (i=1,\dots,n)\} \tag{3.2}$$

where $\{f_1,\dots,f_n\}$ is an arbitrary finite subset of L_1.

Next we apply a well-known theorem, stating that the unit ball $B=\{\{g\};\ \{g\}\in L_\infty,\ \|\{g\}\|_\infty\leqq 1\}$ in the continuous dual $L_1'=L_\infty$ of a normed space L_1 is $\sigma(L_\infty, L_1)$-compact. But $\{\Phi\}=\{\tfrac{1}{2}\}+\tfrac{1}{2}B$, where $\tfrac{1}{2}$ denotes the constant function with $\tfrac{1}{2}(x)=\tfrac{1}{2}$ over $\mathfrak{X}$.

Theorem 3.1. *$\{\Phi\}$ is compact in the weak (-star) topology $\sigma(L_\infty, L_1)$.*

Under certain conditions (the initial domain of definition of μ consists of a finite or countable number of sets as is the case for Lebesgue measure in R^k), one can show that the measure μ is separable, hence $L_1(\mathfrak{X}, \mathfrak{A}, \mu)$ is separable and the $\sigma(L_\infty, L_1)$-topology has a countable base of neighbourhoods (of the origin) (is metrisable). But then compactness implies sequential compactness. Witting [8] and Nölle and Plachky [4] showed that the above-mentioned conditions are not necessary: $\{\Phi\}$ is both compact and sequentially compact in the weak (-star) topology $\sigma(L_\infty, L_1)$, provided that μ is σ-finite. We shall only apply Theorem 3.1.

Theorem 3.2. *There exists a solution (φ^*, γ^*) to the* L.P. *Problem* II *of Section* 2.

Proof. Let $\Gamma(\gamma)$ denote the set of all $\{\varphi\} \in \{\Phi\}$ such that (φ, γ) is feasible (satisfies (2.4), ..., (2.8)) for some $\varphi \in \{\varphi\}$. If $\Gamma(0) \neq \emptyset$ then obviously there exists $\varphi^* \in \Phi$ so that $(\varphi^*, 0)$ is a solution to Problem II. Hence suppose $\Gamma(0) = \emptyset$. We have $\Gamma(1-\alpha) \neq \emptyset$ and $\Gamma(\gamma) \subset \Gamma(\gamma')$ if $0 \leqq \gamma < \gamma' \leqq 1-\alpha$. Consequently there exists $\gamma^* \in [0, 1-\alpha]$ such that $\Gamma(\gamma) = \emptyset$ if $\gamma < \gamma^*$ and $\Gamma(\gamma) \neq \emptyset$ if $\gamma > \gamma^*$. But $\Gamma(\gamma^*) = \bigcap_{\gamma > \gamma^*} \Gamma(\gamma)$ and each $\Gamma(\gamma)$ of the chain $\{\Gamma(\gamma); \gamma > \gamma^*\}$ is a $\sigma(L_\infty, L_1)$-closed non-empty subset of the $\sigma(L_\infty, L_1)$-compact set $\{\Phi\}$. Hence $\Gamma(\gamma^*) \neq \emptyset$. If $\{\varphi^*\} \in \Gamma(\gamma^*)$, then (φ^*, γ^*) is a solution.

In order to prove that $\Gamma(\gamma)$ is $\sigma(L_\infty, L_1)$-closed, we show that $L_\infty - \Gamma(\gamma)$ is open or in other words that for each $\{\varphi\} \in L_\infty - \Gamma(\gamma)$ there exists a neighbourhood U of the origin, of the form (3.2), in such a way that $(\{\varphi\} + U) \cap \Gamma(\gamma) = \emptyset$. If $\{\varphi\} \notin \{\Phi\}$ then the existence of U follows from the compactness of $\{\Phi\}$ (L_∞ is $\sigma(L_\infty, L_1)$-separated). Hence suppose $\{\varphi\} \in (\{\Phi\} - \Gamma(\gamma))$. Then for $\varphi \in \{\varphi\}$ one of the inequalities (2.7), (2.8) does not hold. Suppose that for some $\theta \in \Omega_K$ we have $\gamma + \int \varphi\, p_\theta\, d\mu < \beta_D^*(\theta) - \varepsilon$. Construct U according to (3.2) where $n = 1$ and $f_1 = p_\theta\, \varepsilon^{-1}$. Then $(\{\varphi\} + U) \cap \Gamma(\gamma) = \emptyset$.

4. A Complete Solution to Problem II in the Finite Case

From the theoretical point of view it may be interesting to consider a theory of *finite statistical decision problems* which are defined by a finite sample space $\mathfrak{X}$, a finite parameter space Ω and a finite decision space $\mathfrak{D}$. Such a theory might suggest which results hold in more general situations and the lines along which these results can be proved. Especially for Problem II we need such considerations in order to find the corresponding dual L.P. Problem I (see Section 5).

Accordingly we assume in this section $\mathfrak{X} = \{x_1, \ldots, x_N\}$, $\Omega_H = \{\theta_1, \ldots, \theta_{M_1}\}$, $\Omega_K = \{\theta_{M_1+1}, \ldots, \theta_{M_1+M_2}\}$ while of course $\mathfrak{D}$ consists only of two decisions $\mathfrak{D} = \{d_0, d_1\}$ where d_0 is formulated as "neither H nor K are rejected nor accepted" and d_1 is "H is rejected and K is accepted". P_{θ_m} will be characterized by means of the probabilities $p_{mn} = P_{\theta_m}(\{x_n\})$ $(n = 1, \ldots, N)$ of the elementary events in $\mathfrak{X}$ (μ is the counting measure).

Problem II of Lemma 2.1 is reduced to the finite L.P. Problem II where a vector $y = (y_1, \ldots, y_{N+1})$ with $y_n = \varphi(x_n)$ $(n = 1, \ldots, N)$ and $y_{N+1} = \gamma$ has to be determined, minimizing the linear form (y, b), under the restrictions

$$y \geqq 0; \quad yA \geqq c \tag{4.1}$$

defining the set Y of all feasible vectors. Here the $(N+1)\times(N+M_1+M_2)$-matrix A, the $(N+1)$-vector b and the $(N+M_1+M_2)$-vector c are determined by

$$A=\begin{bmatrix} -1 & \dots & 0 & -p_{11} & \cdots & -p_{M_1 1} & p_{M_1+1,1} & \cdots & p_{M_1+M_2,1} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & -1 & -p_{1N} & \cdots & -p_{M_1 N} & p_{M_1+1,N} & \cdots & p_{M_1+M_2,N} \\ 0 & \dots & 0 & 0 & \dots & 0 & 1 & \dots & 1 \end{bmatrix}; \quad \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}=b$$

$$c=[-1 \ \dots \ -1 \quad -\alpha \quad \dots \ -\alpha \quad \beta^*_{M_1+1} \ \cdots \ \beta^*_{M_1+M_2}]$$

where $\beta^*_m=\beta^*_D(\theta_m)$ $(m=M_1+1,\dots,M_1+M_2)$.

In addition to Problem II we consider the L.P. Problem I where a vector $x=(v_1,\dots,v_N;\lambda_1,\dots,\lambda_{M_1};\nu_1,\dots,\nu_{M_2})=(v,\lambda,\nu)$ has to be determined, maximizing the linear form (c,x), under the restrictions

$$x\geqq 0; \quad Ax\leqq b \tag{4.2}$$

defining the set X of all feasible vectors. Problem II is said to be the dual of Problem I. The following well-known results will provide a guide to the following sections.

Lemma 4.1. *If x is feasible for Problem* I *($x\in X$) and y for Problem* II *($y\in Y$) then*

$$(c,x)\leqq(yA,x)=(y,Ax)\leqq(y,b). \tag{4.3}$$

Lemma 4.2. *If $x^*\in X$, $y^*\in Y$ and*

$$(c,x^*)=(y^*,b) \tag{4.4}$$

then (x^*,y^*) constitute a pair of (optimal) solutions.

