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Markov Processes and Harmonic Spaces

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0. Introduction

It is well known by results of Meyer, Boboc-Constantinescu-Cornea, Hansen that for every $\mathfrak{P}$-harmonic space $(X, \mathscr{H}^*)$ with a countable base for which the function 1 is hyperharmonic there exists a Hunt process $\mathfrak{X}$ with state space X and paths continuous on $[0, \zeta[$ such that the set of positive hyperharmonic functions coincides with the set of excessive functions of $\mathfrak{X}$ (see [4]). This paper is devoted to a study of the converse problem. Our main theorem (5.2) gives a complete solution, namely:

For any standard process $\mathfrak{X}$ on a locally compact space X with a countable base and proper potential kernel, the set of excessive function of $\mathfrak{X}$ is the set of positive hyperharmonic functions of a $\mathfrak{P}$-harmonic space $(X, \mathscr{H}^*)$ if and only if the following conditions are satisfied:

(a) Every excessive function is the limit of an increasing sequence of continuous excessive functions.

(b) The paths of $\mathfrak{X}$ are almost surely continuous on $[0, \zeta[$.

(c) $\mathfrak{X}$ has no absorbent points.

The paper is divided in five sections.

Inspired by [4], Chapter IV, we introduce in section one the notion of a balayage space $(X, \mathscr{W})$, i.e. $\mathscr{W}$ is a convex cone of continuous positive numerical functions on a Baire space X such that the axioms of increasing sequences, lower semi-continuous regularization and natural decomposition are satisfied.

In sections two and three we consider the special situation of a convex cone $\mathscr{W}$ of lower semi-continuous numerical functions on a locally compact space X with countable base such that $(X, \mathscr{W})$ is a balayage space when X is endowed with the $\mathscr{W}$-fine topology. Furthermore, we assume that $\mathscr{W}$ contains an adapted linearly separating convex cone $\mathscr{P} \subset \mathscr{C}^+(X)$ such that every function in $\mathscr{W}$ is the limit of an increasing sequence in $\mathscr{P}$. We note that these "standard" balayage spaces give rise to a rich potential theory generalizing [4] and [8]. In this paper we content ourselves with a proof of some basic properties. In particular, the Bauer convergence property turns out to be a consequence of the axiom of l.s.c.

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regularization (see Prop. 3.4). Furthermore, $\mathscr{W}$ is the set of positive hyperharmonic functions of a $\mathfrak{P}$-harmonic space $(X, {}^*\mathscr{H})$ iff the standard balayage space $(X, \mathscr{W})$ satisfies the truncation property and has no finely isolated points.

In section four we give characterizations of standard balayage spaces by means of semi-groups and standard processes.

The final section contains the proof of the main theorem stated at the beginning.

Having prepared the material of this paper we received a preprint of Taylor [13] where the same problem is studied with different methods (see our Remark 5.3).

1. Balayage Spaces

Throughout this section let X be a Baire topological space and $\mathscr{W}$ be a convex cone of continuous positive numerical functions on X containing the constant function $+\infty$.

We shall call $(X, \mathscr{W})$ a *balayage space* if the following axioms are satisfied:

I. *Axiom of Increasing Sequences.* For every increasing sequence (u_n) in $\mathscr{W}$ we have $\sup u_n \in \mathscr{W}$.

II. *Axiom of Lower Semi-Continuous Regularization.* For every non-empty subset $\mathscr{V}$ of $\mathscr{W}$ we have $\widehat{\inf \mathscr{V}} \in \mathscr{W}$.[1]

III. *Axiom of Natural Decomposition.* If $u, v', v'' \in \mathscr{W}$ such that $u \leqq v' + v''$, there exist $u', u'' \in \mathscr{W}$ such that $u = u' + u''$, $u' \leqq v'$, $u'' \leqq v''$.

1.1. *Examples.* 1) Let $\mathscr{W}$ be the convex cone of positive hyperharmonic functions on a harmonic space $(X, {}^*\mathscr{H})$ in the sense of Constantinescu-Cornea [4] or Hansen [6]. Then $(X, \mathscr{W})$ is a balayage space if X is endowed with the fine topology (see [4], Chapter 5).

2) Let $(X, {}^*\mathscr{H})$ be a $\mathfrak{P}$-harmonic space in the sense of Constantinescu-Cornea satisfying the axiom of domination. For every finely open set $U \subset X$, $(U, \mathscr{W})$ is a balayage space where U is endowed with the fine topology, $\mathscr{W}$ is the convex cone of positive finely hyperharmonic functions on U (Fuglede [5], p. 131).

3) Let $\mathscr{E}$ be the set of excessive functions of a standard process with state space X having a reference measure (see [1]). Then $(X, \mathscr{E})$ is a balayage space if X is endowed with the fine topology of the process. Indeed, by [1], page 86, X possesses a base of open sets which are compact in the initial topology. This implies that X is a Baire topological space. By [1], page 72, $(X, \mathscr{E})$ satisfies the axiom of increasing sequences, by [1], page 200ff., the axiom of l.s.c. regularization. Finally, the axiom of natural decomposition is satisfied by [10], page 221.

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[1] As usual $\hat{f}$ denotes for any numerical function f on X the greatest lower semi-continuous (l.s.c.) minorant of f

The proofs of the following properties of balayage spaces can be obtained in the same way as in Chapter 4 of Constantinescu-Cornea [4].

Assume for the present that $(X, \mathscr{W})$ satisfies the axioms I and II. It is obvious that for any non-empty subset $\mathscr{V} \subset \mathscr{W}$ the natural infimum $\Lambda\mathscr{V}$ exists and is equal to $\widehat{\inf \mathscr{V}}$. In particular, $\mathscr{W}$ is min-stable.

1.2. **Proposition.** 1) *If* $u \in \mathscr{W}$ *then* $[u=0]$ *and* $\overline{[u<+\infty]}$ *are open.*

2) *If* $\mathscr{V}', \mathscr{V}'' \subset \mathscr{W}$ *then* $\Lambda(\mathscr{V}'+\mathscr{V}'') = \Lambda\mathscr{V}' + \Lambda\mathscr{V}''$.

3) *Let* $\mathscr{V}_n, \mathscr{V} \subset \mathscr{W}$ *and define* $f_n = \inf \mathscr{V}_n$, $f = \inf \mathscr{V}$. *If* (f_n) *is an increasing sequence converging to* f, *then* $\lim_{n\to\infty} \Lambda\mathscr{V}_n = \Lambda\mathscr{V}$.

