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A Note on Deterministic Equivalents to Stochastic Linear Programming Problems

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Summary. Generalized inverse matrices are used as a tool for a study of two-stage linear program under uncertainty. For a special choice of **M** which represents an extension of the so-called complete problem, a deterministic equivalent is given in the explicite form.

1. We shall deal with the two-stage stochastic linear program (resp. linear program under uncertainty) in its standard form (see e.g. [2, 4])

$$\text{minimize} \quad f(\mathbf{x}) = \mathscr{E}\{\mathbf{c}'\mathbf{x} + \varphi(\mathbf{x}, \mathbf{A}, \mathbf{b})\} \tag{1}$$

subject to

$$\mathbf{A}_1\mathbf{x} = \mathbf{b}_1, \qquad \mathbf{x} \geqq 0, \tag{2}$$

where $\varphi(\mathbf{x}, \mathbf{A}, \mathbf{b})$, for fixed $\mathbf{x}, \mathbf{A}, \mathbf{b}$, denotes the minimal value of

$$\mathbf{q}'\mathbf{y} \tag{3}$$

subject to

$$\mathbf{M}\mathbf{y} = \mathbf{b} - \mathbf{A}\mathbf{x}, \qquad \mathbf{y} \geqq 0. \tag{4}$$

The elements of matrices $\mathbf{A}_1(m_1, n)$, $\mathbf{M}(m, p)$, $\mathbf{b}_1(m_1, 1)$, $\mathbf{q}(p, 1)$ are given constants whereas the elements of $\mathbf{A}(m, n)$ and $\mathbf{b}(m, 1)$ are random variables with a known joint distribution and elements of $\mathbf{c}(n, 1)$ are random variables with known finite mean values.

Usually, the problem (1), (2) is investigated under following assumptions:

(i) the set $\mathfrak{M} = \{\mathbf{x}: \mathbf{x} \geqq 0, \mathbf{A}_1\mathbf{x} = \mathbf{b}_1\}$ is non-empty and bounded,

(ii) the set $\{\mathbf{y}: \mathbf{y} \geqq 0, \mathbf{M}\mathbf{y} = \mathbf{A}\mathbf{x} - \mathbf{b}\}$ is non-empty for all (finite) realizations of $\mathbf{A}, \mathbf{b}$ and for all $\mathbf{x} \in \mathfrak{M}$,

(iii) the function $\varphi(\mathbf{x}, \mathbf{A}, \mathbf{b})$ is defined and finite for all realizations of $\mathbf{A}, \mathbf{b}$ and for all $\mathbf{x} \in \mathfrak{M}$.

Using the generalized inverse $\mathbf{M}^*$ we can write $\varphi(\mathbf{x}, \mathbf{A}, \mathbf{b})$ (see Charnes, Cooper, Thompson [1], Kall [3]) as

$$\varphi(\mathbf{x}, \mathbf{A}, \mathbf{b}) = \max \mathbf{w}'\mathbf{M}^*(\mathbf{A}\mathbf{x} - \mathbf{b}) + \mathbf{q}'\mathbf{M}^*(\mathbf{b} - \mathbf{A}\mathbf{x}) \tag{5}$$

subject to

$$(\mathbf{E} - \mathbf{M}^*\mathbf{M})\mathbf{w} = (\mathbf{E} - \mathbf{M}^*\mathbf{M})\mathbf{q}, \qquad \mathbf{w} \geqq 0. \tag{6}$$

The constraints (6) are deterministic and do not involve $\mathbf{x}$; the constrained maximum of $\mathbf{w}'\mathbf{M}^*(\mathbf{A}\mathbf{x} - \mathbf{b})$ can be found as the unconstrained $\max_{1 \leqq s \leqq S} \mathbf{w}_s'\mathbf{M}^*(\mathbf{A}\mathbf{x} - \mathbf{b})$, where $\mathbf{w}_s$, $s = 1, \ldots, S$, are extreme points of the set $\{\mathbf{w}: \mathbf{w} \geqq 0, (\mathbf{E} - \mathbf{M}^*\mathbf{M})\mathbf{w} = (\mathbf{E} - \mathbf{M}^*\mathbf{M})\mathbf{q}\}$.