Lemma 4.3. *If $x^*\in X$ and $y^*\in Y$ then* (4.4) *holds if and only if*

$$(Ax^*)_j<b_j \quad \textit{implies} \quad y^*_j=0, \tag{4.5}$$

$$(y^*A)_i>c_i \quad \textit{implies} \quad x^*_i=0. \tag{4.6}$$

Proof. (4.4) holds if and only if equality holds everywhere in (4.3) or in other words $(y^*,b-Ax)=(y^*A-c,x^*)=0$. But the latter conditions are equivalent to (4.5) and (4.6).

The above-described sufficient conditions for the optimality of a pair x^*,y^* are also necessary. This follows from the following fundamental duality theorem which applies to general finite L.P. problems just like the above-described lemmas.

Theorem 4.1. *If* (i) *one of the two problems has a solution or* (ii) *both problems have feasible vectors, then both problems have solutions. Each pair x^*,y^* of solutions satisfies* (4.4) *and on account of Lemma* 4.3 *the equivalent conditions* (4.5) *and* (4.6).

For our special L.P. problems with A, b and c described above, we know that there exists a solution y^* to Problem II on account of Theorem 3.2.

Corollary 4.1. *There exists a pair of solutions to our special pair of* L.P. *problems. x^*,y^* with $x^*\in X$ and $y^*\in Y$ is a pair of solutions if and only if* (4.4) *or the equivalent conditions* (4.5) *and* (4.6) *are satisfied.*

5. Sufficient Conditions for Optimality

We generalize the Lemmas 4.1, 4.2 and 4.3 along the lines described in [2] and [8]. For that purpose let $\mathfrak{B}_H(\mathfrak{B}_K)$ be a suitable σ-field of subsets of $\Omega_H(\Omega_K)$ such that (for a certain determination) the p.d.f. $p_\theta(x)$: $\mathfrak{X}\times\Omega_H\to[0,\infty)$ is *measurable* with respect to the σ-field $\mathfrak{A}\times\mathfrak{B}_H$ generated by all sets of the form $A\times B$ where $A\in\mathfrak{A}$ and $B\in\mathfrak{B}_H$ and that $p_\theta(x)$: $\mathfrak{X}\times\Omega_K\to[0,\infty)$ is *measurable* $(\mathfrak{A}\times\mathfrak{B}_K)$. Of course β_D^*: $\Omega_K\to[0,1]$ is assumed to be *measurable* $(\mathfrak{B}_K)$.

Our L.P. Problem I will be to determine a $\mathfrak{A}$-measurable function $v:\mathfrak{X}\to[0,\infty)$, a measure λ over $(\Omega_H,\mathfrak{B}_H)$ and a measure ν over $(\Omega_K,\mathfrak{B}_K)$ such that (see Problem I in Section 4)

$$c(v,\lambda,\nu)=-\int_{\mathfrak{X}} v(x)\,\mathrm{d}\mu(x)-\alpha\lambda(\Omega_H)+\int_K\beta_D^*(\theta)\,\mathrm{d}\nu \tag{5.1}$$

is maximized, under the following feasibility restrictions which define the set X of elements (v,λ,ν):

$$v(x)\geqq 0 \qquad \text{a.e. } (\mu), \tag{5.2}$$

$$\lambda(B)\geqq 0 \qquad \forall B\in\mathfrak{B}_H, \tag{5.3}$$

$$\nu(B)\geqq 0 \qquad \forall B\in\mathfrak{B}_K, \tag{5.4}$$

$$-v(x)-\int_H p_\theta(x)\,\mathrm{d}\lambda+\int_K p_\theta(x)\,\mathrm{d}\nu\leqq 0 \qquad \text{a.e. } (\mu), \tag{5.5}$$

$$\nu(\Omega_K)\leqq 1. \tag{5.6}$$

Problem I has been formulated in such a manner that it reduces in the finite case to Problem I of Section 4. (5.1) and (5.5) contain terms which might be infinite. We can restrict X by requiring definiteness, and without changing $\sup_X c(v,\lambda,\nu)$ or the set of all optimal (v,λ,ν)'s: $X\neq\emptyset$ for X contains (v,λ,ν) with $v=0,\lambda=0,$ $\nu=0$. Hence $\sup_X c(v,\lambda,\nu)\geqq 0$. But $c(v,\lambda,\nu)>c$ for c arbitrary negative, implies $\int v\,\mathrm{d}\mu<\infty$ and $\lambda(\Omega_H)<\infty$, for $0\leqq\int\beta_D^*\,\mathrm{d}\nu\leqq 1$ and $0<\alpha$ (we always assume $0<\alpha<1$). This implies $\int p_\theta(x)\,\mathrm{d}\lambda<\infty$ a.e. (μ), for otherwise $\iint p_\theta(x)\,\mathrm{d}\lambda\,\mathrm{d}\mu=\lambda(\Omega_H)=\infty$ (Fubini). Similarly $\int p_\theta(x)\,\mathrm{d}\nu<\infty$ a.e. (μ).

Lemma 5.1. *If* (v,λ,ν) *is feasible for Problem* I $\big((v,\lambda,\nu)\in X\big)$ *and* (φ,γ) *for Problem* II $\big((\varphi,\gamma)\in Y\big)$, *then*

$$c(v,\lambda,\nu)\leqq b(\varphi,\gamma). \tag{5.7}$$

Proof. We generalize the inequalities (4.3) and apply Fubini.

$$\begin{aligned} c(v,\lambda,\nu)&\leqq-\int\varphi v\,\mathrm{d}\mu-\iint\varphi\,p_\theta\,\mathrm{d}\mu\,\mathrm{d}\lambda+\int\{\gamma+\int\varphi\,p_\theta\,\mathrm{d}\mu\}\,\mathrm{d}\nu\\ &=\int\varphi\{-v-\int p_\theta\,\mathrm{d}\lambda+\int p_\theta\,\mathrm{d}\nu\}\,\mathrm{d}\mu+\gamma\,\nu(\Omega_K)\\ &\leqq\gamma=b(\varphi,\gamma). \end{aligned} \tag{5.8}$$

Consequently proofs in finite statistics may indeed provide a guide to much more general situations.

Lemma 5.2. *If* $(v^*, \lambda^*, \nu^*) \in X$, $(\varphi^*, \gamma^*) \in Y$ *and*

$$c(v^*, \lambda^*, \nu^*) = b(\varphi^*, \gamma^*) \tag{5.9}$$

then (v^*, λ^*, ν^*) *is a solution to Problem* I *and* (φ^*, γ^*) *to Problem* II.

Lemma 5.3. *If* $(v^*, \lambda^*, \nu^*) \in X$ *and* $(\varphi^*, \gamma^*) \in Y$ *then* (5.9) *holds if and only if*

$$\varphi^*(x) = 1 \quad \text{for almost } (\mu) \text{ all } x \text{ with } v^*(x) > 0, \tag{5.10}$$

$$\int \varphi^* p_\theta \, d\mu = \alpha \quad \text{for almost } (\lambda^*) \text{ all } \theta \in \Omega_H, \tag{5.11}$$

$$\beta_D^*(\theta) = \gamma^* + \int \varphi^* p_\theta \, d\mu \quad \text{for almost } (\nu^*) \text{ all } \theta \in \Omega_K, \tag{5.12}$$

$$\varphi^*(x) = 0 \quad \text{for almost } (\mu) \text{ all } x \text{ with } v^*(x) > -\int p_\theta(x) \, d\lambda^* + \int p_\theta(x) \, d\nu^*, \tag{5.13}$$

$$\nu^*(\Omega_K) = 1 \quad \text{if } \gamma^* > 0, \tag{5.14}$$

$$v^*(x) = \max\{0, -\int p_\theta(x) \, d\lambda^* + \int p_\theta(x) \, d\nu^*\} \quad a.e. \ (\mu). \tag{5.15}$$

Proof. (5.9) holds if and only if equality holds everywhere in (5.8) or in other words if and only if (5.10), ..., (5.14) hold. Hence (5.9) will certainly hold if (5.10), ..., (5.15) are true. On the other hand if (5.9) holds then (v^*, λ^*, ν^*) is a solution to Problem I and it follows easily from (5.1), ..., (5.6) that this implies (5.15).

