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Subfair "Red-and-Black" in the Presence of Inflation

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Summary. The gambling problem of "Red-and-Black" casinos in the presence of inflation is introduced. The optimality of the bold strategy is shown when the lottery is subfair or fair. The non-optimality of the bold strategy is also shown when the lottery is superfair.

1. Introduction

Suppose you have x dollars and you want to buy a house which sells today for y dollars, $y>x>0$. Due to inflation, the price of the house tomorrow will be $(1+\alpha)y$ $(\alpha \geqq 0)$, and will continue to go up at the same rate, so as to become $(1+\alpha)^n y$ on the n^{th} day. Once each day, you can stake any amount of money in your possession, but no more than you possess, on a "Red-and-Black" lottery. If you do so, you gain the amount of your stake with probability w and lose your stake with the complementary probability $\bar{w}=1-w$. How should you gamble so as to maximize your chance of eventually catching up with inflation and being able to buy the house?

Plainly, if the rate of inflation is 100 % or more (i.e. $\alpha \geqq 1$), you will never catch up, so this case is trivial. The other extreme case of no inflation at all (i.e. $\alpha=0$) was settled by Dubins and Savage (Chapter 5 of [5]), who established the optimality of the bold play, when the lottery is subfair or fair, i.e. when $w \leqq \frac{1}{2}$. In this paper, we will show that when the lottery is subfair or fair, the bold play continues to be optimal for all $\alpha>0$. However, it is also shown that when the lottery is superfair, the bold play is no longer optimal, even if $w/(1+\alpha)<\frac{1}{2}$, that is, even if the inflation rate is high enough to make the situation subfair in the sense that the ratio of your fortune to the price of the house is an expectation decreasing semimartingale.

In [8], Klugman used a somewhat different (less direct) method to account for inflation and he also obtained a similar result. In [3], Chen used Klugman's method to account for inflation in primitive casinos and he proved that, in general, the bold play is no longer optimal. In the fall of 1975, the author of this paper discussed with L.E. Dubins the curiosity of the non-optimality of the

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bold play for subfair primitive casinos by using Klugman's method to handle inflation. Dubins suggested that with a more direct method of handling inflation, as used by him and Teicher in [6], it might be shown that the bold play is always optimal for subfair "Red-and-Black" lotteries and subfair primitive casinos in the presence of inflation. This is the motivation of the current research. However, the optimality (or non-optimality) of the bold play for subfair primitive casinos in this new model has not been obtained yet and the problem is still open.

2. Formulation of the Problem

In order to make our gambling problem fit easily into the gambling framework of Dubins and Savage (1965), we can consider that the price of the house is fixed at 1 but the value of fortunes will be discounted step by step by the fixed discount rate $(1+\alpha)^{-1}$, i.e., the current one dollar will be worth only $(1+\alpha)^{-1}$ dollars on the next step. Therefore our gambling problem can be formally formulated as a game whose set of fortunes, utility function, and set of available gambles are respectively as follows: $F=[0, \infty)$; $u(f)=0$ or 1 according as $0 \leqq f<1$ or $f \geqq 1$; $\Gamma(f)=\{\gamma(f, s) | \gamma(f, s)=w\delta\{(f+s)/(1+\alpha)\}+\bar{w}\delta\{(f-s)/(1+\alpha)\},\ 0 \leqq s \leqq f\}$ if $0 \leqq f<1$ and $\Gamma(f)=\{\delta(f), \gamma(f, s) | \gamma(f, s)=w\delta\{(f+s)/(1+\alpha)\}+\bar{w}\delta\{(f-s)/(1+\alpha)\},$ $0<s \leqq f\}$ if $1 \leqq f<\infty$. Here $\bar{w}=1-w$, $\delta(x)$ denotes the probability measure which assigns probability one to $\{x\}$ for all $0 \leqq x<\infty$, and α is the inflation rate (a device often encountered in dynamic programming models) for the gambler and is used to handle inflation. The reason that $\delta(f)$ is in $\Gamma(f)$ for $f \geqq 1$ is that when the gambler has a fortune $f \geqq 1$, he can buy the house immediately and he has reached his goal already.