Further, using the known property of generalized inverses, $\mathbf{M}^*\mathbf{M}\mathbf{M}^*=\mathbf{M}^*$, and expressing any vector $\mathbf{v}$ as $\mathbf{v}=\mathbf{M}^*\mathbf{M}\mathbf{v}+(\mathbf{E}-\mathbf{M}^*\mathbf{M})\mathbf{v}$, the problem becomes: Find

$$\varphi(\mathbf{x}, \mathbf{A}, \mathbf{b})=\max(\mathbf{M}^*\mathbf{M}\mathbf{w})'\mathbf{M}^*(\mathbf{A}\mathbf{x}-\mathbf{b})+(\mathbf{M}^*\mathbf{M}\mathbf{q})'\mathbf{M}^*(\mathbf{b}-\mathbf{A}\mathbf{x}) \tag{7}$$

subject to

$$\mathbf{M}^*\mathbf{M}\mathbf{w}\geqq-(\mathbf{E}-\mathbf{M}^*\mathbf{M})\mathbf{q}. \tag{8}$$

We shall give a sufficient condition for assumption (iii) to hold. (For a necessary condition see Kall [2], Theorem 5.)

Theorem 1. *Let the set* $\{\mathbf{y}: \mathbf{y}\geqq 0, \mathbf{M}\mathbf{y}=\mathbf{z}\}$ *be non-empty for all* $\mathbf{z}\in E_m$ *and let* $(\mathbf{E}-\mathbf{M}^*\mathbf{M})\mathbf{q}\geqq 0$. *Then assumption* (iii) *holds.*

Proof. The assumption $\{\mathbf{y}: \mathbf{y}\geqq 0, \mathbf{M}\mathbf{y}=\mathbf{z}\}\neq\emptyset$ for all $\mathbf{z}\in E_m$ implies the existence of a vector $\mathbf{t}$ such that $\mathbf{t}\neq 0$, $\mathbf{t}\geqq 0$, $\mathbf{M}\mathbf{t}=0$ (see Kall [2], Theorem 3).

The function $\mathbf{q}'\mathbf{y}$ attains its minimum subject to (4) if $\mathbf{q}'\mathbf{t}\geqq 0$ holds for all $\mathbf{t}\neq 0$, $\mathbf{t}\geqq 0$, satisfying $\mathbf{M}\mathbf{t}=0$. Using the symmetry of $\mathbf{M}^*\mathbf{M}$, we get for all $\mathbf{t}$ possessing the mentioned properties that

$$\mathbf{q}'\mathbf{t}=\mathbf{q}'\mathbf{M}^*\mathbf{M}\mathbf{t}+[(\mathbf{E}-\mathbf{M}^*\mathbf{M})\mathbf{q}]'\mathbf{t}=[(\mathbf{E}-\mathbf{M}^*\mathbf{M})\mathbf{q}]'\mathbf{t}\geqq 0$$

according to the assumption $(\mathbf{E}-\mathbf{M}^*\mathbf{M})\mathbf{q}\geqq 0$.

Corollary. *Under the assumptions of Theorem* 1, *the set*

$$\{\mathbf{w}:\ \mathbf{M}^*\mathbf{M}\mathbf{w}\geqq-(\mathbf{E}-\mathbf{M}^*\mathbf{M})\mathbf{q}\}$$

is non-empty.

2. For a special choice of $\mathbf{M}$, the explicite form $\varphi(\mathbf{x}, \mathbf{A}, \mathbf{b})$ will be given. That choice of $\mathbf{M}$ represents an extension of the so-called *complete problem* (see Wets [4]).

Let $\mathbf{M}=(\mathbf{P}\,\vdots\,-\mathbf{P})$, where $\mathbf{P}$ is a (m, p) matrix of rank m. In this case, the set $\{\mathbf{y}: \mathbf{M}\mathbf{y}=\mathbf{z}, \mathbf{y}\geqq 0\}$ is non-empty for arbitrary $\mathbf{z}\in E_m$ if and only if the set $\{\mathbf{u}: \mathbf{P}\mathbf{u}=\mathbf{z}\}$ is non-empty for arbitrary $\mathbf{z}\in E_m$; but the latter assertion is entailed by the assumption $h(\mathbf{P})=m$.

