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Invariance Principles for Rank Discounted Partial Sums and Averages*

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Summary. For rank-discounted partial sums and averages, forward and backward invariance principles are established through the use of the Bahadur-Kiefer representation of sample quantiles and the Kiefer process approximation of the sample distributions.

1. Introduction

Let $\{X_i, i \geqq 1\}$ be a sequence of independent and identically distributed random variables (i.i.d.r.v.) with a continuous distribution function (df) F, defined on the real line $(-\infty, \infty)$. For every $n(\geqq 1)$, let R_{ni} be the rank of X_i among $X_1, \ldots, X_n$ for $i = 1, \ldots, n$; by virtue of the assumed continuity of F, ties among the X_i may be neglected, in probability, so that $\underset{\sim}{R}_n = (R_{n1}, \ldots, R_{nn})$ is some (random) permutation of $(1, \ldots, n)$. We conceive of a triangular array $\{a_n(i), 1 \leqq i \leqq n; n \geqq 1\}$ of real scores and define a rank-discounted (partial) sum by

$$T_n = \sum_{k=1}^{n} a_k(R_{kk}) g(X_k), \qquad n \geqq 1; \; T_0 = 0, \tag{1.1}$$

where $g(.)$ is a suitable function. Basically, at each stage k, the r.v. $g(X_k)$ is discounted by the factor $a_k(R_{kk})$ depending on the rank (R_{kk}) of X_k among $X_1, \ldots, X_k$, $k \geqq 1$, and the ranking is made sequentially.

Note that if the discounting factor $a_n(i)$ does not depend on i, viz.,

$$a_n(i) = a_n \qquad \text{for every } i = 1, \ldots, n; \; n \geqq 1, \tag{1.2}$$

then T_n reduces to $T_n^o = \sum_{k=1}^{n} a_k g(X_k)$ and involves independent summands so that the classical invariance principles hold under very general conditions on g and $\{a_n\}$.

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Secondly, if $g(x)=g(+0)$, except perhaps on a set of measure 0, then T_n reduces to $\tilde{T}_n = g\left(\sum_{k=1}^{n} a_k(R_{kk})\right)$ almost everywhere (a.e.), where by Lemma 3.2 (to follow), R_{jj}, $j \geqq 1$ are all independent, so that $\tilde{T}_n$ again involves independent summands and the classical invariance principles hold under very general conditions on the scores $\{a_n(i)\}$. Here, we are primarily concerned with the non-degenerate case where neither (1.2) hold nor $g(x)$ is a constant a.e. It may be noted that R_{kk} is independent of R_{jj}, $j<k$ and also $g(X_k)$ is independent of $g(X_j)$, $j<k$, but $[g(X_k), R_{kk}]$ is not independent of $([g(X_j), R_{jj}], j<k)$, and hence, in general, T_n does not involve independent summands. In view of the nature of the dependence of the R_{kk}, $g(X_k)$, $k \geqq 1$, invariance principles for martingales (or related sequences) are difficult to apply here. Let $Z_{k,1} < \cdots < Z_{k,k}$ be the ordered r.v. corresponding to $X_1, \ldots, X_k$, for $k \geqq 1$. Then, T_n may be expressed in either of the following two equivalent forms:

$$T_n = \sum_{k=1}^{n} a_k(R_{kk})\, g(Z_{k,R_{kk}}) \quad \text{or} \quad \sum_{k=1}^{n} a_k(R_{kk})\, g(Z_{n,R_{nk}}), \qquad n \geqq 1. \tag{1.3}$$

Though there is an one-to-one corresponding between $\underset{\sim}{R}_n$ and $\underset{\sim}{R}_n^* = (R_{11}, R_{22}, \ldots, R_{nn})$, for $n \geqq 2$, they are defined on different spaces and are not identical. Thus, in either form, T_n is different from a conventional linear combination of functions of order statistics, and the invariance principles for the latter [viz., Sen (1978)] may not yield the desired results for $\{T_n\}$. These call for a somewhat different approach to the study of invariance principles for the partial sequence $\{T_k, k \leqq n\}$ or the tail sequence $\{T_k, k \geqq n\}$.

Our task is accomplished here by some particular decomposition of T_n and by an appeal to the Bahadur-Kiefer representation of sample quantiles and the Kiefer-process approximation for the empirical distributions. Along with the preliminary notions, the main results are stated in Section 2. Section 3 is devoted to the proofs of the main theorems. The concluding Section deals with some general remarks, including the scope of this approach in some related problems.