The gambling problem formulated above is a modification of the "Red-and-Black" casinos considered by Dubins and Savage in [5] (1965). The modification is designed to handle inflation and to motivate the gambler recognizing the time value of money and completing the game as quickly as reasonably consistent with reaching the goal. To distinguish it from the "Red-and-Black", this modified game will be simply called "Red-and-Black in the presence of inflation" in this paper. When $\alpha=0$, the modified game is identical to the "Red-and-Black" considered by Dubins and Savage in [5].

Since the possibilities $1 \leqq \alpha<\infty$, or $w=0$ or $=1$, would not be interesting, we will always assume that $0 \leqq \alpha<1$ and $0<w<1$ in this paper.

3. The Utility of Modified Bold Strategies

As in "Red-and-Black" casinos, in a "Red-and-Black in the presence of inflation" gambling system, the gambler desires to reach his goal, i.e., the interval $[1, \infty)$, he can play only one "Red-and-Black" lottery with win probability w (w is fixed) in each step, and he can stake any or all of his current fortune on each game. The only difference is that the value of his current fortune f will be worth only $(1+\alpha)^{-n} f$ on the n^{th} step for all $n \geqq 0$ (his fortune will be discounted step by step by the fixed discount rate $(1+\alpha)^{-1}$).

The gambler stakes s when he uses the game $\gamma(f, s)$ at f. The modified bold stake at f is defined by $s(f)=\min\{f, 2r-f\}$ if $0 \leqq f < 1$ and $s(f)=0$ if $f \geqq 1$, where $r=\frac{1+\alpha}{2}$ is in $[\frac{1}{2}, 1)$. A gambler uses the modified bold strategy if he stakes the modified bold stake $s(f)$ whenever he has fortune f.

Let $Q(f)$ be the probability that the gambler starting from f and using the modified bold strategy reaches his goal $[1, \infty)$. It is obvious that $Q(0)=0$, $Q(f)=1$ if $f \geqq 1$, and if $0<f<1$, then, after the first game, the gambler's fortune is $\frac{f+s(f)}{1+\alpha}$ or $\frac{f-s(f)}{1+\alpha}$ with probability w or $\bar{w}$ respectively, hence

$$Q(f)=wQ\left(\frac{f+s(f)}{1+\alpha}\right)+\bar{w}Q\left(\frac{f-s(f)}{1+\alpha}\right) \quad \text{if } 0<f<1.$$

Therefore

$$Q(f)=\begin{cases} wQ\left(\frac{f}{r}\right) & \text{if } 0 \leqq f < r \\ w+\bar{w}Q\left(\frac{f-r}{r}\right) & \text{if } r \leqq f < 1 \\ 1 & \text{if } f \geqq 1. \end{cases} \tag{1}$$

Lemma 1. *The bounded solution of* (1) *is unique.*

Proof. The proof is essentially the same as that in page 99 of [5] and is omitted.

Let $\{X_n\}_{n \geqq 1}$ be a sequence of i.i.d. random variables such that $P(X_1=0) = w = 1-P(X_1=1)$. For each integer $n \geqq 1$ and each integer $1 \leqq k < n$, let $S_{n,n}=rX_n$ and $S_{n,k}=rX_k+r\min\{1, S_{n,k+1}\}$. Let $G(x)=\lim_{n\to\infty} P\{S_{n,1} \leqq x\}$ for all $x<1$ and let $G(x)=1$ for $x \geqq 1$. Since $S_{n,1}$ is non-decreasing in n, $G(x)$ is well-defined for all x.

Lemma 2. *The distribution function G satisfies* (1) *on the interval* $[0, \infty)$, *i.e.,* $G(f)=wG\left(\frac{f}{r}\right)$ *if* $0 \leqq f < r$, $G(f)=w+\bar{w}G\left(\frac{f-r}{r}\right)$ *if* $r \leqq f < 1$, *and* $G(f)=1$ *if* $f \geqq 1$.

Proof. (a) By the definition of G, $G(f)=1$ if $f \geqq 1$.