Let f be a numerical function on X. We set

$$R_f = {}^{\mathscr{W}}R_f = \inf\{u \in \mathscr{W} : f \leqq u\}.$$

Then $\hat{R}_f \in \mathscr{W}$. If moreover f is l.s.c., then $R_f = \hat{R}_f$. We have

$$f \leqq g \Rightarrow R_f \leqq R_g; \quad R_{f+g} \leqq R_f + R_g, \ R_{\alpha f} = \alpha R_f \ (\alpha \in \mathbb{R}_+).$$

1.3. **Proposition.** *The axiom of natural decomposition is equivalent to the following property: for any* $u, v \in \mathscr{W}$ *we have* R_f, $u - R_f \in \mathscr{W}$ *where*

$$f = \begin{cases} u - v & \text{on } \{v < +\infty\} \\ 0 & \text{on } \{v = +\infty\} \end{cases}.$$

For the remainder of this section let $(X, \mathscr{W})$ be a balayage space. Let $u \in \mathscr{W}$ and $A \subset X$. Define the *reduit* of u on A by

$$\begin{aligned} R_u^A &= \inf\{v \in \mathscr{W} : v \geqq u \text{ on } A\} \\ &= \inf\{v \in \mathscr{W} : v = u \text{ on } A\}. \end{aligned}$$

The function $\hat{R}_u^A \in \mathscr{W}$ is called the *balayage* of u on A.

Obviously,

$$R_u^A \leqq u; \quad R_u^A = u \text{ on } A;$$

if $A \subset B$, $u \leqq v$ then $R_u^A \leqq R_v^B$;

$$R_{u+v}^A \leqq R_u^A + R_v^A; \quad R_u^{A\cup B} \leqq R_u^A + R_u^B;$$

if A is open then $\hat{R}_u^A = R_u^A$.

1.4. **Proposition.** *Let* $A \subset X$ *and* $u \in \mathscr{W}$ *finite on* A. *Then*

$$R_u^A = \inf\{R_u^U : U \text{ open}, A \subset U, u \text{ finite on } U\}.$$

1.5. **Theorem.** 1) *For any* $A \subset X$ *and any* $u, v \in \mathscr{W}$ *we have*

$$R_{u+v}^A = R_u^A + R_v^A; \quad \hat{R}_{u+v}^A = \hat{R}_u^A + \hat{R}_v^A.$$

2) *For any* $A, B \subset X$ *and any* $u \in \mathscr{W}$ *we have*

$$R_u^{A \cup B} + R_u^{A \cap B} \leqq R_u^A + R_u^B; \quad \hat{R}_u^{A \cup B} + \hat{R}_u^{A \cap B} \leqq \hat{R}_u^A + \hat{R}_u^B.$$

1.6. **Theorem.** 1) *Let* $(u_n) \subset \mathscr{W}$ *be increasing and let* $u = \lim_{n \to \infty} u_n$. *Then, for any* $A \subset X$ *we have* $\lim_{n \to \infty} \hat{R}_{u_n}^A = \hat{R}_u^A$.

2) *Let* (A_n) *be an increasing sequence of subsets of* X *and* $A = \bigcup_{n \in \mathbb{N}} A_n$. *Then for any* $u \in \mathscr{W}$, *we have*

$$\lim \hat{R}_u^{A_n} = \hat{R}_u^A.$$

2. Standard Balayage Spaces

In the following let X be a locally compact space with a countable base. Let $\mathscr{B}$ denote the set of all Borel measurable numerical functions on X, and let $\mathscr{C}$ be the set of all continuous real-valued functions on X. For any set $\mathscr{A}$ of numerical functions on X, we shall denote by $\mathscr{A}^+$, $\mathscr{A}_b$, $\mathscr{A}_r$, $\mathscr{A}_c$ the set of all functions in $\mathscr{A}$ which are positive, bounded, real-valued and which have compact support respectively. The support of a numerical function f on X will be denoted by $S(f)$.

Let $\mathscr{W}$ be a convex cone of l.s.c. positive numerical functions on X. The coarsest topology on X which is finer than the initial topology and for which all the functions of $\mathscr{W}$ are continuous will be called the ($\mathscr{W}$ −) *fine topology* on X. Every point of X possesses a fundamental system of fine neighborhoods which are compact in the initial topology; in particular, X is a Baire topological space with respect to the fine topology (Brelot [2], p. 5).

For every subset $\mathscr{F}$ of $\mathscr{B}^+$ let

$$\mathscr{S}(\mathscr{F}) = \{f\colon \exists (f_n) \subset \mathscr{F} \text{ such that } f_n \uparrow f\}$$

and let $\mathscr{G}(\mathscr{F})$ be the smallest subset of $\mathscr{B}^+$ having the following properties:

a) $\mathscr{F} \subset \mathscr{G}(\mathscr{F})$.

b) $\mathscr{S}(\mathscr{G}(\mathscr{F})) \subset \mathscr{G}(\mathscr{F})$.

c) If $\mathscr{M}, \mathscr{M}' \subset \mathscr{G}(\mathscr{F})$ such that $\inf \mathscr{M}$, $\inf \mathscr{M}' \in \mathscr{B}_r^+$ and $\inf \mathscr{M} + \inf \mathscr{M}' \in \mathscr{G}(\mathscr{F})$ then $\inf \mathscr{M} \in \mathscr{G}(\mathscr{F})$.

We say that $\mathscr{G}(\mathscr{F})$ is *generated* by $\mathscr{F}$.

A convex cone $\mathscr{P} \subset \mathscr{C}^+$ will be called *admissable*, if

a) $\mathscr{P}$ is adapted;

b) $\mathscr{P}$ is linearly separating;

c) $\mathscr{P}$ contains a strictly positive function.

If an admissable cone $\mathscr{P}$ is min-stable then $\mathscr{P} - \mathscr{P}$ is dense in $\mathscr{C}_{\mathscr{P}} = \{f \in \mathscr{C}(X)\colon \exists\, p \in \mathscr{P} \text{ with } |f| \leqq p\}$ with respect to the order topology (see [11], p. 57), especially for every $f \in \mathscr{C}_c^+$ there exists $p_0 \in \mathscr{P}$ such that for every $\varepsilon > 0$ there exist $p, q \in \mathscr{P}$ satisfying

$$0 \leqq p - q \leqq f \leqq p - q + \varepsilon p_0.$$

We note that for every admissable cone $\mathscr{P}$ on X and every $\mathscr{P}$-bounded u.s.c. function f on X

$$\inf\{p\in\mathscr{P}\colon p\geqq f\}=\inf\{u\in\mathscr{S}(\mathscr{P})\colon u\geqq f\}.$$

An admissable cone $\mathscr{P}$ on X will be called a *potential cone* if the following holds:

a) ${}^{\mathscr{P}}R_{\varphi}\in\mathscr{P}$ for every $\varphi\in\mathscr{C}^{+}_{\mathscr{P}}$.

b) $p-{}^{\mathscr{P}}R_{p-q}\in\mathscr{P}$ for every $p,q\in\mathscr{P}$.

In particular, $\mathscr{P}$ is min-stable and closed with respect to the order topology. Let $\mathscr{W}$ be a convex cone of l.s.c. positive numerical functions on X containing the constant function $+\infty$. $(X,\mathscr{W})$ will be called a *standard balayage space* if

(i) $(X,\mathscr{W})$ is a balayage space where X is endowed with the $\mathscr{W}$-fine topology.

(ii) There exists an admissable cone $\mathscr{P}$ on X such that $\mathscr{S}(\mathscr{P})=\mathscr{W}$.