2. The Main Theorems

We conceive of a *score-function* $\phi = \{\phi(u), 0<u<1\}$ and define the *scores* as

$$a_n(i) = \phi\left(\frac{i}{n+1}\right) \quad \text{or} \quad E\phi(U_{n,i}), \qquad i=1, \ldots, n;\ n \geqq 1, \tag{2.1}$$

where $U_{n,1} < \cdots < U_{n,n}$ are the ordered rv's of a sample of size n from the rectangular (0, 1) df. Also, note that $U_i = F(X_i)$, $i \geqq 1$ are i.i.d.rv with the rectangular (0, 1) df, and we may write

$$g(X_i) = g(F^{-1}(U_i)) = h(U_i), \qquad i \geqq 1. \tag{2.2}$$

We assume that ϕ is absolutely continuous inside $I=[0, 1]$ and h has a continuous first derivative $h^{(1)}$ inside I. Denoting $h(u)$ by $h^{(0)}(u)$, $0<u<1$, we assume that there exist non-negative numbers α, β and positive constants δ and $K(<\infty)$, such that

$$0 \leqq \alpha+\beta=\tfrac{1}{2}-\delta, \qquad \delta>0; \tag{2.3}$$

$$|h^{(r)}(u)| \leqq K[u(1-u)]^{-\alpha-r}, \qquad 0<u<1, \; r=0,1; \tag{2.4}$$

$$|\phi(u)| \leqq K[u(1-u)]^{-\beta}, \qquad 0<u<1. \tag{2.5}$$

For every $n(\geqq 1)$, let us define

$$\mu_{n,i}=Eh(U_{n,i}), \qquad 1 \leqq i \leqq n, \tag{2.6}$$

$$h_n=\left\{h_n(u)=\mu_{n,i}, \frac{i-1}{n}<u \leqq \frac{i}{n}, \; 1 \leqq i \leqq n\right\}, \tag{2.7}$$

$$\phi_n=\left\{\phi_n(u)=a_n(i), \frac{i-1}{n}<u \leqq \frac{i}{n}, \; 1 \leqq i \leqq n\right\}. \tag{2.8}$$

Also, let

$$\mu=\int_0^1 h(u)\,du=\int_0^1 h_n(u)\,du=n^{-1}\sum_{i=1}^n \mu_{n,i}, \tag{2.9}$$

$$\sigma_h^2=\int_0^1 h^2(u)\,du-\mu^2, \qquad \sigma_{h_n}^2=\frac{1}{n}\sum_{i=1}^n \mu_{n,i}^2-\mu^2, \tag{2.10}$$

$$\tau=\int_0^1 h(u)\,\phi(u)\,du, \qquad \sigma_1^2=\int_0^1 h^2(u)\,\phi^2(u)\,du-\tau^2, \tag{2.11}$$

$$\bar{\phi}=\int_0^1 \phi(u)\,du, \qquad \gamma(u)=[\phi(u)-\bar{\phi}]\,h^{(1)}(u), \qquad 0<u<1, \tag{2.12}$$

$$\sigma_2^2=4\iint_{0<u<v<1} \gamma(u)\,\gamma(v)\,u(1-v)\,du\,dv, \tag{2.13}$$

$$\sigma^2=\sigma_1^2+\sigma_2^2. \tag{2.14}$$

By virtue of (2.3)–(2.5), it can be shown that

$$\lim_{n\to\infty}\int_0^1 \{\phi_n(u)-\phi(u)\}^2\,du=0, \tag{2.15}$$

$$\lim_{n\to\infty}\int_0^1 \{h_n(u)-h(u)\}^2\,du=0, \tag{2.16}$$

$$\lim_{n\to\infty}\int_0^1 h_n^a(u)\,\phi_n^b(u)\,du=\int_0^1 h^a(u)\,\phi^b(u)\,du, \qquad a\geqq 0, \; b\geqq 0, \; a+b\leqq 2. \tag{2.17}$$

Further, let

$$\tau_n=\int_0^1 h_n(u)\,\phi_n(u)\,du=\frac{1}{n}\sum_{i=1}^n a_n(i)\,\mu_{n,i}, \qquad n\geqq 1, \tag{2.18}$$

$$\tau_n^* = \sum_{k=1}^{n} \tau_k, \quad n \geqq 1 \quad \text{and} \quad \tau_0 = \tau_0^* = 0. \tag{2.19}$$

Then, we note that

$$ET_n = \tau_n^*, \quad \forall\, n \geqq 0. \tag{2.20}$$

First, we consider the forward invariance principle, and for every $n(\geqq 1)$, we introduce a stochastic process $W_n = \{W_n(t), t \in I\}$ by letting

$$W_n(t) = \sigma^{-1} n^{-\frac{1}{2}} \{T_{k_n(t)} - \tau^*_{k_n(t)}\}, \quad t \in I; \tag{2.21}$$

$$k_n(t) = \max\{k: k/n \leqq t\}, \quad t \in I. \tag{2.22}$$

Then, W_n belongs to the $D[0,1]$ space endowed with the Skorokhod J_1-topology. Further, let $W = \{W(t), t \in I\}$ be a standard Wiener process on I.

Theorem 1. *Under* (2.1) *through* (2.5),

$$W_n \xrightarrow{\mathscr{D}} W, \quad \textit{in the } J_1\textit{-topology on } D[0,1]. \tag{2.23}$$

Let us next consider a backward invariance principle, and define

$$S_n = n^{-1} T_n, \quad \bar{\tau}_n = n^{-1} \tau_n^*, \quad n \geqq 1. \tag{2.24}$$

Then, we introduce another stochastic process $W_n^* = \{W_n^*(t), t \in I\}$ by letting

$$W_n^*(t) = \sigma^{-1} n^{\frac{1}{2}} \{S_{k_n^*(t)} - \bar{\tau}_{k_n^*(t)}\}, \quad t \in I, \tag{2.25}$$

$$k_n^*(t) = \min\{k: n/k \leqq t\}, \quad t \in I. \tag{2.26}$$

Thus, W_n^* is constructed from the tail-sequence $\{n^{\frac{1}{2}}(S_k - \bar{\tau}_k), k \geqq n\}$.