We get

$$\mathbf{M}^*=\mathbf{M}'(\mathbf{M}\mathbf{M}')^{-1}=\tfrac{1}{2}\begin{pmatrix}\mathbf{P}'(\mathbf{P}\mathbf{P}')^{-1}\\ -\mathbf{P}'(\mathbf{P}\mathbf{P}')^{-1}\end{pmatrix}=\tfrac{1}{2}\begin{pmatrix}\mathbf{P}^*\\ -\mathbf{P}^*\end{pmatrix},$$

$$\mathbf{M}^*(\mathbf{A}\mathbf{x}-\mathbf{b})=\tfrac{1}{2}\begin{pmatrix}\mathbf{P}^*(\mathbf{A}\mathbf{x}-\mathbf{b})\\ -\mathbf{P}^*(\mathbf{A}\mathbf{x}-\mathbf{b})\end{pmatrix},$$

$$\mathbf{M}^*\mathbf{M}=\tfrac{1}{2}\begin{pmatrix}\mathbf{P}^*\mathbf{P} & -\mathbf{P}^*\mathbf{P}\\ -\mathbf{P}^*\mathbf{P} & \mathbf{P}^*\mathbf{P}\end{pmatrix},$$

$$\mathbf{M}^*\mathbf{M}\mathbf{w}=\tfrac{1}{2}\begin{pmatrix}\mathbf{P}^*\mathbf{P}\mathbf{w}_1-\mathbf{P}^*\mathbf{P}\mathbf{w}_2\\ -\mathbf{P}^*\mathbf{P}\mathbf{w}_1+\mathbf{P}^*\mathbf{P}\mathbf{w}_2\end{pmatrix}, \qquad \text{where } \mathbf{w}=\begin{pmatrix}\mathbf{w}_1\\ \mathbf{w}_2\end{pmatrix},\ \mathbf{w}_1, \mathbf{w}_2\in E_p.$$

Denoting $\tfrac{1}{2}(\mathbf{P}^*\mathbf{P}\mathbf{w}_1-\mathbf{P}^*\mathbf{P}\mathbf{w}_2)=\mathbf{w}_0$ we have $\mathbf{M}^*\mathbf{M}\mathbf{w}=\begin{pmatrix}\mathbf{w}_0\\ -\mathbf{w}_0\end{pmatrix}$ and condition (8) becomes

$$-\tilde{\mathbf{q}}_1\leqq\mathbf{w}_0\leqq\tilde{\mathbf{q}}_2 \tag{9}$$

where

$$\tilde{\mathbf{q}} = (\mathbf{E} - \mathbf{M}^* \mathbf{M})\,\mathbf{q} = \begin{pmatrix} \tilde{\mathbf{q}}_1 \\ \tilde{\mathbf{q}}_2 \end{pmatrix}, \qquad \tilde{\mathbf{q}}_1, \tilde{\mathbf{q}}_2 \in E_p.$$

The problem is to find

$$\begin{aligned} \varphi(\mathbf{x}, \mathbf{A}, \mathbf{b}) &= \max (\mathbf{M}^* \mathbf{M}\,\mathbf{w})' \mathbf{M}^* (\mathbf{A}\,\mathbf{x} - \mathbf{b}) + (\mathbf{M}^* \mathbf{M}\,\mathbf{q})' \mathbf{M}^* (\mathbf{b} - \mathbf{A}\,\mathbf{x}) \\ &= \max \mathbf{w}_0' \,\mathbf{P}^* (\mathbf{A}\,\mathbf{x} - \mathbf{b}) + \mathbf{q}_0' \,\mathbf{P}^* (\mathbf{b} - \mathbf{A}\,\mathbf{x}) \end{aligned}$$

subject to (9); here

$$\mathbf{q} = \begin{pmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{pmatrix}, \qquad \mathbf{q}_1, \mathbf{q}_2 \in E_p,$$

$$\mathbf{q}_0 = \tfrac{1}{2}(\mathbf{P}^* \mathbf{P}\,\mathbf{q}_1 - \mathbf{P}^* \mathbf{P}\,\mathbf{q}_2) \quad \text{and} \quad \mathbf{M}^* \mathbf{M}\,\mathbf{q} = \begin{pmatrix} \mathbf{q}_0 \\ -\mathbf{q}_0 \end{pmatrix}.$$