2.1. *Examples.* 1) If $(X,{}^{*}\mathscr{H})$ is a $\mathfrak{P}$-harmonic space in the sense of Constantinescu-Cornea or Hansen, then $(X,{}^{*}\mathscr{H}^{+}(X))$ is a standard balayage space.

2) Let X be a denumerable set (with the discrete topology) and $\mathscr{W}$ be the set of all positive numerical functions on X. Then $(X,\mathscr{W})$ is a standard balayage space.

2.2. **Proposition.** *For any standard balayage space $(X,\mathscr{W})$ there exists a potential cone $\mathscr{P}$ on X such that $\mathscr{W}=\mathscr{S}(\mathscr{P})=\mathscr{G}(\mathscr{P})$.*

Proof. Let $\mathscr{P}$ be an admissable cone on X such that $\mathscr{S}(\mathscr{P})=\mathscr{W}$ and define $\mathscr{P}'=\mathscr{W}\cap\mathscr{C}_{\mathscr{P}}$. Then $\mathscr{P}'$ is a min-stable convex cone such that $\mathscr{P}\subset\mathscr{P}'\subset\mathscr{C}_{\mathscr{P}}$, hence $\mathscr{P}'$ is admissable.

Furthermore, for any $f\in\mathscr{C}^{+}_{\mathscr{P}}$ we have

$$R_f={}^{\mathscr{W}}R_f={}^{\mathscr{P}'}R_f={}^{\mathscr{P}}R_f;$$

in particular R_f is upper semi-continuous (u.s.c.) and thus $R_f\in\mathscr{P}'$. Take now $f=(p-q)^{+}$ for $p,q\in\mathscr{P}'$. Then

$$R_{p-q}=R_{(p-q)^{+}}\in\mathscr{P}'.$$

By (1.3) we obtain

$$p-R_{p-q}\in\mathscr{W}\cap\mathscr{C}_{\mathscr{P}}=\mathscr{P}',$$

hence $\mathscr{P}'$ is a potential cone.

It remains to show that $\mathscr{W}=\mathscr{S}(\mathscr{P}')=\mathscr{G}(\mathscr{P}')$.

First note that $\mathscr{G}(\mathscr{W})=\mathscr{W}$. Indeed, obviously $\mathscr{S}(\mathscr{W})\subset\mathscr{W}$. Let $\mathscr{M},\mathscr{M}'\subset\mathscr{W}$ such that $u=\inf\mathscr{M}\in\mathscr{B}^{+}_{r}$, $v=\inf\mathscr{M}'\in\mathscr{B}^{+}_{r}$ and $u+v\in\mathscr{W}$. By (1.2/2) $u+v=\widehat{u+v}=\hat{u}+\hat{v}$, hence $u=\hat{u}\in\mathscr{W}$, $v=\hat{v}\in\mathscr{W}$. Since $\mathscr{P}\subset\mathscr{P}'\subset\mathscr{W}$ we obtain

$$\mathscr{W}=\mathscr{S}(\mathscr{P})\subset\mathscr{S}(\mathscr{P}')\subset\mathscr{G}(\mathscr{P}')\subset\mathscr{G}(\mathscr{W})=\mathscr{W}$$

which implies $\mathscr{S}(\mathscr{P}')=\mathscr{G}(\mathscr{P}')=\mathscr{W}$.

In Section 5 we shall prove the converse of Proposition 2.2., i.e. for every potential cone $\mathscr{P}$ on X such that $\mathscr{S}(\mathscr{P})=\mathscr{G}(\mathscr{P})$, the pair $(X, \mathscr{S}(\mathscr{P}))$ is a standard balayage space.

For the remainder of this section let $(X, \mathscr{W})$ be a standard balayage space. By Proposition (2.2) there exists a potential cone $\mathscr{P}$ such that

$$\mathscr{P}=\mathscr{W}\cap\mathscr{C}_{\mathscr{P}}, \qquad \mathscr{W}=\mathscr{S}(\mathscr{P})=\mathscr{G}(\mathscr{P}).$$

Let $A\subset X$ be closed and $p\in\mathscr{P}$. Then

$$R_p^A=\inf\{q\in\mathscr{P}\colon q\geqq p \text{ on } A\}$$

is u.s.c., and the mapping

$$p\mapsto R_p^A \text{ on } \mathscr{P}$$

is additive, increasing and positively homogeneous. Hence there exists a unique kernel R_A on X such that

$$R_A p=R_p^A$$

for every $p\in\mathscr{P}$. Since $\mathscr{W}=\mathscr{S}(\mathscr{P})$, we obtain $R_A u=R_u^A$ for every $u\in\mathscr{W}$. Furthermore, for any $x\in X$, the measure $R_A(x,\cdot)$ is supported by A, for $R_p^A=R_q^A$ if $p=q$ on A.

2.3. **Lemma.** *For any closed $A\subset X$ and any $p\in\mathscr{P}$, we have*

$$\lim_{\substack{U\uparrow X\\ U \text{ open,}\\ \text{rel. comp.}}} R_{A\cup\complement U}\,p=R_A p.$$

In particular, for any $x\in X$, $(R_{A\cup\complement U}(x,\cdot))$ is vaguely convergent to $R_A(x,\cdot)$ as U tends to X.

Proof. For every open subset U of X,

$$R_A p\leqq R_{A\cup\complement U}\,p\leqq R_A p+R_{\complement U}\,p.$$

Let $x\in X$ and $\varepsilon>0$. Since $\mathscr{P}$ is adapted there exists a function $q\in\mathscr{P}$ and a compact subset K of X such that $q(x)<\varepsilon$ and $q\geqq p$ on $\complement K$. Then for every open neighborhood U of K we obtain $R_{\complement U}\,p\leqq R_{\complement K}\,p\leqq q$, hence

$$R_{\complement U}\,p(x)<\varepsilon.$$

So the statement follows.

2.4. **Proposition.** *For every l.s.c. function $u\geqq 0$ on X the following statements are equivalent:*

1. *$u\in\mathscr{W}$.*
2. *$R_K u\leqq u$ for every compact subset K of X.*
3. *$R_{\complement V}\,u\leqq u$ for every relatively compact open subset V of X.*

Proof. (1) $\Rightarrow$ (3): Trivial.

(3)$\Rightarrow$(2): Let K be a compact subset of X. Let $\varphi\in\mathscr{C}_c$, $0\leqq\varphi\leqq u$. Then for every relatively compact open subset U of X

$$R_{K\cup\complement U}\varphi=R_{\complement(U\setminus K)}\varphi\leqq R_{\complement(U\setminus K)}u\leqq u,$$

hence

$$R_K\varphi\leqq u$$

by (2.3). So we obtain $R_K u\leqq u$.
(2)$\Rightarrow$(1): We may assume that u is $\mathscr{P}$-bounded since $\mathscr{S}(\mathscr{W})=\mathscr{W}$. By [11], chap. III. C, it suffices to show that u in $\mathscr{P}$-concave, i.e. $\mu(u)\leqq u(x)$ for any $x\in X$ and (positive) measure μ on X satisfying $\mu(p)\leqq p(x)$ for every $p\in\mathscr{P}$. But this is an immediate consequence of the assumption (2) and [9], page 5–11.