Theorem 2. *Under* (2.1) *through* (2.5),

$$W_n^* \xrightarrow{\mathscr{D}} W, \quad \textit{in the } J_1\textit{-topology on } D[0,1]. \tag{2.27}$$

It may be remarked that in the so called degenerate case where either $\phi(u)$ or $h(u)$ is a constant on I, $\gamma(u) \equiv 0$, and hence, $\sigma_2^2 = 0$. Thus, in such a case, σ^2 may be replaced by σ_1^2.

3. Proofs of the Theorems

The following basic decomposition is considered

$$\begin{aligned} T_n - \tau_n^* &= \sum_{k=1}^{n} [a_k(R_{kk})\,\mu_{k,R_{kk}} - \tau_k] + \sum_{k=1}^{n} \bar{a}_k [h(U_k) - \mu_{k,R_k}] \\ &\quad + \sum_{k=1}^{n} [a_k(R_{kk}) - \bar{a}_k][h(U_k) - \mu_{k,R_{kk}}] \\ &= B_{1n} + B_{2n} + B_{3n}, \quad \text{say, } n \geqq 1, \end{aligned} \tag{3.1}$$

where

$$\bar{a}_n = \frac{1}{n} \sum_{i=1}^{n} a_n(i) = \int_0^1 \phi_n(u)\,du, \qquad n \geqq 1. \tag{3.2}$$

Before proceeding to the derivation of (2.23) and (2.27), we consider the following.

Lemma 3.1. *R_{kk}, $k \geqq 1$ are all stochastically independent and $P\{R_{kk}=s\}=k^{-1}$, for $1 \leqq s \leqq k$, $k \geqq 1$.*

Proof. Since the X_i are i.i.d.rv,

$$P\{R_{k+1\,k+1}=s \mid R_k^*\} = P\{Z_{k,s-1} < X_k < Z_{k,s}\} = 1/(k+1), \qquad \forall\ 1 \leqq s \leqq k+1$$

(where $Z_{k,0} = -\infty$ and $Z_{k,k+1} = +\infty$). Q.E.D.

Lemma 3.2. *$\underset{\sim}{R}_n$ and $\underset{\sim}{Z}_n = (Z_{n,1}, \ldots, Z_{n,n})$ are stochastically independent.*

Proof. The conditional distribution of $(X_1, \ldots, X_n)$ given $\underset{\sim}{Z}_n$ is uniform over the $n!$ permutations of the coordinates of $\underset{\sim}{Z}_n$ yielding a uniform distribution of R_n (given Z_n) over the $n!$ permutations of $(1, \ldots, n)$. Since the unconditional df of R_n is also uniform over the $n!$ permutations of $(1, \ldots, n)$, the lemma follows.

Lemma 3.3. *$n^{-\frac{1}{2}}\{\max_{1 \leqq k \leqq n} |B_{2k}|\} \xrightarrow{p} 0$, as $n \to \infty$.*

Proof. Let us define $L_0 = L_0^* = 0$ and for $n \geqq 1$,

$$\begin{aligned} L_n &= \sum_{k=1}^{n} \bar{a}_k [h(U_k) - \mu], \qquad L_n^* = \sum_{k=1}^{n} \bar{a}_k [\mu_{k,R_{kk}} - \mu], \\ \bar{L}_n &= n^{-1} L_n \quad \text{and} \quad \bar{L}_n^* = n^{-1} L_n^*. \end{aligned} \tag{3.3}$$

Then, $L_n^* = E(L_n \mid R_n^*)$, $\forall\, n \geqq 1$, so that for every $m \leqq n$,

$$\begin{aligned} n^{-1} E(L_m - L_m^*)^2 &= n^{-1}\{E(L_m^2) - E(L_m^{*2})\} \\ &= n^{-1} \sum_{k=1}^{m} \bar{a}_k^2 \left\{ \int_0^1 (h(u) - h_k(u))^2\,du \right\} \to 0 \qquad \text{as } n \to \infty, \end{aligned} \tag{3.4}$$

by (2.1) and (2.17). Hence, by the Čebyšev inequality, as $n \to \infty$,

$$n^{-\frac{1}{2}} |L_{[nt]} - L_{[nt]}^*| = n^{-\frac{1}{2}} |B_{2[nt]}| \xrightarrow{p} 0, \qquad \forall\ t \in I. \tag{3.5}$$

Let us now introduce two sequences of stochastic processes $\{W_n^{(i)} = [W_n^{(i)}(t), t \in I],\ i = 1, 2\}$, by letting

$$\begin{aligned} W_n^{(1)}(t) &= n^{-\frac{1}{2}} \sigma_n^{-1} \{(k+1-nt) L_k + (nt-k) L_{k+1}\}, \\ &\frac{k}{n} \leqq t \leqq \frac{k+1}{n}, \qquad k = 0, \ldots, n-1; \end{aligned} \tag{3.6}$$

$$\begin{aligned} W_n^{(2)}(t) &= n^{-\frac{1}{2}} \sigma_n^{-1} \{(k+1-nt) L_k^* + (nt-k) L_{k+1}^*\}, \\ &\frac{k}{n} \leqq t \leqq \frac{k+1}{n}, \qquad k = 0, \ldots, n-1. \end{aligned} \tag{3.7}$$