Formula (5.15) shows that we can reduce Problem I to the following *reduced Problem* I where a measure λ over $(\Omega_H, \mathfrak{B}_H)$ and a measure ν over $(\Omega_K, \mathfrak{B}_K)$ have to be determined such that

$$f(\lambda, \nu) = -\int_{\mathfrak{X}} \max\{0, -\int_H p_\theta(x) \, d\lambda + \int_K p_\theta(x) \, d\nu\} \, d\mu - \alpha \lambda(\Omega_H) + \int_K \beta_D^*(\theta) \, d\nu \tag{5.16}$$

is maximized under the restrictions

$$\lambda(B) \geqq 0 \qquad \forall B \in \mathfrak{B}_H, \tag{5.17}$$

$$\nu(B) \geqq 0 \qquad \forall B \in \mathfrak{B}_K, \tag{5.18}$$

$$\nu(\Omega_K) \leqq 1. \tag{5.19}$$

Of course we have

$$\sup_{\lambda, \nu} f(\lambda, \nu) = \sup_X c(v, \lambda, \nu) \geqq 0. \tag{5.20}$$

Moreover the following Lemmas can be proved easily.

Lemma 5.4. (λ^*, ν^*) *is a solution to the reduced Problem* I, *if and only if* (v^*, λ^*, ν^*) *is a solution to Problem* I *when* v^* *is determined by* (5.15).

Lemma 5.5. *If* (λ^*, ν^*) *is a solution to the reduced Problem* I *with* $\lambda^*(\Omega_H) > 0$ *(and* $\alpha > 0$*) then* $\nu^*(\Omega_K) > 0$. *If* (λ^*, ν^*) *is a solution to the reduced Problem* I *and* $f(\lambda^*, \nu^*) > 0$ *then* $\lambda^*(\Omega_H) > 0$ *and* $\nu^*(\Omega_K) = 1$. *The supremum of* (5.16) *under the restrictions* (5.17), (5.18) *and* (5.19) *is equal to the supremun under the restrictions* (5.17), (5.18) *and* $\nu^*(\Omega_K) = 1$.

6. A Duality Theorem Showing that the Sufficient Conditions of Lemma 5.3 are Necessary for Optimality

Lemma 5.3 provides necessary and sufficient conditions for the equality (5.9) which implies that $(v^*, \lambda^*, \nu^*), (\varphi^*, \gamma^*)$ is a pair of (optimal) solutions. We shall prove a fundamental (weak) Duality Theorem 6.2, which generalizes Theorem 4.1 and Corollary 4.1 to some extent, and which states that

$$0 \leqq \sup_{(v,\lambda,\nu)\in X} c(v,\lambda,\nu) = \min_{(\varphi,\gamma)\in Y} b(\varphi,\gamma) = \gamma^* \leqq 1-\alpha. \tag{6.1}$$

Hence if there exists a solution (v^*, λ^*, ν^*) to Problem I and (φ^*, γ^*) is one of the solutions of Problem II, then (5.9) holds.

Theorem 6.1. *(v^*, λ^*, ν^*), (φ^*, γ^*) constitute a pair of solutions to our pair of* L.P. *problems if and only if the conditions* (5.2), ..., (5.6) $((v^*, \lambda^*, \nu^*)\in X)$, (2.4), ..., (2.8) $((\varphi^*, \gamma^*)\in Y)$ *and* (5.10), ..., (5.15) *are satisfied* (*we assume* $0<\alpha<1$).

We shall prove Theorem 6.2 by generalizing the elegant considerations of Krafft and Witting in [2] Section 3. First we prove strong duality in the case of a finite parameter space Ω (strong duality means that both L.P. problems admit solutions). Next we use this result for showing weak duality (6.1) in the case of an arbitrary Ω. The latter proof uses the compactness of $\{\Phi\}$ and is based on the fundamental result that a subset A of a topological space is compact, if and only if for each family $\{\Psi_j; j\in J\}$ of closed non-empty subsets of A with the finite intersection property holds that $\bigcap_{j\in J} \Psi_j \neq \emptyset$.

Lemma 6.1 *(Strong duality if Ω is finite). If $\alpha>0$, $\Omega_H=\{\theta_1, \dots, \theta_{M_1}\}$, $\Omega_K=\{\theta_{M_1+1}, \dots, \theta_{M_1+M_2}\}$ and $\beta_D^*\colon \Omega_K \to [0,1]$ is an arbitrary function, then there exists a solution (v^*, λ^*, ν^*) to Problem* I, *and* (5.9) *holds for each pair of solutions (v^*, λ^*, ν^*), (φ^*, γ^*) to our* L.P. *problems.*

Proof Define $Q=\{q(\varphi); \varphi\in\Phi\}\subset R^{M_1+M_2}$ where

$$q_i(\varphi)=\mathrm{E}_{\theta_i}(\varphi)=\int \varphi \, dP_{\theta_i} \qquad (i=1,\dots,M_1)$$

and $q_i(\varphi)=\beta_D^*(\theta_i)-\mathrm{E}_{\theta_i}(\varphi)$ $(i=M_1+1,\dots,M_1+M_2)$. Q is convex, for $\varphi_1, \varphi_2\in\Phi$ and $0\leqq\rho\leqq 1$ implies that $\rho\,\varphi_1+(1-\rho)\,\varphi_2\in\Phi$ and hence that $\rho\, q(\varphi_1)+(1-\rho)\, q(\varphi_2)=q\{\rho\,\varphi_1+(1-\rho)\,\varphi_2\}\in Q$. Moreover Q is a compact subset of $R^{M_1+M_2}$ on account of the weak compactness of $\{\Phi\}$ (the weak-star $\sigma(L_\infty, L_1)$-topology makes each $q_i\colon L_\infty\to R$ with $q_i(\{\varphi\})=q_i(\varphi)$ continuous; accordingly $q\colon L_\infty\to R^{M_1+M_2}$ is $\sigma(L_\infty, L_1)$-continuous; the continuous image $q(\{\Phi\})=Q$ of the compact set $\{\Phi\}$ is compact).

Let S_γ denote the open negative orthant, shifted so that the origin is at $q_{\alpha,\gamma}=(\alpha,\dots,\alpha;\gamma,\dots,\gamma)$. Hence

$$S_\gamma=\{q;\, q_i<\alpha \ (i=1,\dots,M_1),\ q_i<\gamma \ (i=M_1+1,\dots,M_1+M_2)\}$$

and $[S_\gamma]$ will denote the closure of S_γ. The family $\{[S_\gamma]\cap Q; [S_\gamma]\cap Q\neq\emptyset\}$ is a chain of closed non-empty subsets of the compact set Q. Hence the intersection is non-empty and there exists a number γ^* such that $[S_\gamma]\cap Q=\emptyset$ for $\gamma<\gamma^*$ and $[S_\gamma]\cap Q\neq\emptyset$ for $\gamma\geqq\gamma^*$. We distinguish two cases: (i) $\gamma^*<0$ and (ii) $0\leqq\gamma^*\leqq 1-\alpha$ (the trivial test φ_α with $\varphi_\alpha(x)=\alpha$ a.e. (μ) satisfies $q(\varphi_\alpha)\in[S_{1-\alpha}]\cap Q$ and hence $\gamma^*\leqq 1-\alpha$).

Case (i). There exists $\varphi \in \Phi$ with $q(\varphi) \in [S_0] \cap Q$. Hence $(\varphi, 0)$ satisfies (2.4), ..., (2.8). On account of (2.5) we obtain $\min_Y b(\varphi, \gamma) = 0$. By applying (5.7) and $\sup_X c(v, \lambda, \nu) \geqq 0 = c(0, 0, 0)$ we obtain (6.1) with $\gamma^* = 0$, while (0, 0, 0) is a solution to Problem I.