2.5. **Proposition.** *Let U be an open subset of X and let (U_n) be a sequence of open sets such that $U=\bigcup_{n=1}^{\infty}U_n$. Let (k_n) be a sequence in $\mathbb{N}$ such that for every $m\in\mathbb{N}$ the set $\{n\in\mathbb{N}: k_n=m\}$ is infinite. Then for every $p\in\mathscr{P}$*

$$\lim_{n\to\infty}R_{\complement U_{k_n}}R_{\complement U_{k_{n-1}}}\dots R_{\complement U_{k_1}}p=R_{\complement U}p.$$

In particular, $(R_{\complement U_{k_n}}\dots R_{\complement U_{k_1}}(x,\cdot))$ is vaguely convergent to $R_{\complement U}(x,\cdot)$ for every $x\in X$.

Proof. We may choose open sets V_n such that $\bar{V}_n\subset U_n$ and $U=\bigcup_{n=1}^{\infty}V_n$. There are functions $\varphi_n\in\mathscr{C}(X)$ such that $0\leqq\varphi_n\leqq 1$, $\varphi_n=0$ on $\bar{V}_n$ and $\varphi_n=1$ on $\complement U_n$.

Let $U_n'=U_{k_n}$, $V_n'=V_{k_n}$ and $\varphi_n'=\varphi_{k_n}$ and let $p\in\mathscr{P}$. We define (u_n) and (q_n) recursively by

$$u_0=p,\qquad q_0=p,$$
$$u_n=R_{\complement U_n'}u_{n-1},\qquad q_n=R_{\varphi_n' q_{n-1}}.$$

Evidently, $(q_n)\subset\mathscr{P}$. Assuming $u_{n-1}\leqq q_{n-1}$ the inequality $1_{\complement U_n'}\leqq\varphi_n'$ implies

$$u_n=R_{\complement U_n'}u_{n-1}\leqq R_{\complement U_n'}q_{n-1}\leqq R_{\varphi_n' q_{n-1}}=q_n.$$

Assuming $u_{n-1}\geqq R_{\complement U}p$ we obtain

$$\begin{aligned}u_n=R_{\complement U_n'}u_{n-1}&\geqq R_{\complement U_n'}(R_{\complement U}p)\\&=\inf\{R_{\complement U_n'}q\colon q\in\mathscr{P},\ q\geqq p \text{ on } \complement U\}\geqq R_{\complement U}p.\end{aligned}$$

Hence for all $n\in\mathbb{N}$

$$R_{\complement U}p\leqq u_n\leqq q_n.$$

Evidently, the sequences (u_n) and (q_n) are decreasing. We obtain

$$R_{\complement U}p\leqq\lim_{n\to\infty}u_n\leqq\lim_{n\to\infty}q_n=:q.$$

Let $m \in \mathbb{N}$. Take $n \in \mathbb{N}$ such that $k_n = m$. Since $q_n = R_{\varphi_m q_{n-1}} \in \mathscr{P}$ and $\varphi_m = 0$ on V_m we have

$$R_{\complement V_m} q_n = R_{q_n}^{\complement V_m} = q_n.$$

The set $\{n \in \mathbb{N} : k_n = m\}$ is infinite and (q_n) is decreasing. Thus

$$R_{\complement V_m} q = q.$$

Therefore by (2.4), for every $u \in \mathscr{W}$

$$R_{\complement V_m}(u - q) \leqq u - q,$$

hence the Choquet boundary of X with respect to the convex cone $\mathscr{W} - \mathbb{R}_+ q$ is contained in $\complement U$.

Let $u \in \mathscr{W}$ such that $u \geqq p$ on $\complement U$. Then $u - q \geqq 0$ on $\complement U$, hence $u - q \geqq 0$ by the minimum principle. This implies $R_{\complement U} p \geqq q$, hence

$$R_{\complement U} p = q = \lim_{n \to \infty} u_n.$$

3. The Associated Harmonic Structure

We shall now proceed to construct a harmonic structure on a given standard balayage space $(X, \mathscr{W})$. Again let $\mathscr{P}$ be a potential cone such that $\mathscr{P} = \mathscr{W} \cap \mathscr{C}_{\mathscr{P}}$ and $\mathscr{S}(\mathscr{P}) = \mathscr{W}$.

For every open subset U of X let ${}^*\mathscr{H}_{\mathscr{P}}(U)$ denote the set of all lower $\mathscr{P}$-bounded functions $u \in \mathscr{B}$ such that u is l.s.c. in U and

$$R_{\complement V} u \leqq u$$

for every open set V with $\bar{V} \subset U$. We note that ${}^*\mathscr{H}_{\mathscr{P}}^+(X) = \mathscr{W}$ by (2.4).

3.1. **Proposition.** *Let U be an open set in X and let $\mathfrak{B}$ be a base of U consisting of open sets W satisfying $\bar{W} \subset U$. Let $u \in \mathscr{B}$ be lower $\mathscr{P}$-bounded and l.s.c. on U and let $R_{\complement W} u \leqq u$ for every $W \in \mathfrak{B}$. Then $u \in {}^*\mathscr{H}_{\mathscr{P}}(U)$.*

Proof. Let $p \in \mathscr{P}$ such that $-p \leqq u$ and let $\varphi \in \mathscr{C}_c^+$ such that $\varphi \leqq u + p$. Let V be an open set such that $\bar{V} \subset U$ and let (W_n) be a sequence in $\mathfrak{B}$ such that $\bar{W}_n \subset V$ for every n. Then for every n

$$\begin{aligned} -p &\leqq -R_{\complement W_n} \cdots R_{\complement W_1} p + R_{\complement W_n} \cdots R_{\complement W_1} \varphi \\ &\leqq \ \ R_{\complement W_n} \cdots R_{\complement W_1} u \leqq u. \end{aligned}$$

So we conclude from (2.5) that

$$R_{\complement V}(-p + \varphi) \leqq u.$$

Hence

$$R_{\complement V} u \leqq u.$$

3.2. **Corollary.** *Let $(U_i)_{i\in I}$ be a family of open subsets of X. Then*

$$^*\mathscr{H}_{\mathscr{P}}(\bigcup_{i\in I} U_i)=\bigcap_{i\in I} {}^*\mathscr{H}_{\mathscr{P}}(U_i).$$

3.3. **Proposition.** *Let U be an open set and let V be a relatively compact open set such that $\bar{V}\subset U$. Let $p\in\mathscr{P}$. Then there exists a function $q\in\mathscr{P}$ and an increasing mapping $v\mapsto q_v$ from $\{v\in{}^*\mathscr{H}_{\mathscr{P}}(U)\colon 0\leqq v\leqq p\}$ into $\mathscr{W}$ such that*

$$q_v=v+q$$

on V. For every v which is continuous on U we may choose a continuous q_v.