Then both $W_n^{(1)}$ and $W_n^{(2)}$ belong to the $C[0,1]$ space. Since, both L_n and L_n^* involve independent summands, the classical Donsker theorems holds, and hence, as $n\to\infty$,

$$W_n^{(i)} \xrightarrow{\mathscr{D}} \bar{\phi} W^{(i)}, \quad \text{in the uniform topology on } C[0,1];\ i=1,2, \tag{3.8}$$

where $W^{(i)}$ is a standard Wiener process on I, $i=1,2$. Let us define for every $x\in C[0,1]$ and $\delta\in$ I,

$$\omega_\delta(x)=\sup\{|x(t)-x(s)|:\ 0\leqq s<t\leqq s+\delta\leqq 1\}. \tag{3.9}$$

Then, (3.8) insures that for every $\varepsilon>0$ and $\eta>0$, there exist a δ: $0<\delta<1$ and an n_0, such that

$$P\{\omega_\delta(W_n^{(i)})>\tfrac{1}{2}\varepsilon\}<\tfrac{1}{2}\eta \qquad \text{for } n\geqq n_0;\ i=1,2. \tag{3.10}$$

This, in turn, implies that for $n\geqq n_0$,

$$P\{\omega_\delta(W_n^{(1)}-W_n^{(2)})>\varepsilon\}\leqq \sum_{i=1}^{2} P\{\omega_\delta(W_n^{(i)})>\tfrac{1}{2}\varepsilon\}<\eta. \tag{3.11}$$

On the other hand, (3.5) insures that for every (fixed) $m(\geqq 1)$ and $0\leqq t_1<\dots<t_m\leqq 1$, $\{W_n^{(1)}(t_1)-W_n^{(2)}(t_1),\dots,W_n^{(1)}(t_m)-W_n^{(2)}(t_m)\}\xrightarrow{P}\{0,\dots,0\}$, so that their joint distribution is asymptotically degenerate at the origin. Hence, by (3.11), as $n\to\infty$,

$$\sup_{t\in I}|W_n^{(1)}(t)-W_n^{(2)}(t)|\xrightarrow{P} 0, \tag{3.12}$$

and the lemma follows directly by noting that $n^{-\frac{1}{2}}\{\max_{kn}|B_{2k}|\}$ is equal to the left hand side (lhs) of (3.12). Q.E.D.

Lemma 3.4. *$n^{\frac{1}{2}}\{\sup_{k\geqq n} k^{-1}|B_{2k}|\}\xrightarrow{P} 0$, as $n\to\infty$.*

The proof follows on parallel lines. We need to replace L_n and L_n^* by $\bar{L}_n$ and $\bar{L}_n^*$ and $[nt]$ by $[n/t]$, while for the *tightness* part, we need to construct tail-sequence processes $\{n^{\frac{1}{2}}\bar{L}_k,\ k\geqq n\}$ and $\{n^{\frac{1}{2}}\bar{L}_k^*,\ k\geqq n\}$ which by (3.8) and Sen (1976) also converge weakly to $\bar{\phi} W^{(i)}$, $i=1,2$. Hence, the details are omitted.

For every ε: $0<\varepsilon<\frac{1}{2}$, let us define

$$Y_{k,\varepsilon}=\begin{cases}[a_k(R_{kk})-\bar{a}_k][h(U_k)-\mu_{k,R_{kk}}], & \text{if } k^{-1}R_{kk}\in[\varepsilon,1-\varepsilon],\\ 0, \quad \text{otherwise, for } k\geqq 1;\end{cases} \tag{3.13}$$

$$\tilde{B}_{3n,\varepsilon}=\sum_{k=1}^{n} Y_{k,\varepsilon}, \qquad n\geqq 1. \tag{3.14}$$

Lemma 3.5. *Under the regularity conditions of Section 2, for every $\eta_1>0$ and $\eta_2>0$, there exist an $\varepsilon>0$ and an n_0, such that*

$$P\{\max_{k\leqq n}\{n^{-\frac{1}{2}}|B_{3k}-\tilde{B}_{3k,\varepsilon}|\}>\eta_1\}<\eta_2, \qquad \forall\ n\geqq n_0. \tag{3.15}$$

Proof. Note that, by definition, for every $n \geqq 1$,

$$\max_{k \leq k} \{n^{-\frac{1}{2}} |B_{3k} - \tilde{B}_{3k,\varepsilon}|\} \leqq B^*_{3n,\varepsilon}$$
$$= n^{-\frac{1}{2}} \sum_{k=1}^{n} I(k^{-1} R_{kk} \notin [\varepsilon, 1-\varepsilon]) |[a_k(R_{kk}) - \bar{a}_k][h(U_{k,R_{kk}}) - \mu_{k,R_{kk}}]|. \tag{3.16}$$