(b) If $0 \leqq f < r$, then

$$\begin{aligned} G(f) &= \lim_{n\to\infty} P\{S_{n,1} \leqq f\} = \lim_{n\to\infty} P\{S_{n,1} \leqq f, X_1=0\} + \lim_{n\to\infty} P\{S_{n,1} \leqq f, X_1=1\} \\ &= w \lim_{n\to\infty} P\{r\min\{1, S_{n,2}\} \leqq f\} = \lim_{n\to\infty} wP\left\{S_{n,2} \leqq \frac{f}{r}\right\} \end{aligned}$$

(since $S_{n,1}=rX_1+r\min\{1, S_{n,2}\}$ and $f<r$). Since $\{X_n\}$ is independent and identically distributed, the distribution of $S_{n,2}$ is the same as of $S_{n-1,1}$. Hence

$$\begin{aligned} G(f) &= \lim_{n\to\infty} P\{S_{n,1} \leqq f\} = w \lim_{n\to\infty} P\left\{S_{n,2} \leqq \frac{f}{r}\right\} = w \lim_{n\to\infty} P\left\{S_{n-1,1} \leqq \frac{f}{r}\right\} \\ &= wG\left(\frac{f}{r}\right) \quad \text{if } 0 \leqq f < r. \end{aligned}$$

(c) If $r \leqq f < 1$, then

$$G(f) = \lim_{n\to\infty} P\{S_{n,1} \leqq f\} = \lim_{n\to\infty} P\{S_{n,1} \leqq f, X_1 = 0\} + \lim_{n\to\infty} P\{S_{n,1} \leqq f, X_1 = 1\}$$
$$= w + \bar{w} \lim_{n\to\infty} P\{r \min\{1, S_{n,2}\} \leqq f - r\}$$

(since $\frac{1}{2} \leqq r < 1$, $r \leqq f < 1$, and $S_{n,1} = rX_1 + r\min\{1, S_{n,2}\}$). Since $\{X_n\}$ is independent and identically distributed, the distribution of $S_{n,2}$ is the same as of $S_{n-1,1}$. Hence

$$G(f) = \lim_{n\to\infty} P\{S_{n,1} \leqq f\} = w + \bar{w} \lim_{n\to\infty} P\{r \min\{1, S_{n,2}\} \leqq f - r\}$$
$$= w + \bar{w} \lim_{n\to\infty} P\left\{S_{n,2} \leqq \frac{f-r}{r}\right\} = w + \bar{w} \lim_{n\to\infty} P\left\{S_{n-1,1} \leqq \frac{f-r}{r}\right\}$$
$$= w + \bar{w} G\left(\frac{f-r}{r}\right) \quad \text{if } r \leqq f < 1.$$

By (a), (b), and (c), hence the distribution function G satisfies (1) on the interval $[0, \infty)$.

In view of Lemmas 1 and 2, we have the following theorem.

Theorem 1. *There is one and only one bounded function Q on the interval $[0, \infty)$ that satisfies (1). Moreover the function Q is right continuous on the interval $[0, \infty)$ and strictly increasing on the interval $[0, 1]$.*

Remark. The function Q is discontinuous on a subset A of $[0, 1]$, which is defined in the next section.

4. For $w \leqq \frac{1}{2}$, the Modified Bold Strategy is Optimal

As defined in [5], an available strategy (in our "Red-and-Black" in the presence of inflation gambling problem) for the gambler is a sequence $\sigma = (\sigma_0, \sigma_1, \sigma_2, \ldots)$ such that σ_0 is a gamble in the set $\Gamma(f)$ of available gambles and, for each positive integer n, σ_n is a gamble in the set $\Gamma(f_n)$ of available gambles, where f is the gambler's initial fortune and, for each positive integer n, f_n is the gambler's fortune on the n^{th} step.

The worth of a particular available strategy (in our "Red-and-Black" in the presence of inflation gambling system), σ, is given by its utility, $u(\sigma)$, the probability that the gambler reaches his goal by using the strategy σ and an available strategy for the gambler is optimal if no other available strategy has a higher utility.