Proof. Let K be a compact subset of U such that $\bar{V}\subset\mathring{K}$. Since $\mathscr{P}$ is a potential cone there exist $q,q'\in\mathscr{P}$ such that $q\leqq q'$, $q=q'$ on $\complement K$ and $q'-q\geqq p$ on $\bar{V}$. For any $v\in{}^*\mathscr{H}_{\mathscr{P}}(U)$ such that $0\leqq v\leqq p$ define

$$q_v=(v+q)\wedge q'.$$

Evidently, $q_v\in{}^*\mathscr{H}_{\mathscr{P}}(U)$. Furthermore, $q_v\leqq q'$ and $q_v=q'$ on $\complement K$, hence $q_v\in{}^*\mathscr{H}_{\mathscr{P}}(\complement K)$. Since $U\cup\complement K=X$ we obtain by (3.2) that $q_v\in{}^*\mathscr{H}_{\mathscr{P}}^+(X)=\mathscr{W}$. It is obvious that the mapping $v\mapsto q_v$ is increasing and that q_v is continuous if v is continuous on U.

For every open U in X we define

$$\mathscr{H}_{\mathscr{P}}(U)={}^*\mathscr{H}_{\mathscr{P}}(U)\cap(-{}^*\mathscr{H}_{\mathscr{P}}(U)).$$

3.4. **Proposition.** *Let U be an open subset of X and let (h_n) be a decreasing sequence in $\mathscr{H}_{\mathscr{P}}^+(U)$. Then $\inf h_n\in\mathscr{H}_{\mathscr{P}}^+(U)$.*

Proof. We have to show that $h=\inf h_n$ is l.s.c. on U. Let V be a relatively compact open subset of U such that $\bar{V}\subset U$. Let $p\in\mathscr{P}$ such that $h_1\leqq p$. We choose a function $q\in\mathscr{P}$ and a mapping $v\mapsto q_v$ according to the preceding proposition. We first consider the differences

$$v_n=p-h_n.$$

The sequence (v_n) is increasing, hence (q_{v_n}) is an increasing sequence in $\mathscr{P}$. So $\sup q_{v_n}\in\mathscr{W}$. On the set V we have

$$h_n=p-v_n=p+q-q_{v_n},$$

for every n, hence

$$h=p+q-\sup q_{v_n}.$$

Therefore h is finely continuous on V.

Furthermore, we have on V

$$q_{h_n}=h_n+q$$

for every n, hence

$$\inf q_{h_n}=h+q.$$

We know that $u=\widehat{\inf q_{h_n}}\in\mathscr{W}$. Since $h+q$ is finely continuous on V we obtain

$$u=h+q \quad \text{on } V.$$

In particular, h is l.s.c. on V.

3.5. **Proposition.** *Let $\varphi\in\mathscr{C}_{\mathscr{P}}$. Then $R_\varphi\in\mathscr{H}_{\mathscr{P}}^+(\complement S(\varphi))$.*

Proof. We have $R_\varphi\in\mathscr{P}$. Let V be a relatively compact open subset of $\complement S(\varphi)$. Then $R_{R_\varphi}^{\complement V}=R_\varphi$. Thus $R_\varphi\in\mathscr{H}_{\mathscr{P}}^+(\complement S(\varphi))$.

3.6. **Corollary.** *For every closed subset A of X and every $p\in\mathscr{P}$, $R_p^A\in\mathscr{H}_{\mathscr{P}}^+(\complement A)$.*

Proof. There exists a decreasing sequence (p_n) in $\mathscr{P}$ such that $p_n\leqq p$, $p_n=p$ on A and

$$\hat{R}_p^A=\widehat{\inf p_n}.$$

Let (U_n) be a sequence of open sets in X such that $\overline{U}_{n+1}\subset U_n$ and $A=\bigcap_{n=1}^{\infty} U_n$. Let (φ_n) be a sequence in $\mathscr{C}$ such that $0\leqq\varphi_n\leqq 1$, $\varphi_n=1$ on $\overline{U}_{n+1}$, $\varphi_n=0$ on $\complement U_n$. For every n, take

$$q_n=R_{\varphi_n p_n}.$$

Then $R_p^A\leqq q_n\leqq p_n$, hence

$$R_p^A\leqq\inf q_n,\qquad \hat{R}_p^A=\widehat{\inf q_n}.$$

Since (p_n) and (φ_n) are decreasing, the sequence (q_n) is decreasing. For every m, $(q_n)_{n\geq m}\subset\mathscr{H}_{\mathscr{P}}^+(\complement\overline{U}_m)$ by (3.5), hence $\inf q_n\in\mathscr{H}_{\mathscr{P}}^+(\complement\overline{U}_m)$ by (3.4). Thus $\inf q_n\in\mathscr{H}_{\mathscr{P}}^+(\complement A)$ by (3.2).

Therefore we obtain for every $x\in\complement A$

$$\inf q_n(x)=\widehat{\inf q_n}(x)=\hat{R}_p^A(x)\leqq R_p^A(x).$$

Since evidently $\inf q_n=p=R_p^A$ on A, we conclude that

$$R_p^A=\inf q_n\in\mathscr{H}_{\mathscr{P}}^+(\complement A).$$

For every open subset U of X and every numerical function f on X let $\overline{\mathscr{U}}_f^U$ denote the set of all l.s.c. functions $u\in{}^*\mathscr{H}_{\mathscr{P}}(U)$ such that $u\geqq f$ on $\complement U$ and let $\underline{\mathscr{U}}_f^U=-\overline{\mathscr{U}}_{-f}^U$. Defining

$$\overline{H}_f^U=\inf\overline{\mathscr{U}}_f^U,\qquad \underline{H}_f^U=\sup\underline{\mathscr{U}}_f^U$$

we have $\underline{H}_f^U\leqq\overline{H}_f^U$ by the minimum principle, for the Choquet boundary of X with respect to ${}^*\mathscr{H}_{\mathscr{P}}(U)$ is contained in $\complement U$ by the definition of ${}^*\mathscr{H}_{\mathscr{P}}(\mathrm{U})$.

3.7. **Proposition.** *Let U be an open subset of X. Then for every $f\in\mathscr{C}_{\mathscr{P}}$*

$$\overline{H}_f^U=\underline{H}_f^U=R_{\complement U}f\in\mathscr{H}_{\mathscr{P}}(U).$$

Proof. Let $p\in\mathscr{P}$. Evidently, $\overline{H}_p^U\leqq R_p^{\complement U}$ since $\{u\in\mathscr{W}: u\geqq p$ on $\complement U\}\subset\overline{\mathscr{U}}_p^U$. On the other hand $R_p^{\complement U}$ is u.s.c., $R_p^{\complement U}\in\mathscr{H}_{\mathscr{P}}(U)$ and $R_p^{\complement U}\leqq p$. Hence $R_p^{\complement U}\in\underline{\mathscr{U}}_p^U$ and therefore $R_p^{\complement U}\leqq\underline{H}_p^U$. So

$$\overline{H}_p^U=\underline{H}_p^U=R_p^{\complement U}\in\mathscr{H}_{\mathscr{P}}(U).$$

The statement now follows readily since $\mathscr{P}-\mathscr{P}$ is dense in $\mathscr{C}_{\mathscr{P}}$ with respect to the order topology.