Also, by (2.4), for every $1 \leqq i \leqq k$; $k \geqq 1$,

$$|\mu_{k,i}| \leqq E|h(U_{k,i})| = k \binom{k-1}{i-1} \int_0^1 |h(u)| u^{i-1} (1-u)^{k-i} du$$
$$\leqq k \binom{k-1}{i-1} K \int_0^1 u^{i-1-\alpha} (1-u)^{k-i-\alpha} du$$
$$= K\{\overline{|k+1}\, \overline{|i-\alpha}\, \overline{|k+1-i-\alpha} / \overline{|i}\, \overline{|k-i+1}\, \overline{|k+1-2\alpha}\}$$
$$\leqq K_1 \{i(k+1-i)/(k+1)^2\}^{-\alpha}, \quad \text{where } K_1 < \infty. \tag{3.17}$$

Similarly, by (2.1) and (2.5), for every $1 \leqq i \leqq k$; $k \geqq 1$,

$$|a_k(i) - \bar{a}_k| \leqq K_2 \{i(k+1-i)/(k+1)^2\}^{-\beta} \quad \text{where } K_2 < \infty. \tag{3.18}$$

Further, for any a: $0 < a < 1$ and $n^a \leqq i \leqq n - n^a$, it can be shown that

(i) for every $c > 0$, $P\left\{n^{\frac{1}{2}} \left|U_{n,i} - \frac{i}{n+1}\right| > c(\log n)[i(n+1-i)/(n+1)^2]^{\frac{1}{2}}\right\}$ exponentially converges to 0,

(ii) $E\left|U_{n,i} - \frac{i}{n+1}\right| \leqq n^{-\frac{1}{2}} \{i(n+1-i)/(n+1)^2\}^{\frac{1}{2}}$, and hence, by using (2.4), it follows by some standard steps that

$$n^{\frac{1}{2}} E|h(U_{n,i}) - \mu_{n,i}| \leqq K_3 [i(n+1-i)/(n+1)^2]^{\frac{1}{2}}, \quad \forall\, n^a \leqq i \leqq n - n^a, \tag{3.19}$$

where $K_3 < \infty$. Then, on letting $a = \delta/(1+2\delta) (>0)$, where δ is defined by (2.3), we have

$$EB^*_{3n,\varepsilon} = n^{-\frac{1}{2}} \sum_{k=1}^{n} \frac{1}{k} \{\sum_{i<k\varepsilon} + \sum_{i>k-k\varepsilon} |a_k(i) - \bar{a}_k| E|h(U_{k,i}) - \mu_{k,i}|\}$$
$$\leqq n^{-\frac{1}{2}} \sum_{k=1}^{n} \frac{1}{k} \{(\sum_{i<k^a} + \sum_{i>k-k^a} 2E|h(U_{k,i})| |a_k(i) - \bar{a}_k|)$$
$$+ (\sum_{k^a \leqq i < k\varepsilon} + \sum_{k-k\varepsilon < i \leqq k-k^a} E|h(U_{k,i}) - \mu_{k,i}| |a_k(i) - \bar{a}_k|)\}$$
$$\leqq n^{-\frac{1}{2}} \sum_{k=1}^{n} \frac{1}{k} \left\{2K_1 K_2 \left(\sum_{i<k^a} + \sum_{i>k-k^a} \left[\frac{i(k+1-i)}{(k+1)^2}\right]^{-\alpha-\beta}\right)\right.$$
$$\left. + k^{-\frac{1}{2}} K_1 K_3 \left(\sum_{k^a \leqq i < k\varepsilon} + \sum_{k-k\varepsilon < i \leqq k-k^a} \left[\frac{i(k+1-i)}{(k+1)^2}\right]^{-\alpha-\beta-\frac{1}{2}}\right)\right\}$$
$$= K^*_1 \varepsilon^{\delta} + K^*_2 n^{-\delta/2}; \quad (K^*_i < \infty,\ i = 1, 2) \tag{3.20}$$

where both K_1^* and K_2^* depend only on α and β. Now, (3.15) follows from (3.16), (3.20) and the Chebychev inequality. Q.E.D.

Lemma 3.6. *Under the regularity conditions of Section 2, for every $\eta_1>0$ and $\eta_2>0$, there exist an $\varepsilon>0$ and an n_0, such that*

$$P\{\sup_{k\geq n}\{n^{\frac{1}{2}}k^{-1}|B_{3k}-\tilde{B}_{3k,\varepsilon}|\}>\eta_1\}<\eta_2, \quad \forall\, n\geqq n_0. \tag{3.21}$$

The proof follows on parallel lines, and hence, is omitted.

Lemma 3.7 [Kiefer (1970)]. *As $n\to\infty$,*

$$\sup_{p\in I} n^{\frac{1}{2}}|U_{n,[np]}-p+F_n^*(p)-p|=O([n^{-1}(\log n)^2\log\log n]^{\frac{1}{4}}) \qquad a.s., \tag{3.22}$$

where $F_n^*(u)=n^{-1}\sum_{i=1}^{n} I(U_i\leqq u)$, $0\leqq u\leqq 1$ is the sample df and $U_{n,0}=0$.

Lemma 3.8 [Komlös, Major and Tusnâdy (1975)]. *As $n\to\infty$,*

$$\sup_{0\leqq t\leqq 1}|n\{F_n^*(t)-t\}-K(t,n)|=O((\log n)^2) \qquad a.s., \tag{3.23}$$

where $\{K(t,s), 0\leqq t\leqq 1, 0\leqq s<\infty\}$ is a Kiefer process which is Gaussian with $EK(t,s)=0\ \forall\, s,t$ and $EK(t,x)K(t',s')=(t\wedge t'-tt')(s\wedge s')$.