Case (ii). We have $S_{\gamma^*} \cap Q = \emptyset$ for γ^* is the smallest number with $[S_{\gamma^*}] \cap Q \neq \emptyset$. But S_{γ^*} is open convex and Q is compact convex. Hence there exists a hyperplane separating S_{γ^*} and Q. It can be shown (Karlin [1] p. 25) that this hyperplane contains q_{α, γ^*} while the direction coefficients of a normal are all non-negative. Hence there exist $\lambda_1^* \ldots \lambda_{M_1}^*; \nu_1^* \ldots \nu_{M_2}^*$ all $\geqq 0$ such that some of these coefficients are > 0 and

$$\sum_{i=1}^{M_1} \lambda_i^* q_i(\varphi) + \sum_{i=1}^{M_2} \nu_i^* q_{M_1+i}(\varphi) \geqq \alpha \sum_{i=1}^{M_1} \lambda_i^* + \gamma^* \sum_{i=1}^{M_2} \nu_i^* \tag{6.2}$$

holds for all $\varphi \in \Phi$, while equality holds in (6.2) for all $\varphi \in \{\varphi;\, q(\varphi) \in [S_{\gamma^*}] \cap Q(\neq \emptyset)\}$, or in other words for all φ^* such that (φ^*, γ^*) is a solution to Problem II.

We remark that $\sum_{i=1}^{M_2} \nu_i^* > 0$, for suppose $\sum_{i=1}^{M_2} \nu_i^* = 0$ then (6.2) implies $\sum_{i=1}^{M_1} \lambda_i^* q_i(\varphi) \geqq \alpha \sum_{i=1}^{M_1} \lambda_i^*$ for all $\varphi \in \Phi$ and in particular for $\varphi = 0$. Hence $0 \geqq \alpha \sum_{i=1}^{M_1} \lambda_i^*$ and $\alpha > 0$ implies that all coefficients λ_i^* and ν_i^* are equal to zero. Thus we obtain a contradiction.

We can normalize the coefficients λ_i^*, ν_i^* in such a way that $\sum_{i=1}^{M_2} \nu_i^* = 1$. In that case the vector ν^* can be regarded as a probability measure over Ω_K while λ^* may be considered to be a measure over Ω_H. Next we define the $\mathfrak{A}$-measurable function $v^*\colon \mathfrak{X} \to R$ according to

$$v^*(x) = \max \left\{ 0,\, -\sum_{i=1}^{M_1} \lambda_i^* p_{\theta_i}(x) + \sum_{i=1}^{M_2} \nu_i^* p_{\theta_{M_1+i}}(x) \right\} \tag{6.3}$$

where p_{θ_i} is the (measurable) Radon-Nikodym derivative $\mathrm{d}P_{\theta_i}/\mathrm{d}\mu$.

Then (v^*, λ^*, ν^*) satisfies the feasibility restrictions (5.2), ..., (5.6) and if I denotes the indicator function of $\{x,\, x \in \mathfrak{X},\, v^*(x) > 0\}$, then

$$\begin{aligned} c(v^*, \lambda^*, \nu^*) &= -\int_{\mathfrak{X}} I(x) \left\{ -\sum_{i=1}^{M_1} \lambda_i^* p_{\theta_i}(x) + \sum_{i=1}^{M_2} \nu_i^* p_{\theta_{M_1+i}}(x) \right\} \mathrm{d}\mu \\ &\quad - \alpha \lambda^*(\Omega_H) + \int_K \beta_D^*(\theta)\, \mathrm{d}\nu^* \\ &= \sum_{i=1}^{M_1} \lambda_i^* \mathrm{E}_{\theta_i}(I) + \sum_{i=1}^{M_2} \nu_i^* \{\beta_D^*(\theta_i) - \mathrm{E}_{\theta_{M_1+i}}(I)\} - \alpha \sum_{i=1}^{M_1} \lambda_i^*. \end{aligned}$$

But $I \in \Phi$ and we can apply (6.2), thus obtaining

$$c(v^*, \lambda^*, \nu^*) \geqq \gamma^* \sum_{i=1}^{M_2} \nu_i^* = \gamma^* = b(\varphi^*, \gamma^*).$$

By applying Lemma 5.1 we obtain (5.9).

Theorem 6.2 *(Weak duality for arbitrary Ω). If $\alpha\in(0,1)$, $p_\theta(x)$ is measurable $(\mathfrak{B}_H\times\mathfrak{A})$ over $\Omega_H\times\mathfrak{X}$ and measurable $(\mathfrak{B}_K\times\mathfrak{A})$ over $\Omega_K\times\mathfrak{X}$ while β_D^*: $\Omega_K\to[0,1]$ is arbitrary but measurable $(\mathfrak{B}_K)$, then* (6.1) *holds.*

Proof. Lemma 5.1 shows that

$$c^*=\sup_X c(v,\lambda,\nu)\leqq\min_Y b(\varphi,\gamma)=\gamma^*.$$

Moreover if $\varphi_\alpha(x)=\alpha$ for all $x\in\mathfrak{X}$ then $(\varphi_\alpha, 1-\alpha)$ satisfies (2.4), ..., (2.8) and hence $\gamma^*\leqq 1-\alpha$. Further $v=0$, $\lambda=0$, $\nu=0$ satisfies (5.2), ..., (5.6) and hence $c^*\geqq 0$. Thus it is sufficient to show that $c^*\geqq\gamma^*$ or in other words that the subset $\{\Psi\}$ of $\{\Phi\}\subset L_\infty$ is non-empty when

$$\Psi=\{\varphi;\ \varphi\in\Phi;\ \mathrm{E}_\theta(\varphi)\leqq\alpha(\theta\in\Omega_H);\ \beta_D^*(\theta)-\mathrm{E}_\theta(\varphi)\leqq c^*(\theta\in\Omega_K)\}. \tag{6.4}$$

It is straightforward to write

$$\Psi=\bigcap\Psi(\theta_1,\ldots,\theta_{M_1};\ \theta_{M_1+1},\ldots,\theta_{M_1+M_2}) \tag{6.5}$$

where the intersection is taken over all pairs of a finite subset $\{\theta_1,\ldots,\theta_{M_1}\}$ of Ω_H and $\{\theta_{M_1+1},\ldots,\theta_{M_1+M_2}\}\subset\Omega_K$, while $\Psi(\theta_1,\ldots,\theta_{M_1};\ \theta_{M_1+1},\ldots,\theta_{M_1+M_2})$ denotes the following subset of Φ

$$\begin{aligned}&\{\varphi;\ \varphi\in\Phi;\ \mathrm{E}_{\theta_i}(\varphi)\leqq\alpha\ \ (i=1,\ldots,M_1);\\ &\quad\beta_D^*(\theta_i)-\mathrm{E}_{\theta_i}(\varphi)\leqq c^*\ \ (i=M_1+1,\ldots,M_1+M_2)\}.\end{aligned} \tag{6.6}$$

We shall show that the subset of $\{\Phi\}\subset L_\infty$, corresponding with (6.6) is (i) non-empty and (ii) $\sigma(L_\infty, L_1)$-closed. The family consisting of all such subsets has the finite intersection property, for the intersection of a finite number of sets of the form (6.6) is again a set of the form (6.6). But $\{\Phi\}$ is $\sigma(L_\infty, L_1)$-compact (Theorem 3.1). Hence $\{\Psi\}\neq\emptyset$ and $\Psi\neq\emptyset$, so that the proof is complete when we have dealt with (i) and (ii).