In [5], Dubins and Savage showed that the bold strategy is optimal for a subfair "Red-and-Black" (Theorem 5-3-1 of [5]). In [8], Klugman showed that the bold play is still optimal for a subfair "Red-and-Black" with a discount factor (Theorem 2-4 of [8]). In this section, we will show that the modified bold strategy is optimal for a subfair "Red-and-Black" lottery in the presence of inflation.

For each integer $n \geqq 1$, let $A_n = \left\{ f \mid 0 < f \leqq 1, f = \sum_{j=1}^{n} c_j r^j, \text{ where } c_j = 0 \text{ or } 1 \text{ for all } 1 \leqq j \leqq n, \text{ and, for each } k = 1, 2, \ldots, n-1, \text{ if } \sum_{j=k+1}^{n} c_j r^j \geqq r^k \text{ then } c_k = 1 \right\}$ and let

$A=\bigcup_{n=1}^{\infty} A_n$. Any number f in A is called a finite order (in r) number and the smallest positive integer n such that $f \in A_n$ is called the order (in r) of f.

Lemma 3. *If* $\frac{1}{2} \leqq r < 1$, *then A is a dense subset of* $[0, 1]$.

Proof. Let B be the set of all binary rationals in $[0, 1]$, Q_r be the utility function of the bold strategy in a "Red-and-Black" with win probability r (see [5, pp. 84–86]), and $C_r = Q_r(B)$ the image of B under Q_r. Since B is a dense subset of $[0, 1]$ and Q_r is a homeomorphizm from $[0, 1]$ into $[0, 1]$, C_r is a dense subset of $[0, 1]$. Now it is easy to check that, for each number x in C_r and each $\varepsilon > 0$, there is a number f in A such that $|f-x| < \varepsilon$. Hence A is a dense subset of $[0, 1]$.

Now we are in the position to show that the modified bold strategy is optimal in a subfair "Red-and-Black" in the presence of inflation gambling system.

Theorem 2 (The Optimality of the Modified Bold Strategy). *If* $\alpha \geqq 0$ *and* $0 \leqq w \leqq \frac{1}{2}$, *then the modified bold strategy is optimal.*

Proof. Since the case that $\alpha \geqq 1$ is obvious, we will always assume that $0 \leqq \alpha < 1$ in the proof.

In view of Theorem 2-12-1 of [5], it suffices to show that Q is excessive, i.e.,

$$Q(f) \geqq w\,Q((f+s)/2r) + \bar{w}\,Q((f-s)/2r), \tag{2}$$

for $0 \leqq f-s \leqq f \leqq 1$, $0 \leqq \alpha < 1$, and $0 \leqq w \leqq \frac{1}{2}$, where $\frac{1}{2} \leqq r = \dfrac{1+\alpha}{2} < 1$ and it is to be understood that $Q((f+s)/2r) = 1$ if $f+s > 2r$.

The possibility that $f+s > 2r$ can be set aside. For in view of the monotony of Q (Theorem 1 of Section 3), if (2) were to fail for some $f+s > 2r$, it would also fail for $f+s = 2r$ since $Q(f) = 1$ if $f \geqq 1$. Since $1 \leqq 2r$, $0 \leqq s \leqq f$, and Q is non-decreasing (Theorem 1 of Section 3), (2) holds if $s=0$, or $s=f$, or $f=1$, or $f+s=2r$, or $f+s=2rf$. Therefore, throughout the rest of the proof, we will always assume that $0 < s < f < 1$, $f+s < 2r$, and $2rf < f+s$.

Now we show that (2) holds in the following two steps.

Step 1. For all f and s in the set A, (2) holds if $0 < s < f < 1$, $f+s < 2r$, and $2rf < f+s$.

Now we prove Step 1 by induction on the orders of f and s (in r).

(i) If both f and s are of order 1 (in r), then $f = s = r$ and (2) holds.