Let us now return to the proof of Theorem 1. Since $h^{(1)}(u)$ is continuous and bounded inside $[\varepsilon, 1-\varepsilon]$, by (3.13), (3.14), (3.21) and (3.22), it follows that $n^{-\frac{1}{2}}\tilde{B}_{3n,\varepsilon}$ is asymptotically equivalent, in probability, to

$$-n^{-\frac{1}{2}}\sum_{k=1}^{n} h^{(1)}((k+1)^{-1}R_{kk})[a_k(R_{kk})-\bar{a}_k]k^{-1}K((k+1)^{-1}R_{kk},k),$$

and hence, given R_n^*, it is asymptotically normal with 0 mean and variance

$$\begin{aligned} V_n=n^{-1}\sum_{k=1}^{n}\sum_{q=1}^{n}&(kVq)^{-1}[a_k(R_{kk})-\bar{a}_k][a_q(R_{qq})-\bar{a}_q]\\ &\cdot h^{(1)}((k+1)^{-1}R_{kk})h^{(1)}((q+1)^{-1}R_{qq})\\ &\cdot[\{(k+1)^{-1}R_{kk}\}\wedge\{(q+1)^{-1}R_{qq}\}-(k+1)^{-1}R_{kk}(q+1)^{-1}Rqq]\\ &\cdot I((k+1)^{-1}R_{kk}\in[\varepsilon,1-\varepsilon],(q+1)^{-1}R_{qq}\in[\varepsilon,1-\varepsilon]). \end{aligned} \tag{3.24}$$

In view of the fact that the summands in (3.24) are all bounded and by Lemma 3.1, the R_{jj} are all independent, it follows by some standard steps that

$$V_n\to\sigma_2^2 \quad \text{in probability (actually, in } L_1\text{-norm), as } n\to\infty. \tag{3.25}$$

Thus, for any real (and finite) θ and almost everywhere in $\underset{\sim}{R}_n^*$,

$$E\{\exp(i\theta n^{-\frac{1}{2}}B_{3n,\varepsilon})|R_n^*\}\to\exp\{-\tfrac{1}{2}\theta^2\sigma_2^2\}, \quad \text{as } n\to\infty. \tag{3.26}$$

On the other hand, given $\underset{\sim}{R}_n^*$, B_{1n} is also held fixed, so that

$$\begin{aligned}&E\{\exp(i\theta n^{-\frac{1}{2}}(B_{1n}+\tilde{B}_{3n,\varepsilon}))\}\\&\quad=E[E\{\exp(i\theta n^{-\frac{1}{2}}(B_{1n}+\tilde{B}_{3n,\varepsilon}))|R_n^*\}]\\&\quad=\exp(-\tfrac{1}{2}\theta^2\sigma_2^2)E\{\exp(i\theta n^{-\frac{1}{2}}B_{1n})\}+o(1).\end{aligned}\tag{3.27}$$

Further, by Lemma 3.1, B_{1n} involves independent summands with 0 mean and variances $\int_0^1 \phi_k^2(u)h_k^2(u)\,du-\left(\int_0^1 \phi_k(u)h_k(u)\,du\right)^2$ ($\to\sigma_1^2$ as $k\to\infty$), and by the classical central limit theorem, as $n\to\infty$

$$E\{\exp(i\theta n^{-\frac{1}{2}}B_{1n})\}\to\exp(-\tfrac{1}{2}\theta^2\sigma_1^2),\qquad \forall \text{ real and finite } \theta. \tag{3.28}$$

From (3.1), Lemmas 3.3 and 3.5, (3.27) and (3.28), we conclude that

$$\mathscr{L}(n^{-\frac{1}{2}}(T_n-\tau_n^*))\to\mathscr{N}(0,\sigma^2);\qquad \sigma^2=\sigma_1^2+\sigma_2^2. \tag{3.29}$$

The same treatment holds for any $m(\geqq 1)$ and $\{n_1,\ldots,n_m\}$ where $n_j/n\to t_j$, $1\leqq j\leqq m$, $0\leqq t_1<\cdots<t_m\leqq 1$ and we have

$$\mathscr{L}(n^{-\frac{1}{2}}(T_{n_j}-\tau_{n_j}^*),\ j=1,\cdots,m)\to\mathscr{N}_m(\mathbf{0},\sigma^2((t_j\wedge t_{j'}))), \tag{3.30}$$

insuring the convergence of the finite-dimensional distributions (f.d.d.) of $\{W_n\}$ in (2.21)–(2.22) to those of W. Hence, to prove Theorem 1, we need to show only that $\{W_n\}$ is *tight*. Since $W_n(0)=0$ with probability 1, by virtue of Lemmas 3.3 and 3.5, it suffices to show that for every $\varepsilon'>0$ and $\eta>0$, there exist ρ: $0<\rho<1$ and an n_0, such that for $n\geqq n_0$ and every $t\in I$, $m=[nt]$,

$$P\{\max_{m\leqq k\leqq(m+n\rho)\wedge n} n^{-\frac{1}{2}}|B_{1k}-B_{1m}|>\tfrac{1}{2}\varepsilon\}<\tfrac{1}{2}\eta\rho, \tag{3.31}$$

$$P\{\max_{m\leqq k\leqq(m+n\rho)\wedge n} n^{-\frac{1}{2}}|\tilde{B}_{3k,\varepsilon}-\tilde{B}_{2m,\varepsilon}|>\tfrac{1}{2}\varepsilon'\}<\tfrac{1}{2}\eta\rho. \tag{3.32}$$