We shall say that $(X,\mathscr{W})$ has the *truncation property* if the following holds: For every open U in X and every $u,v\in\mathscr{W}$ such that $u\geqq v$ on U^* the function w defined by

$$w=\begin{cases}\inf(u,v) & \text{on } U\\ v & \text{on } \complement U\end{cases}$$

is contained in $\mathscr{W}$.

3.8. **Proposition.** *The following properties are equivalent:*

(1) *$(X,\mathscr{W})$ has the truncation property.*

(2) *For every open V in X and every $x\in V$, $R_{\complement V}(x,\complement V^*)=0$.*

(3) *There exists a base $\mathfrak{B}$ of open sets V such that $R_{\complement V}(x,\complement V^*)=0$ for every $x\in V$.*

Proof. (1)$\Rightarrow$(2): Let V be an open subset of X. Let $p,q\in\mathscr{P}$ such that $q\leqq p$ and $q=p$ on V^*. Let $u\in\mathscr{W}$ such that $u\geqq q$ on $\complement V$. Then $u\geqq q=p$ on V^*, hence by the truncation property the function w defined by

$$w=\begin{cases}\inf(u,p) & \text{on } V\\ p & \text{on } \complement V\end{cases}$$

is contained in $\mathscr{W}$. Therefore $w\geqq R_p^{\complement V}$. In particular, $u\geqq R_p^{\complement V}$ on V. So we obtain that $R_q^{\complement V}\geqq R_p^{\complement V}$ on V, i.e. $R_q^{\complement V}=R_p^{\complement V}$ on V since $q\leqq p$. Thus for every $x\in V$,

$$0=R_p^{\complement V}(x)-R_q^{\complement V}(x)=R_{\complement V}(p-q)(x).$$

Since $q\leqq p$ we conclude that

$$R_{\complement V}(x,\{p-q>0\})=0.$$

This yields $R_{\complement V}(x,\complement V^*)=0$.

(2)$\Rightarrow$(3): Trivial.

(3)$\Rightarrow$(1): Let U be an open subset, and $u,v\in\mathscr{W}$ such that $u\geqq v$ on U^*. Define

$$w=\begin{cases}\inf(u,v) & \text{on } U,\\ v & \text{on } \complement U.\end{cases}$$

Since $\mathscr{W}=\mathscr{S}(\mathscr{P})$ we may assume $v\in\mathscr{P}$. Suppose first that $u>v$ on U^*. Then $w=\inf(u,v)$ on some open neighborhood W of $\overline{U}$, hence $R_{\complement V}w\leqq w$ for every $V\in\mathfrak{B}$ such that $\overline{V}\subset W$. For every open set V such that $\overline{V}\subset\complement U$ we have $R_{\complement V}w\leqq R_{\complement V}v\leqq v=w$ on V, hence $R_{\complement V}w\leqq w$. Therefore $w\in\mathscr{W}$ by (3.1).

Suppose now $u \geqq v$ on U^*. Choose $p_0 \in \mathscr{P}$, $p_0 > 0$. Replacing u by $u + \frac{1}{n} p_0$ we obtain a sequence (w_n) in $\mathscr{W}$ such that $w_n \downarrow w$. Hence the l.s.c. function w is in $\mathscr{W}$ by the axiom of l.s.c. regularization.

From now on we shall assume that $(X, \mathscr{W})$ has the truncation property.

For every open subset U of X let ${}^*\mathscr{H}(U)$ be the set of all l.s.c. functions u: $U \to]-\infty, +\infty]$ such that for every relatively compact open V satisfying $\bar{V} \subset U$ and for every $x \in V$

$$\int u(y)\, R_{\complement V}(x, dy) \leqq u(x).$$

Extending by 0 we may view the set $\{u \in {}^*\mathscr{H}(U)\colon u \text{ lower } \mathscr{P}\text{-bounded}\}$ as a subset of ${}^*\mathscr{H}_{\mathscr{P}}(U)$ whereas on the other hand ${}^*\mathscr{H}_{\mathscr{P}}(U)|_U \subset {}^*\mathscr{H}(U)$. In particular, ${}^*\mathscr{H}^+(U) = {}^*\mathscr{H}_{\mathscr{P}}^+(U)|_U$, ${}^*\mathscr{H}^+(X) = \mathscr{W}$.

3.9. **Theorem.** *${}^*\mathscr{H}$ is a hyperharmonic sheaf satisfying the axioms of convergence, resolutivity and completeness* ([4]).

Proof. Evidently $U' \subset U$ implies ${}^*\mathscr{H}(U') \subset {}^*\mathscr{H}(U)$. Let $(U_i)_{i \in I}$ be a family of open sets, let $U = \bigcup_{i \in I} U_i$ and let v: $U \to \bar{\mathbb{R}}$ such that $v|_{U_i} \in {}^*\mathscr{H}(U_i)$ for every $i \in I$. Let V be a relatively compact open set in X such that $\bar{V} \subset U$. Let W be relatively compact open such that $\bar{V} \subset W$ and $\bar{W} \subset U$. Then $v|_W$ (extended by 0) is a function in $\bigcap_{i \in I} {}^*\mathscr{H}_{\mathscr{P}}(W \cap U_i) = {}^*\mathscr{H}_{\mathscr{P}}(W \cap U)$. Hence $R_{\complement V}\, v \leqq v$. Thus $v \in {}^*\mathscr{H}(U)$.

Using (3.4) we easily obtain the axiom of convergence.

Let U be a relatively compact open subset of X and let $g \in \mathscr{C}(U^*)$. Let $f \in \mathscr{C}_c$ such that $f = g$ on U^*. Then

$$(\bar{H}_f^U)|_U = \inf\{u \in {}^*\mathscr{H}(U)\colon \liminf_{x \to z} u(x) \geqq g(z) \text{ for every } z \in U^*\}.$$

Indeed, if v is a l.s.c. function in ${}^*\mathscr{H}_{\mathscr{P}}(U)$ such that $v \geqq f$ on $\complement U$ then $v|_U \in {}^*\mathscr{H}(U)$ and $\liminf_{\substack{x \to z \\ x \in U}} v(x) \geqq v(z) \geqq f(z) = g(z)$ for every $z \in U^*$.

Conversely, let $u \in {}^*\mathscr{H}(U)$ such that $\liminf_{x \to z} u(x) \geqq g(z)$ for every $z \in U^*$. Define v by

$$v = \begin{cases} u & \text{on } U \\ f & \text{on } \complement U. \end{cases}$$

Then v is a l.s.c. function in ${}^*\mathscr{H}_{\mathscr{P}}(U)$ such that $v \geqq f$ on $\complement U$.

So we conclude from (3.7) that U is resolutive. Hence the axiom of resolutivity is satisfied.

Furthermore, (3.7) yields that for every $x \in U$ the measure $R_{\complement U}(x, \cdot)$ is the harmonic measure with respect to the sheaf ${}^*\mathscr{H}$. So by definition of ${}^*\mathscr{H}$ the axiom of completeness is satisfied.