(i) *The set* (6.6) *is non-empty.* This will be proved by applying Lemma 6.1 to the testing problem (H', K') when $\Omega_{H'}=\{\theta_1,\ldots,\theta_{M_1}\}$ and $\Omega_{K'}=\{\theta_{M_1+1},\ldots,\theta_{M_1+M_2}\}$. Let I′ and II′ denote the corresponding L.P. problems and X' and Y' the corresponding sets of feasible elements. Each pair of measures over $\Omega_{H'}$ and $\Omega_{K'}$ respectively can be identified with a pair of measures over $(\Omega_H, \mathfrak{B}_H)$ and $(\Omega_K, \mathfrak{B}_K)$. Hence on account of Lemma 6.1 and the fact that X' may be regarded as a subset of X,

$$c^*=\sup_X c(v,\lambda,\nu)\geqq\max_{X'} c(v',\lambda',\nu')=\min_{Y'} b(\varphi',\gamma')=\gamma',$$

so that there exists a solution (φ',γ') to Problem II′ where φ' belongs to the set (6.6).

(ii) *The set* (6.6) *is* $\sigma(L_\infty, L_1)$*-closed.* This can be shown along the same lines as the $\sigma(L_\infty, L_1)$-closedness of $\Gamma(\gamma)$ at the end of Section 3.

7. Characterizing the Solution (φ^*, γ^*) to Problem II by Means of the Solution (λ^*, ν^*) to the Reduced Problem I

Theorem 6.1 provides necessary and sufficient conditions for the optimality of a pair (v^*, λ^*, ν^*), (φ^*, γ^*). We can give various modifications of the formulation of this theorem.

Theorem 7.1. *Sufficient for the optimality of* (φ^*, γ^*) *with* $\varphi^* \in \Phi$ *as a solution to Problem* II *is that there exists a pair* $\tilde{\lambda}, \tilde{\nu}$ *of probability measures over* $(\Omega_H, \mathfrak{B}_H)$ *and* $(\Omega_K, \mathfrak{B}_K)$ *respectively and a number k such that*

$$\varphi^*(x) = 1 \qquad \textit{for almost } (\mu) \textit{ all } x \textit{ with } \int_K p_\theta(x)\,\mathrm{d}\tilde{\nu} > k \int_H p_\theta(x)\,\mathrm{d}\tilde{\lambda}, \tag{7.1}$$

$$\varphi^*(x) = 0 \qquad \textit{for almost } (\mu) \textit{ all } x \textit{ with } \int_K p_\theta(x)\,\mathrm{d}\tilde{\nu} < k \int_H p_\theta(x)\,\mathrm{d}\tilde{\lambda}, \tag{7.2}$$

$$\sup_H \mathrm{E}_\theta(\varphi^*) = \alpha, \tag{7.3}$$

$$\mathrm{E}_\theta(\varphi^*) = \alpha \qquad \textit{for almost } (\tilde{\lambda}) \textit{ all } \theta \in \Omega_H, \tag{7.4}$$

$$\sup_K \gamma_{\varphi^*, D}(\theta) = \sup_K \{\beta_D^*(\theta) - \mathrm{E}_\theta(\varphi^*)\} = \gamma^* \geqq 0, \tag{7.5}$$

$$\gamma_{\varphi^*, D}(\theta) = \gamma^* \qquad \textit{for almost } (\tilde{\nu}) \textit{ all } \theta \in \Omega_K. \tag{7.6}$$

If there exists a solution (λ^*, ν^*) *to the reduced Problem* I *with* $\lambda^*(\Omega_H) > 0$ *(and consequently* $\nu^*(\Omega_K) > 0$*) then each solution* (φ^*, γ^*) *to Problem* II *(such solutions exist on account of Theorem* 3.2*) satisfies* $\varphi^* \in \Phi$ *and* (7.1), ..., (7.6)*; where* $\tilde{\lambda} = \lambda^*/\lambda^*(\Omega_H)$; $\tilde{\nu} = \nu^*/\nu^*(\Omega_K)$ *and* $k = \lambda^*(\Omega_H)/\nu^*(\Omega_K)$.

Proof. (i) *Sufficiency of the conditions.* Define $\lambda^* = k\,\tilde{\lambda}$ and $\nu^* = \tilde{\nu}$. Then (v^*, λ^*, ν^*) is feasible for Problem I when v^* is determined according to (5.15). Moreover (φ^*, γ^*) is feasible for Problem II on account of $\varphi^* \in \Phi$, (7.3) and (7.5). But (7.1) → (5.10), (7.4) → (5.11), (7.6) → (5.12), (7.2) → (5.13) and Lemma 5.3 together with Lemma 5.2 show that (φ^*, γ^*) is a solution to Problem II (and that (λ^*, ν^*) is a solution to the reduced Problem I: if Problem I does not admit a solution then it is not possible to find $\tilde{\lambda}, \tilde{\nu}$ such that the conditions are satisfied).

(ii) *Necessity of the conditions.* If (λ^*, ν^*) is a solution to the reduced Problem I then (v^*, λ^*, ν^*) is a solution to Problem I when v^* is determined according to (5.15). But if (v^*, λ^*, ν^*), (φ^*, γ^*) is a pair of solutions then on account of Theorem 6.1 the conditions (5.10), ..., (5.15) hold together with the feasibility of (φ^*, γ^*) and $\gamma^* = f(\lambda^*, \nu^*) \geqq 0$. This implies $\varphi^* \in \Phi$ and (7.1), ..., (7.6).

By applying Theorem 7.1 to the special case $D = \Phi$, corresponding with the construction of a maximin size-α test, and applying Lemma 2.1 we obtain the following result.

Corollary 7.1. *Sufficient in order that* φ^* *is a maximin size-*α *test for Problem* (H, K), *is that there exists a pair* $\tilde{\lambda}, \tilde{\nu}$ *of probability measures over* $(\Omega_H, \mathfrak{B}_H)$ *and* $(\Omega_K, \mathfrak{B}_K)$ *respectively and a number k such that* $\varphi^* \in \Phi_\alpha$, (7.1), (7.2) *and* (7.4) *are satisfied together with*

$$\mathrm{E}_\theta(\varphi^*) = \inf_K \mathrm{E}_\theta(\varphi^*) \qquad \textit{for almost } (\tilde{\nu}) \textit{ all } \theta \in \Omega_K. \tag{7.7}$$

If there exists a solution λ^*, ν^* *to the reduced Problem* I *with* $\lambda^*(\Omega_H)>0$ (*and consequently* $\nu^*(\Omega_K)>0$) *then these conditions are also necessary for optimality of* φ^* *if* $\tilde{\lambda}=\lambda^*/\lambda^*(\Omega_H)$; $\tilde{\nu}=\nu^*/\nu^*(\Omega_K)$ *and* $k=\lambda^*(\Omega_H)/\nu^*(\Omega_K)$.

Corollary 7.1 corresponds with Satz 7 in Krafft und Witting [2] and with Theorem 1 in Lehmann [3]. Dealing with the maximin size-α test, these authors characterize $(\tilde{\lambda}, \tilde{\nu})$ as a least favorable pair of probability measures. We must try to generalize this notion to our general pair of L.P. problems. But for that purpose we need a formulation of the Neyman-Pearson fundamental lemma which will be obtained by applying Corollary 7.1 to the case of a simple hypothesis $H: \theta=\theta_0$ and a simple alternative $K: \theta=\theta_1$ (see Lehmann [3] p. 65).

Corollary 7.2. *Sufficient in order that* $\varphi^*\in\Phi$ *is* M.P. *size-*α *for testing* $H: \theta=\theta_0$ *against* $K: \theta=\theta_1$ *is that there exists a number k such that*

$$\varphi^*(x)=1 \qquad \text{for almost } (\mu) \text{ all } x \text{ with } p_{\theta_1}(x)>k\,p_{\theta_0}(x), \tag{7.8}$$

$$\varphi^*(x)=0 \qquad \text{for almost } (\mu) \text{ all } x \text{ with } p_{\theta_1}(x)<k\,p_{\theta_0}(x), \tag{7.9}$$

$$\mathrm{E}_{\theta_0}(\varphi^*)=\alpha. \tag{7.10}$$

These conditions are also necessary provided that there does not exist a test φ' *with* $\mathrm{E}_{\theta_0}(\varphi')<\alpha$ *and* $\mathrm{E}_{\theta_1}(\varphi')=1$.