Now, B_{1k} involves independent summands with zero means and finite variances (converging to σ_1^2), and hence, (3.31) follows readily by using the lemma on page 69 of Billingsley (1968). Also, note that by (2.3) and (2.4) (where $\alpha\leqq\frac{1}{2}-\delta$),

$$\max_{1\leqq i\leqq m}|h(U_i)|/m^{\frac{1}{2}-\delta}=o(1)\qquad\text{a.s., as } m\to\infty, \tag{3.33}$$

and hence, on letting $k_n\sim n^{\frac{1}{3}}$, we obtain from (3.1), (3.13), (3.17) and (3.18) that as $n\to\infty$

$$n^{-\frac{1}{2}}\{\max_{1\leqq k\leqq k_n}|\tilde{B}_{3k,\varepsilon}|\}=O(n^{-\delta/2})\qquad\text{a.s., as } n\to\infty. \tag{3.34}$$

So, to prove (3.32), it suffices to work with any $m\geqq k_n$. Further, note that by (3.13), (3.18), (3.19), (3.22) and (3.23),

$$n^{-\frac{1}{2}}\{\max_{k_n\leqq m\leqq k\leqq n}|\tilde{B}_{3k,\varepsilon}-\tilde{B}_{3m,\varepsilon}-Y_{km}^*|\}\to 0\qquad\text{a.s., as } n\to\infty, \tag{3.35}$$

where

$$Y_{km}^{*}=\sum_{s=m+1}^{k}[a_s(R_{ss})-\bar{a}_s]h^{(1)}((s+1)^{-1}R_{ss})\cdot\frac{1}{s}K((s+1)^{-1}R_{ss},s),\qquad k\geqq m\geqq 1. \tag{3.36}$$

Hence, it is sufficient to show that for every $m\geqq k_n$,

$$P\{\max_{m\leqq k\leqq(m+n\rho)\wedge n} n^{-\frac{1}{2}}|Y_{km}^{*}|>\tfrac{1}{2}\varepsilon'\}<\tfrac{1}{2}\eta\rho,\qquad \forall\, n\geqq n_0. \tag{3.37}$$

By virtue of Lemma 3.2 and the construction of the Kiefer process in (3.23), we have for every $s,s'\geqq 1$,

$$\begin{aligned}&\left|E\left\{\frac{1}{s}K((s+1)^{-1}R_{ss},s)\frac{1}{s'}K((s'+1)^{-1}R_{s's'},s')\,\middle|\,R_n^{*}\right\}\right|\\&=|(s\vee s')^{-1}[\{(s+1)^{-1}R_{ss}\}\wedge\{(s'+1)^{-1}R_{s's'}\}\\&\quad-(s+1)^{-1}R_{ss}(s'+1)^{-1}R_{s's'}]|\\&\leqq(s\vee s')^{-1}\{[(s+1)^{-1}R_{ss}(1-(s+1)^{-1}R_{ss})]\\&\quad\cdot[(s'+1)^{-1}R_{s's}(1-(s'+1)^{-1}R_{s's})]\}^{\frac{1}{2}},\end{aligned} \tag{3.38}$$

and the product moments of order 3 and 4 can be directly obtained by using the fact that the joint distributions of the $K(a,b)$ are multinormal. Finally, by (2.3)–(2.5) and (3.18),

$$\begin{aligned}&E|[a_s(R_{ss})-\bar{a}_s]h^{(1)}((s+1)^{-1}R_{ss})[(s+1)^{-1}R_{ss}(1-(s+1)^{-1}R_{ss})]^{\frac{1}{2}}|\\&\leqq K^2\left\{s^{-1}\sum_{k=1}^{s}[k(s+1-k)/(s+1)^2]^{-\alpha-\beta-\frac{1}{2}}\right\}\\&\leqq\left\{\int_0^1\{u(1-u)\}^{-1+\delta}du\right\}<\infty,\qquad\forall s\geqq 1.\end{aligned} \tag{3.39}$$

Hence, it follows by some routine steps that

$$n^{-1}E\{Y_{mm_1}^{*2}Y_{m_1m_2}^{*2}\}\leqq K_4[(m_2-m_1)(m_1-m)]/n^2,\qquad K_4<\infty, \tag{3.40}$$

for every $m\leqq m_1\leqq m_2$ where K_4 does not depend on (m,m_1,m_2). Then, (3.37) follows from (3.40) and Theorem 12.1 of Billingsley (1968, p. 89), and this completes the proof of Theorem 1.