We shall say that a point $x \in X$ is *absorbent* if $R_{\complement \{x\}}(x, X) = 0$.

3.10. **Proposition.** *For every $x \in X$ the following statements are equivalent:*

1. *x is absorbent.*
2. *$R_{\complement V}(x, X) = 0$ for every open neighborhood V of x.*

3. $h(x)=0$ *for every function h which is harmonic in a neighborhood of* x.
4. x *is finely isolated.*

Proof. Let $p\in\mathscr{P}$, $p>0$.

(1)$\Rightarrow$(2): For every open neighborhood V of x,

$$R_p^{\complement V}(x)\leqq R_p^{\complement\{x\}}(x)=0,$$

hence $R_{\complement V}(x,X)=0$.

(2)$\Rightarrow$(1): Let (V_n) be a sequence of neighborhoods of x such that $V_n\downarrow\{x\}$. Then

$$R_p^{\complement\{x\}}(x)=\sup R_p^{\complement V_n}(x)=0,$$

hence $R_{\complement\{x\}}(x,X)=0$.

(2)$\Rightarrow$(3): Let V be an open neighborhood of x and let $h\in\mathscr{H}(V)$. Let W be open such that $x\in W$ and $\overline{W}\subset V$. Then

$$h(x)=R_{\complement W}h(x)=0.$$

(3)$\Rightarrow$(2): Let V be an open neighborhood of x. Then $(R_p^{\complement V})|_V\in\mathscr{H}(V)$, hence $R_p^{\complement V}(x)=0$, i.e. $R_{\complement V}(x,X)=0$.

(1)$\Rightarrow$(4): $\{x\}=\{R_p^{\complement\{x\}}<p\}$ is finely open.

(4)$\Rightarrow$(3): By the definition of finely open sets there exists an open set U in X, a function $v\in\mathscr{W}$ and a real β such that

$$x\in U\cap\{v<\beta\}\subset\{x\}.$$

Suppose that there exists an open neighborhood V of x and a function $h\in\mathscr{H}(V)$ such that $h(x)=1$. Let $\dfrac{v(x)}{\beta}<\alpha<1$ and let W be an open set such that $\overline{W}\subset U\cap\{y\in V\colon \alpha h(y)<1\}$. Then we obtain the contradiction

$$\begin{aligned}v(x)&\geqq R_{\complement W}v(x)\geqq\beta R_{\complement W}(x,W^*)\\&\geqq\alpha\beta R_{\complement W}h(x)=\alpha\beta>v(x).\end{aligned}$$

So (3) holds.

3.11. *Example* (see [3], p. 171). Let $X=[0,1[$ and $\mathscr{W}$ be the convex cone of (l.s.c.) positive concave functions on X. Then $(X,\mathscr{W})$ is a standard balayage space having the truncation property such that the point 0 is an absorbent point.

3.12. **Theorem.** *For any convex cone $\mathscr{W}$ of l.s.c. positive numerical functions on a locally compact space X with countable base the following statements are equivalent:*

(1) *$(X,\mathscr{W})$ is a standard balayage space such that the truncation property is satisfied and no point of X is finely isolated.*

(2) *There exists a hyperharmonic sheaf $^*\mathscr{H}$ on X such that $(X,{}^*\mathscr{H})$ is a $\mathfrak{P}$-harmonic space (in the sense of [4]) and $^*\mathscr{H}^+(X)=\mathscr{W}$.*

Proof. (1)$\Rightarrow$(2): (3.9) and (3.10).

(2)$\Rightarrow$(1): (2.1), (3.8) and (3.10).

3.13. *Remark.* We note that the hyperharmonic sheaf $^*\mathscr{H}$ of (3.12) is uniquely determined by $\mathscr{W}$ since on every $\mathfrak{P}$-harmonic space $(X, {}^*\mathscr{H})$ the extension theorem ([4], p. 46) yields a characterization of every cone $^*\mathscr{H}(U)$ in terms of $^*\mathscr{H}^+(X)$.

4. Balayage Spaces and Standard Processes

Let X be a locally compact space with a countable base. A *quasi-Feller semigroup* $P=(P_t)_{t>0}$ on X is a semigroup of sub-Markovian kernels on $(X, \mathscr{B})$ having the following properties:

(i) $P_t(\mathscr{C}_0) \subset \mathscr{C}_b$ for all $t>0$.

(ii) For all $f \in \mathscr{C}_0$, the function $P_t f$ converges locally uniformly to f as t tends to zero.

(iii) There exist strictly positive real continuous excessive functions p, q such that $\frac{p}{q} \in \mathscr{C}_0$.

We denote by $\mathscr{E}(P)$ the set of excessive functions with respect to the semigroup P.

4.1. **Theorem.** *Let $\mathscr{P}$ be an admissable cone on X and let $\mathscr{W}=\mathscr{S}(\mathscr{P})$ such that $1 \in \mathscr{W}$. Then the following statements are equivalent:*

(1) $(X, \mathscr{W})$ is a standard balayage space.

(2) There exists a potential cone $\mathscr{P}'$ on X such that $\mathscr{G}(\mathscr{P}')=\mathscr{W}$.

(3) There exists a quasi-Feller semigroup $P=(P_t)_{t>0}$ on X such $\mathscr{W}=\mathscr{E}(P)$. Furthermore, the potential kernel V of P satisfies $V(\mathscr{B}_b) \subset \mathscr{C}_b$.

(4) $\mathscr{W}$ is the set of excessive functions of a standard process $\mathfrak{X}$ with state space X having a proper potential kernel.

Proof. (1) $\Rightarrow$ (2): (2.2).
(2) $\Rightarrow$ (3): By [7], page 342, there exists a quasi-Feller semigroup $P=(P_t)_{t>0}$ on X such that $\mathscr{W}=\mathscr{E}(P)$ and $V(\mathscr{C}_b) \subset \mathscr{C}_b$. Since every excessive function is l.s.c., we obtain $V(\mathscr{B}_b) \subset \mathscr{C}_b$.
(3) $\Rightarrow$ (4): [6], page 208.
(4) $\Rightarrow$ (1): Since the potential kernel V of $\mathfrak{X}$ is proper, the fine topology is generated by $\mathscr{W}$. Furthermore, every excessive function is l.s.c., hence the process $\mathfrak{X}$ has a reference measure. Therefore, $(X, \mathscr{W})$ is a standard balayage space by (1.1/3).

4.2. *Remark.* [6], page 208, shows that in (4.1), we can replace "standard process" by "Hunt process".

4.3. **Corollary.** *Let $\mathscr{W}$ be a family of positive numerical functions on X. Then the following statements are equivalent:*

(1) $(X, \mathscr{W})$ is a standard balayage space.

(2) There exists a potential cone $\mathscr{P}$ on X such that $\mathscr{W}=\mathscr{S}(\mathscr{P})=\mathscr{G}(\mathscr{P})$.