Denote $\mathbf{P}_i^*$ the i-th row of $\mathbf{P}^*$ and $z_i = \mathbf{P}_i^* (\mathbf{A}\,\mathbf{x} - \mathbf{b})$. Now, the announced result is given by

Theorem 2. *Let* $\mathbf{M} = (\mathbf{P} \,\vdots\, -\mathbf{P})$, *where* $\mathbf{P}$ *is a* (m, p) *matrix of rank* m, *let* $q_i + q_{p+i} \geqq 0$, $i = 1, \ldots, p$. *Then*

$$\begin{aligned} \varphi(\mathbf{x}, \mathbf{A}, \mathbf{b}) &= \sum_{i=1}^{p} q_{p+i} [\mathbf{P}_i^* (\mathbf{A}\,\mathbf{x} - \mathbf{b})]^+ + \sum_{i=1}^{p} q_i [\mathbf{P}_i^* (\mathbf{A}\,\mathbf{x} - \mathbf{b})]^- \\ &= \sum_{i=1}^{p} q_{p+i}\, z_i^+ + \sum_{i=1}^{p} q_i\, z_i^- \end{aligned} \tag{10}$$

which is a convex separable function in $z_1, \ldots, z_p$.

Proof. The condition $q_i + q_{p+i} \geqq 0$, $i = 1, \ldots, p$, (resp. $\mathbf{q}_1 + \mathbf{q}_2 \geqq 0$) is necessary and sufficient for the set $\{\mathbf{w}_0 \in E_p\colon -\tilde{\mathbf{q}}_1 \leqq \mathbf{w}_0 \leqq \tilde{\mathbf{q}}_2\}$ be non-empty. The maximum of $\mathbf{w}_0' \mathbf{P}^* (\mathbf{A}\,\mathbf{x} - \mathbf{b})$ is attained when the i-th component of the vector $\mathbf{w}_0$ equals to $\tilde{q}_{p+i}$ for $\mathbf{P}_i^* (\mathbf{A}\,\mathbf{x} - \mathbf{b}) \geqq 0$ and equals to $-\tilde{q}_i$ for $\mathbf{P}_i^* (\mathbf{A}\,\mathbf{x} - \mathbf{b}) \leqq 0$, $i = 1, 2, \ldots, p$, what gives the desired form of $\varphi(\mathbf{x}, \mathbf{A}, \mathbf{b})$. The condition $q_i = q_{p+i} \geqq 0$, $i = 1, \ldots, p$, secures the convexity of $\varphi(\mathbf{x}, \mathbf{A}, \mathbf{b})$ with respect to $\mathbf{x}$, too.

Especially, for the *complete problem* with $\mathbf{M} = (\mathbf{E} \,\vdots\, -\mathbf{E})$ ($\mathbf{E}$ is the identity matrix), we have

$$\mathbf{M}^* = \tfrac{1}{2} \begin{pmatrix} \mathbf{E} \\ -\mathbf{E} \end{pmatrix}, \quad \mathbf{M}^* \mathbf{M} = \tfrac{1}{2} \begin{pmatrix} \mathbf{E} & -\mathbf{E} \\ -\mathbf{E} & \mathbf{E} \end{pmatrix}, \quad (\mathbf{E} - \mathbf{M}^* \mathbf{M})\,\mathbf{q} = \begin{pmatrix} \tilde{\mathbf{q}}_1 \\ \tilde{\mathbf{q}}_1 \end{pmatrix}$$

where the i-th component of the vector $\tilde{\mathbf{q}}_1$ equals to $\tfrac{1}{2}(q_i + q_{m+i})$ and for arbitrary $\mathbf{v} \in E_{2m}$ we have $\mathbf{M}^* \mathbf{M}\,\mathbf{v} = \begin{pmatrix} \mathbf{v}_0 \\ -\mathbf{v}_0 \end{pmatrix}$ where the i-th component of the vector $\mathbf{v}_0$ equals to $\tfrac{1}{2}(v_i - v_{m+i})$.