Let us now sketch the proof of Theorem 2. Since $S_n-\bar{\tau}_n^{*}=n^{-1}(T_n-\tau_n^{*})$ for every $n\geqq 1$, if we let n_j: $n/n_j\to s_j$, $s_j\in[0,1]$, $1\leqq j\leqq m$, then $n^{\frac{1}{2}}[(S_{n_j}-\bar{\tau}_{n_j}^{*}),1\leqq j\leqq m]=n^{-\frac{1}{2}}[(n/n_j)(T_{n_j}-\tau_{n_j}^{*}),1\leqq j\leqq m]$, so that identifying $s_j=t_j^{-1}$, $1\leqq j\leqq m$, the convergence of f.d.d.'s of $\{W_n^{*}\}$ to those of W follows directly from (3.30). In this context note that $n^{-1}B_{1n}\to 0$ a.s., as $n\to\infty$, and hence, $W_n^{*}(0)=0$, with probability 1. By virtue of Lemmas 3.4 and 3.6, to establish the tightness of $\{W_n^{*}\}$, it suffices to show that for every $\varepsilon'>0$ and $\eta>0$, there exist a $\rho: 0<\rho<1$ and an n_0, such that for $n\geqq n_0$, $m\sim[n/t]$, $m'\sim[n/(\rho+t)]$, $0\leqq t<t+\rho\leqq 1$,

$$P\{\max_{m'\leqq k\leqq m} n^{\frac{1}{2}}|k^{-1}B_{1k}-m^{-1}B_m|>\tfrac{1}{2}\varepsilon'\}<\tfrac{1}{2}\eta\rho, \tag{3.41}$$

$$P\{\max_{m'\leqq k\leqq m} n^{\frac{1}{2}}|k^{-1}\tilde{B}_{3k,\varepsilon}-m^{-1}\tilde{B}_{3m,\varepsilon}|>\tfrac{1}{2}\varepsilon'\}<\tfrac{1}{2}\eta\rho. \tag{3.42}$$

Since $\{B_{1k}, k \geqq 1\}$ is a martingale sequence and the forward invariance principle holds, by Theorem 2.2 of Sen (1976), (3.41) follows from (3.31). To prove (3.42), we note that by virtue of (2.4), (3.13), (3.17), (3.18), (3.22) and (3.23), as $n \to \infty$,

$$\left| n^{-1} \tilde{B}_{3n,\varepsilon} + \frac{1}{n} \sum_{i=1}^{n} h^{(1)}((i+1)^{-1} R_{ii})[a_i(R_{ii}) - \bar{a}_i] \, i^{-1} K((i+1)^{-1} R_{ii}, i) \right|$$
$$= o(n^{-\frac{1}{2}}) \quad \text{a.s.,} \tag{3.43}$$

and hence, we may follow the line of approach in (3.37)–(3.40) and verify an inequality similar to (3.40), which insures (3.42). For brevity, the details are omitted. Q.E.D.

4. Some General Remarks

As has been noted in (3.29), the asymptotic normality of $n^{-\frac{1}{2}}(T_n - \tau_n^*)$ [or $n^{\frac{1}{2}}(S_n - \bar{\tau}_n^*)$] follows directly from Theorem 1 or 2. In addition, either of these theorems also insure the asymptotic normality when the sample size n is itself a (positive integer-valued) rv.

Statistics of the type T_n in (1.1) are useful for testing randomness against trend alternatives when observations are available sequentially. For such a problem, a pure rank statistic is of the form $\sum_{k=1}^{n} a_k(R_{kk})$ [see $\tilde{T}_n$ in Section 1] and for our theorems to hold, we do not need (2.4) and we may even replace (2.3) and (2.5) by:

$\phi(u)$ is the difference of two non-decreasing and square integrable functions inside I. (4.1)

On the other hand, a mixed rank statistic is of the form (1.1) and our Theorems 1 and 2 provide the desired results under the regularity conditions (2.1)–(2.5). Finally, for T_n^0 in Section 1, for the invariance principle to holds, it suffices to assume that

$$\max_{1 \leqq k \leqq n} a_k^2 \Big/ \left\{ \sum_{i=1}^{n} a_i^2 \right\} \to 0 \quad \text{as } n \to \infty, \tag{4.2}$$

$$\int_0^1 h^2(u)\, du < \infty. \tag{4.3}$$

Our Theorems 1 and 2 provide, respectively, the forward and backward invariance principles for $\{T_n\}$. One could have also considered the space $D[0, \infty)$ endowed with the metric

$$\rho^*(x, y) = \sup\{(t+1)^{-1} |x(t) - y(t)| : t \geqq 0\} \tag{4.4}$$

and obtained the weak convergence of $\{T_n - \tau_n^*, n \geqq 0\}$ (on $D[0, \infty)$ in the ρ^*-metric) to a Wiener process on $[0, \infty)$. Since, in practical applications, we mostly face the forward or backward invariance principles, we prefer to use our Theorems 1 and 2, which insure the above weak convergence result. We conclude this section with the following remark on the scope of the techniques developed in this paper in some related problems. For the study of some inavariance principles for linear combination of order statistics, Sen (1978) has developed a reverse martingale approach yielding a simple and direct proof. His regularity conditions are somewhat different from those in here and are, in most cases, relatively less stringent too. It seems that the Kiefer-process representation of empirical distributions can be used in the other problem too. However, certain additional results on the precise order of approximations at the two tails of the empirical distributions are needed to enable the results to be derived under the general set of conditions stated in Sen (1978). The reverse martingale approach in Sen (1978) avoids these extra manipulations. On the other hand, this reverse martingale approach is not very handy in the current problem.

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