(ii) Now assume that (2) holds for all f' and s' in the set A of order less than n (with the stated property) and let f and s be numbers in the set A of order less than or equal to n. To show that (2) holds for these values it will be easier to write (2) as

$$Q(f) - w\,Q((f+s)/2r) - \bar{w}\,Q((f-s)/2r) \geqq 0. \tag{3}$$

There are four cases that must be considered.

Case I. $0 < (f-s)/2r < f < (f+s)/2r \leqq r$.

(a) If $f+s = 2r^2$, then $r^2 < f < r$ (since $0 < s < f$) and $Q(f) - w\,Q((f+s)/2r) - \bar{w}\,Q((f-s)/2r) = w^2 + w\bar{w}\,Q((f-r^2)/r^2) - w^2 - w\bar{w}\,Q((f-s)/2r^2) = 0$ since $f+s = 2r^2$. So (3) holds.

(b) If $f+s<2r^2$, then

$$Q(f)-wQ((f+s)/2r)-\bar{w}Q((f-s)/2r)$$
$$=w\left\{Q(f/r)-wQ\left[\left(\frac{f+s}{r}\right)/2r\right]-\bar{w}Q\left[\left(\frac{f-s}{r}\right)/2r\right]\right\}.$$

Since $0<s<f<r$ and s, f are in the set A, $\frac{f}{r}$ and $\frac{s}{r}$ are in the set A of order less than n. By inductive hypothesis,

$$Q\left(\frac{f}{r}\right)-wQ\left[\left(\frac{f+s}{r}\right)/2r\right]-\bar{w}Q\left[\left(\frac{f-s}{r}\right)/2r\right]\geqq 0.$$

So (3) holds.

Case II. $r\leqq(f-s)/2r<f<(f+s)/2r<1$.
$s<r-r^2\leqq r^2$ since $2r^2<f-s$, $f+s<2r$, and $\frac{1}{2}\leqq r<1$. Hence

$$Q(f)-wQ\left(\frac{f+s}{2r}\right)-\bar{w}Q\left(\frac{f-s}{2r}\right)=w+\bar{w}Q\left(\frac{f-r}{r}\right)-w^2-w\bar{w}Q\left(\frac{f+s-2r^2}{2r^2}\right)$$
$$-w\bar{w}-\bar{w}^2Q\left(\frac{f-s-2r^2}{2r^2}\right)\geqq\bar{w}\{Q[(f-r)/r]-wQ[(f-r+s)/2r^2]$$
$$-\bar{w}Q[(f-r-s)/2r^2]\}$$

(since $r\leqq 2r^2$ and Q is non-decreasing). Since $r<f$, $s<r^2$, $r\leqq 2r^2<f-s$, and f, s are in the set A, $(f-r)/r$ and $\frac{s}{r}$ are in the set A of order less than n such that $\frac{s}{r}\leqq\frac{f-s}{r}$. By inductive hypothesis,

$$Q\left(\frac{f-r}{r}\right)-wQ((f-r+s)/2r^2)-\bar{w}Q((f-r-s)/2r^2)\geqq 0.$$

So (3) holds.

Case III. $0<(f-s)/2r<f<r\leqq(f+s)/2r<1$.
$r^2<f$ since $s<f$ and $f+s\geqq 2r^2$. Hence

$$Q(f)-wQ\left(\frac{f+s}{2r}\right)-\bar{w}Q\left(\frac{f-s}{2r}\right)=w^2+w\bar{w}Q((f-r^2)/r^2)-w^2$$
$$-w\bar{w}Q((f+s-2r^2)/2r^2)-w\bar{w}Q((f-s)/2r^2)=\bar{w}\{Q((f-r^2)/r)$$
$$-wQ((f+s-2r^2)/2r^2)-wQ((f-s)/2r^2)\}=H(s)$$

(since $f-r^2<r^2$, $Q((f-r^2)/r)=wQ((f-r^2)/r^2)$). Since $f+s\geqq 2r^2$, $2r^2-f\leqq s$. It suffices to show that $H(s)\geqq 0$ for $2r^2-f\leqq s\leqq f$. If f, s are in the set A and $r^2\leqq s\leqq f<r$, then $(f-r^2)/r$ and $(s-r^2)/r$ are in the set A of order less than n. Hence, by inductive hypothesis, $H(s)=\bar{w}\{Q((f-r^2)/r)-wQ([(f-r^2)+(s-r^2)]/2r^2)$ $-wQ([(f-r^2)-(s-r^2)]/2r^2)\}\geqq 0$ for $r^2\leqq s\leqq f<r$ since $w\leqq\frac{1}{2}$. Now notice that $H(s)$ is symmetric about $s=r^2$ on the interval $[2r^2-f,f]$, hence $H(s)\geqq 0$ on the interval $[2r^2-f,f]$ and (3) holds.