Proof. We must show that there exists a number $\lambda>0$ such that (see Corollary 7.1)

$$f(\lambda, 1)=-\int_{\mathfrak{X}} \max\{0, -\lambda\, p_{\theta_0}(x)+p_{\theta_1}(x)\}\, \mathrm{d}\mu-\alpha\lambda+1\geqq 0$$

(for then there exists a solution $(\lambda^*, 1)$ to the reduced Problem I with $\lambda^*>0$ – the existence of a solution follows from Lemma 6.1), unless there exists a test φ' with $\mathrm{E}_{\theta_0}(\varphi')<\alpha$ and $\mathrm{E}_{\theta_1}(\varphi')=1$.

First suppose that there exists $\lambda>0$ such that $\mathrm{E}_{\theta_0}(I_\lambda)\geqq\alpha$ where I_λ is the indicator function of $\{x; p_{\theta_1}(x)\geqq\lambda\, p_{\theta_0}(x)\}$. Then

$$f(\lambda, 1)=\lambda\{\mathrm{E}_{\theta_0}(I_\lambda)-\alpha\}+\{1-\mathrm{E}_{\theta_1}(I_\gamma)\}\geqq 0.$$

Next suppose that for each $\lambda>0$ we have $\mathrm{E}_{\theta_0}(I_\lambda)<\alpha$ and $f(\lambda, 1)\leqq 0$. Let I denote the indicator function of $\{x; p_{\theta_1}(x)>0\}$. Then

$$\lambda\{\mathrm{E}_{\theta_0}(I_\lambda)-\alpha\}+\{1-\mathrm{E}_{\theta_1}(I)\}\leqq f(\lambda, 1)\leqq 0$$

and $\lambda\to 0$ shows that $\mathrm{E}_{\theta_1}(I)=1$ and $\mathrm{E}_{\theta_0}(I)\leqq\alpha$ (I_λ is nonnegative and $I_\lambda(x)$ is nondecreasing for $\lambda\to 0$ and $\lim_{\lambda\to 0} I_\lambda(x)=I(x)$; hence the Lebesque monotone convergence theorem shows that $\mathrm{E}_{\theta_0}(I)=\lim_{\lambda\to 0}\mathrm{E}_{\theta_0}(I_\lambda)$).

Hence the conditions (7.8), (7.9) and (7.10) are necessary unless $\mathrm{E}_{\theta_1}(I)=1$ and $\mathrm{E}_{\theta_0}(I)\leqq\alpha$. It can be shown easily that the conditions are also necessary if $\mathrm{E}_{\theta_1}(I)=1$ and $\mathrm{E}_{\theta_0}(I)=\alpha$. If $\mathrm{E}_{\theta_1}(I)=1$ and $\mathrm{E}_{\theta_0}(I)<\alpha$ then the conditions are not necessary, for $\varphi'=I$ is a test with $\mathrm{E}_{\theta_1}(\varphi')=1$ but which does not satisfy (7.10).

Suppose that (φ^*, γ^*) with $\varphi^* \in \Phi$ is such that there exists a pair $\tilde{\lambda}$, $\tilde{\nu}$ of probability measures and a number k such that (7.1), ..., (7.6) are satisfied. Then on account of Corollary 7.2 test φ^* is a M.P. size-α test for testing the simple hypothesis $H_{\tilde{\lambda}}$ that the p.d.f. is given by $p_{\tilde{\lambda}}(x) = \int_H p_\theta(x)\, d\tilde{\lambda}$ against the simple alternative $K_{\tilde{\nu}}$ that the p.d.f. is given by $p_{\tilde{\nu}}(x) = \int_K p_\theta(x)\, d\tilde{\nu}$.

Definition. $(\tilde{\lambda}, \tilde{\nu})$ is said to be a least favorable pair of probability measures over $(\Omega_H, \mathfrak{B}_H)$ and $(\Omega_K, \mathfrak{B}_K)$ respectively, if for each other pair (λ', ν') we have

$$\int_K \gamma_{\varphi', D}(\theta)\, d\nu' \leqq \int_K \gamma_{\tilde{\varphi}, D}(\theta)\, d\tilde{\nu} \tag{7.11}$$

where $\tilde{\varphi}$ is an arbitrary M.P. size-α test for Problem $(H_{\tilde{\lambda}}, K_{\tilde{\nu}})$ and φ' is a M.P. size-α test for $(H_{\lambda'}, K_{\nu'})$.

Theorem 7.2. *If (λ^*, ν^*) with $\lambda^*(\Omega_H) > 0$ (and consequently $\nu^*(\Omega_K) > 0$) is a solution to the reduced Problem* I *and $\tilde{\lambda} = \lambda^*/\lambda^*(\Omega_H)$, $\tilde{\nu} = \nu^*/\nu^*(\Omega_K)$ then $(\tilde{\lambda}, \tilde{\nu})$ is a least favorable pair of prabability measures.*

Proof. Let (φ^*, γ^*) be one of the solutions to Problem II (Theorem 3.2). Then $\varphi^* \in \Phi$ satisfies (7.1), ..., (7.6) (Theorem 7.1) and hence φ^* is M.P. size-α for Problem $(H_{\tilde{\lambda}}, K_{\tilde{\nu}})$ (Corollary 7.2). Moreover

$$\int_K \gamma_{\varphi^*, D}(\theta)\, d\tilde{\nu} = \gamma^* \geqq \int_K \gamma_{\varphi^*, D}(\theta)\, d\nu' \tag{7.12}$$

on account of (7.5) and (7.6), when ν' is an arbitrary probability measure over $(\Omega_K, \mathfrak{B}_K)$. Next suppose that φ' is M.P. size-α for $(H_{\lambda'}, K_{\nu'})$ where (λ', ν') is an arbitrary pair of probability measures. Test φ^* is also of size-α for testing $H_{\lambda'}$ on account of (7.3). Hence

$$\int_K \mathrm{E}_\theta(\varphi')\, d\nu' \geqq \int_K \mathrm{E}_\theta(\varphi^*)\, d\nu'$$

and by applying (7.12) we obtain (7.11) (where φ^* plays the part of $\tilde{\varphi}$).

If there exists a least favorable pair $(\tilde{\lambda}, \tilde{\nu})$ then we can construct a number $\tilde{k}$ and a M.P. size-α test $\tilde{\varphi}$ for Problem $(H_{\tilde{\lambda}}, K_{\tilde{\nu}})$ such that (7.8), (7.9) and (7.10) are satisfied (see Corollary 7.2; if there exists a test φ' with size $< \alpha$ and power 1, then take $\tilde{k} = 0$, $\tilde{\varphi}(x) = 1$ for all x with $p_{\tilde{\nu}}(x) > 0$, and $\tilde{\varphi}(x) = \tilde{c}$ if $p_{\tilde{\nu}}(x) = 0$, with $\tilde{c}$ such that (7.10) holds).

Theorem 7.3. *If $(\tilde{\lambda}, \tilde{\nu})$ is a least favorable pair then $(\tilde{k}\,\tilde{\lambda}, \tilde{\nu})$ is a solution to the reduced Problem* I.

Proof. Let (λ, ν) be an arbitrary pair of measures satisfying (5.17), (5.18) and (5.19). We have to show that $f(\tilde{k}\,\tilde{\lambda}, \tilde{\nu}) \geqq f(\lambda, \nu)$. We shall prove this inequality under the condition that $\lambda(\Omega_H) > 0$ and $\nu(\Omega_K) > 0$. (If $\lambda(\Omega_H) = 0$ or $\nu(\Omega_K) = 0$ then it follows easily from (5.16), ..., (5.19) that $f(\lambda, \nu) \leqq 0$. But on account of (6.1) we have $\sup f(\lambda, \nu) \geqq 0$ and thus $f(\tilde{k}\,\tilde{\lambda}, \tilde{\nu}) \geqq f(\lambda, \nu)$ holds for all (λ, ν).)