Proof. (1) $\Rightarrow$ (2): (2.2).
(2) $\Rightarrow$ (1) Let $p \in \mathscr{P}$, $p>0$ and define

$$\mathscr{P}' = \frac{1}{p} \mathscr{P}.$$

Then $\mathscr{P}'$ is a potential cone such that $\mathscr{S}(\mathscr{P}')=\mathscr{G}(\mathscr{P}')=\frac{1}{p}\mathscr{W}$ and $1\in\mathscr{P}'$. By (4.1), $\left(X,\frac{1}{p}\mathscr{W}\right)$ is a standard balayage space, and it easily follows that $(X,\mathscr{W})$ is a standard balayage space.

4.4. *Remark.* The statements (4.3) and (3.12) include a result of Sieveking [12], stating that every reasonable potential cone generates a harmonic space.

5. Standard Processes and Harmonic Spaces

5.1. **Proposition.** *Let $\mathfrak{X}$ be a standard process on X with paths continuous on $[0,\zeta[$ and proper potential kernel V. Suppose that the set $\mathscr{E}$ of excessive functions of $\mathfrak{X}$ satisfies $\mathscr{E}=\mathscr{S}(\mathscr{E}\cap\mathscr{C})$. Then there exists a (unique) hyperharmonic sheaf $^*\mathscr{H}$ on X satisfying the axioms of convergence, resolutivity and completeness such that $^*\mathscr{H}^+(X)=\mathscr{E}$.*

Proof. By (1.1/3), $(X,\mathscr{E})$ is a balayage space if X is endowed with the fine topology of the process. Since V is proper, the fine toplogy coincides with the $\mathscr{E}$-fine topology. Let

$$\mathscr{P}=\{p\in\mathscr{E}\cap\mathscr{C}:\ \inf_{K\,\text{compact}} R_p^{\complement K}=0\}.$$

It suffices to show that $\mathscr{P}$ is an admissible cone on X such that $\mathscr{S}(\mathscr{P})=\mathscr{E}$. Indeed, using (3.8) the continuity of the paths yields the truncation property of $(X,\mathscr{E})$, hence the assertion follows from (3.9). The assertions about $\mathscr{P}$ will be proved in several steps:

(a) Obviously, $\mathscr{P}$ is a convex cone such that for every sequence $(p_n)\subset\mathscr{P}$ with $p=\sum_{n=1}^{\infty}p_n\in\mathscr{C}$ we have $p\in\mathscr{P}$.

(b) Let $f\in\mathscr{B}^+$ such that $Vf<\infty$. Then for every compact subset K of X and every $x\in X$,

$$R_{Vf}^{\complement K}(x)=E^x\left(\int_{T_{\complement K}}^{\zeta} f\circ X_t\,dt\right),^2$$

hence

$$\inf_{K\,\text{compact}} R_{Vf}^{\complement K}=0.$$

Let $x\in X$ and $f\in\mathscr{B}^+$ such that $Vf<\infty$ and $Vf(x)>0$. Since by assumption $\mathscr{E}=\mathscr{S}(\mathscr{E}\cap\mathscr{C})$ there exists a function $p\in\mathscr{E}\cap\mathscr{C}$ such that $p\leqq Vf$ and $p(x)>0$. Then obviously $p\in\mathscr{P}$. Using (a) we obtain a strictly positive $p_0\in\mathscr{P}$.

(c) Let $g\in\mathscr{E}$ and $(g_n)\subset\mathscr{E}\cap\mathscr{C}$ such that $g_n\uparrow g$. Then $p_n:=g_n\wedge np_0\in\mathscr{P}$ and $p_n\uparrow g$. Hence $\mathscr{E}=\mathscr{S}(\mathscr{P})$.

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2 As usual $T_{\complement K}$ denotes the first hitting time of $\complement K$ and ζ denotes the life time of $\mathfrak{X}$

(d) In order to show that $\mathscr{P}$ is linearly separating it suffices to show that $\mathscr{E}$ is separating since $\mathscr{E}=\mathscr{S}(\mathscr{P})$ and $1\in\mathscr{E}$. Let $x,y\in X$, $x\neq y$, and $\varphi\in\mathscr{C}_c^+$ such that $\varphi(x)\neq\varphi(y)$, Since $\lim\limits_{\lambda\to\infty}\lambda V_\lambda\varphi=\varphi$ (pointwise) there exist a $\lambda>0$ such that $V_\lambda\varphi(x)\neq V_\lambda\varphi(y)$. The equation

$$V\varphi=V_\lambda\varphi+\lambda VV_\lambda\varphi$$

shows that $V\varphi(x)\neq V\varphi(y)$ or $V(V_\lambda\varphi)(x)\neq V(V_\lambda\varphi)(y)$. Hence $\mathscr{E}$ separates x and y.

(e) Let $p\in\mathscr{P}$, $x\in X$ and $\varepsilon>0$. By definition of $\mathscr{P}$, there exists a compact subset K of X such that

$$R_p^{\complement K}(x)<\frac{\varepsilon}{2}.$$

Let U be a relatively compact open neighborhood of K. Let $\alpha>0$ such that $\alpha p_0(x)<\frac{\varepsilon}{2}$. There exists $q\in\mathscr{P}$ such that $q\leqq R_p^{\complement K}$ and $q+\alpha p_0>p$ on U^*, since $R_p^{\complement K}=p$ on U^*. Define

$$u=\begin{cases}\inf(q+\alpha p_0, p) & \text{on } U\\ p & \text{on } \complement U.\end{cases}$$

Then $u\in\mathscr{E}$ by [1], page 93. Furthermore $u\in\mathscr{C}$ and $u\leqq p$, hence $u\in\mathscr{P}$. Finally, $u\geqq p$ on $\complement U$ and

$$u(x)\leqq q(x)+\alpha p_0(x)<\varepsilon.$$

Thus $\mathscr{P}$ is adapted by (a) and [11], page 34.

5.2. **Theorem.** *For every family $\mathscr{W}$ of numerical functions on X such that $1\in\mathscr{W}$ the following statements are equivalent:*

1. *$\mathscr{W}$ is the set of positive hyperharmonic functions of a $\mathfrak{P}$-harmonic space $(X,{}^*\mathscr{H})$ in the sense of Constantinescu-Cornea* [4].
2. *$\mathscr{W}$ is the set of excessive functions of a standard process $\mathfrak{X}$ on X having the following properties:*
   (a) $\mathscr{W}=\mathscr{S}(\mathscr{W}\cap\mathscr{C})$.
   (b) *The paths of $\mathfrak{X}$ are continuous on* $[0,\zeta[$.
   (c) *$\mathfrak{X}$ has no absorbent points.*
   (d) *The potential kernel V of $\mathfrak{X}$ is proper.*

Proof. (1)$\Rightarrow$(2): [6], page 213, and (3.10).
(2)$\Rightarrow$(1): (5.1) and (3.10).

5.3. *Remark.* If $\mathfrak{X}$ is a "reasonable" standard process in the sense of Taylor [13] and $\mathscr{W}$ denotes the set of its excessive functions then the condition (2) of (5.2) is obviously satisfied. Therefore Taylor's result on the existence of an associated harmonic structure (obtained by different methods) is included in the above theorem.

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