Case IV. $0<(f-s)/2r<r\leqq f<(f+s)/2r<1$.

$r\leqq f<r+r^2$ since $f-s<2r^2$ and $f+s<2r$. Hence

$$Q(f)-wQ\left(\frac{f+s}{2r}\right)-\bar{w}Q\left(\frac{f-s}{2r}\right)=w+w\bar{w}Q((f-r)/r^2)-w^2$$
$$-w\bar{w}Q((f+s-2r^2)/2r^2)-w\bar{w}Q((f-s)/2r^2)=w\bar{w}+w\bar{w}Q((f-r)/r^2)$$
$$-w\bar{w}Q((f+s-2r^2)/2r^2)-w\bar{w}Q((f-s)/2r^2).$$

(a) If $2r-r^2\leqq f$ or $s\leqq 2r^2-r$, then $(f-r)/r^2\geqq(f+s-2r^2)/2r^2$. Hence $w\bar{w}+w\bar{w}Q((f-r)/r^2)-w\bar{w}Q((f+s-2r^2)/2r^2)-w\bar{w}Q((f-s)/2r^2)\geqq 0$ and (3) holds since $Q(f)\leqq 1$ for all $f\geqq 0$.

(b) If $2r^2-r<s<r$ and $f<2r-r^2$ ($s<r$ since $f+s<2r$ and $f\geqq r$), then

$$Q(f)-wQ((f+s)/2r)-\bar{w}Q((f-s)/2r)$$
$$=w\{(\bar{w}-w)+Q((f-r+r^2)/r)-\bar{w}Q((f+s-2r^2)/2r^2)-\bar{w}Q((f-s)/2r^2)\}=H(s).$$

Now it suffices to show that $H(s)\geqq 0$ for $2r^2-r<s<r$ and $r\leqq f<2r-r^2$. Since $H(s)$ is symmetric about $s=r^2$ on the interval $(2r^2-r, r)$, without loss of generality, it is sufficient to show that $H(s)\geqq 0$ for $r^2\leqq s<r$ and $r\leqq f<2r-r^2$. Now, if $r^2\leqq s<r$ and $r\leqq f<2r-r^2$, then

$$H(s)\geqq w\{Q((f-r+r^2)/r)-wQ((f+s-2r^2)/2r^2)-\bar{w}Q((f-s)/2r^2)\}$$
$$\geqq w\{Q((f-r+r^2)/r)-wQ([(f-r+r^2)+(s-r^2)]/2r^2)$$
$$-\bar{w}Q([(f-r+r^2)-(s-r^2)]/2r^2)\}$$

since $\frac{1}{2}\leqq r<1$ and Q is non-decreasing. Since $r\leqq f<2r-r^2$, $r^2\leqq s<r$, f and s are in the set A, $(f-r+r^2)/r$ and $(s-r^2)/r$ are in the set A of order less than n $(f-r+r^2\geqq s-r^2)$. By inductive hypothesis,

$$Q((f-r+r^2)/r)-wQ([(f-r+r^2)+(s-r^2)]/2r^2)$$
$$-\bar{w}Q([(f-r+r^2)-(s-r^2)]/2r^2)\geqq 0.$$

Hence $H(s)\geqq 0$ for $2r^2-r<s<r$ and $r\leqq f<2r-r^2$ and (3) holds.

Step 2. For all $0\leqq s\leqq f<1$, (2) holds.