Hence suppose $\lambda(\Omega_H) > 0$ and $\nu(\Omega_K) > 0$. Let (λ', ν') denote the corresponding pair of probability measures: $\lambda' = \lambda/\lambda(\Omega_H)$, $\nu' = \nu/\nu(\Omega_K)$ and let φ' be a M.P. size-α

test for Problem $(H_{\lambda'}, K_{\nu'})$. Then (7.11) holds and thus we obtain

$$\begin{aligned}
f(\tilde{k}\,\tilde{\lambda}, \tilde{\nu}) &= -\int_{\mathfrak{X}} \tilde{\varphi}(x)\left\{-\int_H p_\theta(x)\,\tilde{k}\,\mathrm{d}\tilde{\lambda} + \int_K p_\theta(x)\,\mathrm{d}\tilde{\nu}\right\}\mathrm{d}\mu - \alpha\,\tilde{k} + \int_K \beta_D^*(\theta)\,\mathrm{d}\tilde{\nu}\\
&= \tilde{k}\left\{\int_{\mathfrak{X}} \tilde{\varphi}(x)\int_H p_\theta(x)\,\mathrm{d}\tilde{\lambda}\,\mathrm{d}\mu - \alpha\right\} + \int_K \gamma_{\tilde{\varphi},D}(\theta)\,\mathrm{d}\tilde{\nu}\\
&= \int_K \gamma_{\tilde{\varphi},D}(\theta)\,\mathrm{d}\tilde{\nu} \geqq \int_K \gamma_{\varphi',D}(\theta)\,\mathrm{d}\nu' \geqq \int_K \gamma_{\varphi',D}(\theta)\,\mathrm{d}\nu\\
&\geqq \lambda(\Omega_H)\left\{\int_{\mathfrak{X}} \varphi'(x)\int_H p_\theta(x)\,\mathrm{d}\lambda'\,\mathrm{d}\mu - \alpha\right\} + \int_K \gamma_{\varphi',D}(\theta)\,\mathrm{d}\nu\\
&= -\int_{\mathfrak{X}} \varphi'(x)\left\{-\int_H p_\theta(x)\,\mathrm{d}\lambda + \int_K p_\theta(x)\,\mathrm{d}\nu\right\}\mathrm{d}\mu - \alpha\,\lambda(\Omega_H) + \int_K \beta_D^*(\theta)\,\mathrm{d}\nu\\
&\geqq f(\lambda, \nu).
\end{aligned}$$

The Theorems 7.1, 7.2 and 7.3 may be regarded as generalizations of results of Lehmann ([3] p. 327) and Krafft und Witting ([2] Satz 7 and Satz 12) concerning the *maximin size-α criterion.* Their results are obtained by taking $D=\Phi$ so that $\beta_D^*(\theta)=1$ for all $\theta\in\Omega_K$. In this case the pair $(\tilde{\lambda}, \tilde{\nu})$ is a least favorable pair if for each other pair (λ', ν') of probability measures over $(\Omega_H, \mathfrak{B}_H)$ and $(\Omega_K, \mathfrak{B}_K)$ the following inequality holds

$$\int_K \mathrm{E}_\theta(\varphi')\,\mathrm{d}\nu' \geqq \int_K \mathrm{E}_\theta(\tilde{\varphi})\,\mathrm{d}\tilde{\nu} \tag{7.13}$$

where φ' is M.P. size-α for Problem $(H_{\lambda'}, K_{\nu'})$ and $\tilde{\varphi}$ is M.P. size-α for Problem $(H_{\tilde{\lambda}}, K_{\tilde{\nu}})$; $(\tilde{\lambda}, \tilde{\nu})$ is least favorable because the power in $K_{\tilde{\nu}}$ for the corresponding M.P. size-α test $\tilde{\varphi}$ is a small as possible.

8. Characterizing the Solution (φ^*, γ^*) to Problem II by Means of a Least Favorable Probability Measure $\tilde{\nu}$ over $(\Omega_K, \mathfrak{B}_K)$

If (λ^*, ν^*) with $\lambda^*(\Omega_H)>0$ (and consequently $\nu^*(\Omega_K)>0$) is a solution to the reduced Problem I then $(\tilde{\lambda}, \tilde{\nu})$ with $\tilde{\lambda}=\lambda^*/\lambda^*(\Omega_H)$ and $\tilde{\nu}=\nu^*/\nu^*(\Omega_K)$ constitute a least favorable pair of probability measures (Theorem 7.2) and each solution (φ^*, γ^*) to Problem II satisfies $\varphi^*\in\Phi$ and (7.1), ..., (7.6) (Theorem 7.1). Hence each φ^* is M.P. size-α for Problem $(H_{\tilde{\lambda}}, K_{\tilde{\nu}})$ (Corollary 7.2), but on account of (7.3) φ^* is also M.P. size-α for Problem $(H, K_{\tilde{\nu}})$ and $\tilde{\lambda}$ is the least favorable a priori distribution over $(\Omega_H, \mathfrak{B}_H)$ for testing the composite hypothesis H against the simple alternative $K_{\tilde{\nu}}$. In Theorem 8.1 we weaken the conditions of Theorem 7.1; it is not necessary that there exists a least favorable $\tilde{\lambda}$.

Theorem 8.1. *Sufficient for the optimality of* (φ^*, γ^*) *with* $\varphi^*\in\Phi$ *as a solution to Problem* II, *is that there exists a probability measure* $\tilde{\nu}$ *over* $(\Omega_K, \mathfrak{B}_K)$ *such that* φ^* *is* M.P. *size-α for Problem* $(H, K_{\tilde{\nu}})$ *and such that the conditions* (7.5) *and* (7.6) *are satisfied.*

If there exists a solution (λ^*, ν^*) *to the reduced Problem* I *with* $\lambda^*(\Omega_H)>0$ *(and consequently* $\nu^*(\Omega_K)>0$*) then for each solution* (φ^*, γ^*) *to Problem* II *the above-described sufficient conditions are satisfied when* $\tilde{\nu}=\nu^*/\nu^*(\Omega_K)$.

Proof. The second part of the theorem follows from the considerations above. In Theorem 8.3 we shall weaken the conditions used in this second part.

In order to prove the sufficiency of the conditions in the first part we show that $\gamma \geqq \gamma^*$ holds for each (φ, γ) satisfying the feasibility conditions (2.4), ..., (2.8). φ is of size-α for testing H and φ^* is M.P. size-α for Problem $(H, K_{\tilde{\nu}})$. Hence

$$\int_K \mathrm{E}_\theta(\varphi^*)\,\mathrm{d}\tilde{\nu} \geqq \int_K \mathrm{E}_\theta(\varphi)\,\mathrm{d}\tilde{\nu}.$$

But by applying (7.6) we obtain

$$\gamma^* = \int_K \gamma_{\varphi^*, D}(\theta)\,\mathrm{d}\tilde{\nu} \leqq \int_K \gamma_{\varphi, D}(\theta)\,\mathrm{d}\tilde{\nu}$$

and consequently $\gamma_{\varphi, D}(\theta) \geqq \gamma^*$ for some $\theta \in \Omega_K$. By applying (2.8) we obtain $\gamma \geqq \gamma^*$.

Definition. $\tilde{\nu}$ is said to be a least favorable probability measure over $(\Omega_K, \mathfrak{B}_K)$ if for each other probability measure ν' Formula (7.11) holds when φ' is a M.P. size-α test for Problem $(H, K_{\nu'})$ and $\tilde{\varphi}$ is a M.P. size-α test for Problem $(H, K_{\tilde{\nu}})$.