Now, the assumption $\tilde{\mathbf{q}} = (\mathbf{E} - \mathbf{M}^* \mathbf{M})\,\mathbf{q} \geqq 0$ of Theorem 1 is precisely the familiar assumption $q_i + q_{m+i} \geqq 0$, $i = 1, \ldots, m$, and it is both necessary and sufficient for (iii). Further,

$$\varphi(\mathbf{x}, \mathbf{A}, \mathbf{b}) = \max \tfrac{1}{2} \sum_{i=1}^{m} (w_i - w_{m+i}) \left(\sum_{j=1}^{n} a_{ij} x_j - b_i \right)$$
$$- \tfrac{1}{2} \sum_{i=1}^{m} (q_i - q_{m+i}) \left(\sum_{j=1}^{n} a_{ij} x_j - b_i \right)$$

subject to

$$-(q_i + q_{m+i}) \leqq w_i - w_{m+i} \leqq q_i + q_{m+i}, \qquad i = 1, \ldots, m.$$

Using (10), we get known result

$$\varphi(\mathbf{x}, \mathbf{A}, \mathbf{b}) = \sum_{i=1}^{m} q_{m+i} \left(\sum_{j=1}^{n} a_{ij} x_j - b_i \right)^{+} + \sum_{i=1}^{m} q_i \left(\sum_{i=1}^{n} a_{ij} x_j - b_i \right)^{-}.$$

The results remain true for matrices $\mathbf{M}$ obtained from $(\mathbf{P} \vdots -\mathbf{P})$ by permutation of columns and by multiplication of columns by possibly different scalars. Especially, the following assertion holds:

Theorem 3. *Let* $\mathbf{M} = (\mathbf{P} \vdots -\mathbf{P}\mathbf{D})$ *where* $\mathbf{P}$ *is a* (m, p) *matrix of rank* m *and* $\mathbf{D}$ *is a diagonal matrix with positive diagonal elements* $d_1, \ldots, d_p$, *let*

$$q_i + \frac{q_{p+i}}{d_i} \geqq 0, \qquad i = 1, \ldots, p.$$

Then

$$\varphi(\mathbf{x}, \mathbf{A}, \mathbf{b}) = \sum_{i=1}^{p} (1 + d_i^2) q_{p+i} [\mathbf{R}_i (\mathbf{A}\mathbf{x} - \mathbf{b})]^{+} + \sum_{i=1}^{p} (1 + d_i^2) q_i [\mathbf{R}_i (\mathbf{A}\mathbf{x} - \mathbf{b})]^{-},$$

where $\mathbf{R}_i$*'s are the rows of the matrix* $\mathbf{R} = \mathbf{P}' [\mathbf{P}(\mathbf{E} + \mathbf{D}^2)\mathbf{P}']^{-1}$.

Proof. The proof is similar to that of Theorem 2. Namely

$$\mathbf{M}^* \mathbf{M} = \begin{pmatrix} \mathbf{RP} & -\mathbf{RPD} \\ -\mathbf{DRP} & \mathbf{DRPD} \end{pmatrix},$$

and for an arbitrary vector

$$\mathbf{v} = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix}, \qquad \mathbf{v}_1, \mathbf{v}_2 \in E_p,$$

we have

$$\mathbf{M}^* \mathbf{M}\, \mathbf{v} = \begin{pmatrix} \mathbf{v}_0 \\ -\mathbf{D}\,\mathbf{v}_0 \end{pmatrix} \qquad \text{with } \mathbf{v}_0 = \mathbf{R}\,\mathbf{P}(\mathbf{v}_1 - \mathbf{D}\,\mathbf{v}_2);$$

the condition (8) becomes

$$\mathbf{w}_0 \geqq -\tilde{\mathbf{q}}_1 (= -\mathbf{q}_1 + \mathbf{q}_0)$$
$$\mathbf{D}\,\mathbf{w}_0 \leqq \tilde{\mathbf{q}}_2 (= \mathbf{q}_2 + \mathbf{D}\,\mathbf{q}_0).$$

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