To see Step 2, we choose two non-increasing sequences $\{f_m\}$ and $\{s_m\}$ from the set A (as is possible by Lemma 3) such that

(a) $f_m\geqq f$ for all $m\geqq 1$ and $\lim_{m\to\infty} f_m=f$.

(b) $s_m\geqq s$ for all $m\geqq 1$ and $\lim_{m\to\infty} s_m=s$.

(c) $s_m\leqq f_m$ for all $m\geqq 1$.

(d) $f-s\leqq f_{m+1}-s_{m+1}\leqq f_m-s_m$ for all $m\geqq 1$.

Now, by the right continuity of Q (Theorem 1 of Section 3) and Step 1 above, (2) holds for all f and s such that $0\leqq s\leqq f<1$. The proof of Theorem 2 now is complete.

Remark. By a similar argument used in [5], we can also show that the optimal strategy is not unique, i.e., there are some other strategies that are also optimal for a subfair "Red-and-Black" in the presence of inflation.

5. For $\frac{1}{2}<w<1$ and $0\leqq\alpha<1$, the Modified Bold Strategy is not Necessarily Optimal even if $0<w/1(1+\alpha)\leqq 1/2$

As in [8], the case discussed in Section 4 was clearly subfair. However, with $\frac{1}{2}<w$ it is no longer clear how the game should be described. Each step the gambler's expected fortune is increasing (when the game is viewed as a "Red-and-Black" with moving goal). But when $w/(1+\alpha)\leqq\frac{1}{2}$ the process of fortunes will be a supermartingale since $E(f_{n+1}|f_n)=w(1+\alpha)^{-1}(f_n+s)+\bar{w}(1+\alpha)^{-1}(f_n-s)\leqq 2w(1+\alpha)^{-1}f_n\leqq f_n$ and the optimal sampling theorem (Theorem 5-10 of [2], Breiman (1968)) gives the result $U(f)<f$ for all $0<f<1$ indicating that the game is subfair (see page 74 of [5]), where $U(f)$ is the utility function of the game (see page 25 of [5]). One would suspect that if $w/(1+\alpha)\leqq\frac{1}{2}$, then the modified bold strategy should be still optimal even if $0\leqq\alpha<1$ and $\frac{1}{2}<w<1$. However, unlike the extremely subfair game of Section 4, if $0\leqq\alpha<1$ and $\frac{1}{2}<w<1$, then the modified bold strategy is no longer optimal even if $w/(1+\alpha)\leqq\frac{1}{2}$. The following theorem justifies this statement, which also provides us with a counterexample to an early gambling result obtained by Coolidge (1908–9). Coolidge [4] stated:

"The player's best chance of winning a certain sum at a disadvantageous game is to stake the sum that will bring him that return in one play, or, if that be not allowed, to make always the largest stake which the banker will accept."

In [9], Klugman also provided a counterexample to this early gambling result. A different counterexample has been provided in [7] by Heath, Pruitt, and Sudderth. By these counterexamples, we may have a better understanding of subfair games.

Theorem 3. *If* $0\leqq\alpha<1$ *and* $\frac{1}{2}<w<1$, *then the modified bold strategy is no longer optimal even if* $w/(1+\alpha)\leqq\frac{1}{2}$.

Proof. In view of Theorem 2-14-1 of [5], it sufficies to show that Q is not excessive, i.e.,

$$Q(f)-wQ\left(\frac{f+s}{2r}\right)-\bar{w}Q\left(\frac{f-s}{2r}\right)<0 \quad \text{for some } 0\leqq s\leqq f\leqq 1. \tag{4}$$

Since $\frac{1}{2}\leqq r<1$, there exists a positive integer $n\geqq 2$ such that $r+r^n+r^{n+1}<1$. Now let $f=r^2+r^{n+1}+r^{2n}$ and $s=r^2+r^{n+1}-r^{2n}$, then $0<s<f<1$, $Q(f)-wQ\left(\frac{f+s}{2r}\right)-\bar{w}Q\left(\frac{f-s}{2r}\right)=\bar{w}w^{2n-2}(1-2w)<0$ since $\frac{1}{2}<w<1$. Hence the modified bold strategy is not optimal.

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