Theorem 8.2. *If there exists a probability measure $\tilde{\nu}$ over $(\Omega_K, \mathfrak{B}_K)$ such that for a* M.P. *size-α test φ^* for Problem $(H, K_{\tilde{\nu}})$ the conditions* (7.5) *and* (7.6) *are satisfied, then $\tilde{\nu}$ is a least favorable probability measure over $(\Omega_K, \mathfrak{B}_K)$.*

Proof. Let ν' be an arbitrary distribution over $(\Omega_K, \mathfrak{B}_K)$ and let φ' be a M.P. size-α test for Problem $(H, K_{\nu'})$. By using (7.6), (7.5) and that φ' is M.P. for $(H, K_{\nu'})$ respectively, we obtain

$$\gamma^* = \int_K \gamma_{\varphi^*, D}(\theta)\,\mathrm{d}\tilde{\nu} \geqq \int_K \gamma_{\varphi^*, D}(\theta)\,\mathrm{d}\nu' \geqq \int_K \gamma_{\varphi', D}(\theta)\,\mathrm{d}\nu'$$

and hence $\tilde{\nu}$ is least favorable.

Theorem 8.3. *If $\tilde{\nu}$ is least favorable over $(\Omega_K, \mathfrak{B}_K)$ then for each solution (φ^*, γ^*) to Problem* II *we have that φ^* is* M.P. *size-α for Problem $(H, K_{\tilde{\nu}})$ and that the conditions* (7.5) *and* (7.6) *are satisfied.*

Proof. Let ν' be an arbitrary probability measure over $(\Omega_K, \mathfrak{B}_K)$. We shall show in Lemma 8.1 that it follows from the Weak Duality Theorem 6.2 that

$$\sup_\lambda f(\lambda, \nu') = \int_K \gamma_{\varphi', D}(\theta)\,\mathrm{d}\nu' \tag{8.1}$$

holds for each of the M.P. size-α tests φ' for Problem $(H, K_{\nu'})$. But $\tilde{\nu}$ is least favorable and hence on account of (7.11) and the Weak Duality Theorem 6.2

$$\begin{aligned}\gamma^* &= \sup_{\lambda, \nu} f(\lambda, \nu) = \max_\nu \sup_\lambda f(\lambda, \nu)\\ &= \sup_\lambda f(\lambda, \tilde{\nu}) = \int_K \gamma_{\tilde{\varphi}, D}(\theta)\,\mathrm{d}\tilde{\nu}\end{aligned}$$

holds for each M.P. size-α test $\tilde{\varphi}$ for Problem $(H, K_{\tilde{\nu}})$.

Hence φ^* is M.P. size-α for Problem $(H, K_{\tilde{\nu}})$ for otherwise

$$\int_K \gamma_{\varphi^*, D}(\theta)\,\mathrm{d}\tilde{\nu} > \gamma^*$$

and $\gamma_{\varphi^*, D}(\theta) > \gamma^*$ for some $\theta \in \Omega_K$ with as a result that (φ^*, γ^*) is not feasible for Problem II. Hence

$$\int_K \gamma_{\varphi^*, D}(\theta)\,\mathrm{d}\tilde{\nu} = \gamma^*.$$

This together with the feasibility of (φ^*, γ^*) shows that (7.5) and (7.6) are satisfied.

Lemma 8.1. *If* φ' *is* M.P. *size-*α *for testing* H *against the simple alternative that the* p.d.f. *is given by* p_{θ_1}, *then*

$$-\mathrm{E}_{\theta_1}(\varphi') = \sup_{\lambda} \left[-\int_{\mathfrak{X}} \max\left\{0, -\int_H p_\theta(x)\,\mathrm{d}\lambda + p_{\theta_1}(x)\right\} \mathrm{d}\mu - \alpha\,\lambda(\Omega_H)\right]. \tag{8.2}$$

Hence if φ' *is* M.P. *size-*α *for Problem* $(H, K_{\nu'})$ *then* (8.1) *holds.*

Proof. One can apply Witting [8] p. 72 together with the Weak Duality Theorem 6.2. We shall apply the theory of the preceding sections. φ' is a maximin size-α test for testing against $\{\theta_1\}$. If $\gamma' = 1 - \mathrm{E}_{\theta_1}(\varphi')$ then (φ', γ') is a solution to the corresponding Problem II where $D = \Phi$ and $\beta_D^*(\theta_1) = 1$. For the corresponding reduced Problem I we may restrict the attention to $\nu(\Omega_K) = 1$ in which case ν is the singular probability measure taking $\{\theta_1\}$ with probability 1. By applying Theorem 6.2 we obtain

$$\sup_{\lambda} \left[-\int_{\mathfrak{X}} \max\left\{0, -\int_H p_\theta(x)\,\mathrm{d}\lambda + p_{\theta_1}(x)\right\} \mathrm{d}\mu - \alpha\,\lambda(\Omega_H)\right] + 1 = \gamma'$$

and we obtain (8.2) as a result. The second part of Lemma 8.1 is a simple consequence which is obtained when $p_{\theta_1}(x) = \int_K p_\theta(x)\,\mathrm{d}\nu'$ so that

$$\sup_{\lambda} f(\lambda, \nu') = \gamma' - 1 + \int_K \beta_D^*(\theta)\,\mathrm{d}\nu' = \int_K \gamma_{\varphi', D}(\theta)\,\mathrm{d}\nu'.$$

We have seen that if $(\tilde{\lambda}, \tilde{\nu})$ is a least favorable pair of probability measures in the sense of Section 7, then $\tilde{\nu}$ is least favorable in the sense of Section 8. Accordingly the results of this section can be applied to more problems. We can weaken the conditions of Lemma 6.1 for example.

Theorem 8.4. *If* Ω_K *is finite then there exists a least favorable probability measure over* $(\Omega_K, \mathfrak{B}_K)$.

This is a new formulation of Characterization Theorem 2 in [6].

Remark. The criterion most stringent size-α was introduced by Wald. In "Tests of statistical hypotheses concerning several parameters when the number of observations is large" (1943) (see [7]), he constructed most stringent size-α tests for problems with linear hypotheses concerning the mean vector of a multivariate normal distribution with known covariance matrix. He showed that for certain problems the likelihood-ratio test is "asymptotically" most stringent size-α. Theorem 8.1 provides an abstract formulation of the arguments applied by Wald and in [6] Section 3. Similar arguments were applied by van Zwet and Oosterhoff [9]. The testing problems with restricted alternative of [6] and [9] are interesting because different criterions provide different optimum solutions. One can compare the most stringent size-α, the M.S.S.M.P. size-α and the likelihood-ratio size-α tests for such problems (see [6] and [9]).

It will be clear from the preceding sections that the results obtained in these sections have been made possible by the Krafft-Witting paper [2] which applies the linear programming method to the construction of maximin tests and indicates the lines along which the Weak Duality Theorem can be proved.

References

1. Karlin, S.: Mathematical methods and theory in games, programming and economics. London: Pergamon 1959.
2. Krafft, O., u. H. Witting: Optimale Tests und ungünstigste Verteilungen. Z. Wahrscheinlichkeitstheorie verw. Geb. **7**, 289–302 (1967).
3. Lehmann, E.: Testing statistical hypotheses. New York: Wiley 1959.
4. Nölle, G., u. D. Plachky: Zur schwachen Folgenkompaktheit von Testfunktionen. Z. Wahrscheinlichkeitstheorie verw. Geb. **8**, 182–184 (1967).
5. Schaafsma, W.: Hypothesis testing problems with the alternative restricted by a number of inequalities. Groningen: Noordhoff 1966.
6. – A comparison of the most stringent and the most stringent somewhere most powerful test for certain problems with restricted alternative. Ann. math. Statistics **39**, 531–546 (1968).
7. Wald, A.: Selected papers in probability and statistics. New York: McGraw-Hill 1955.
8. Witting, H.: Mathematische Statistik. Eine Einführung in Theorie und Methoden. Stuttgart: Teubner 1966.
9. Zwet, W. R. van, and J. Oosterhoff: On the combination of independent test statistics. Ann. math. Statistics **38**, 659–680 (1967).

Dr. Willem Schaafsma
Mathematisch Instituut
der R. U. te Groningen
Reitdiepskade 4
Groningen, The Netherlands

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