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Controlling of Non-Recurrent Lattice Random Walks

By

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1. Introduction

Let $X_n = (X_{n1}, \ldots, X_{nd})$, $n = 1, 2, \ldots$, be independent, identically distributed, d-dimensional random vectors with integer-valued components. Throughout this paper we assume the existence and finiteness of the vector of expectations $EX_1 = (EX_{11}, \ldots, EX_{1d})$. We shall interpret the stochastic sequence $S = \{S_n\}$, defined by

$$S_n = \sum_{\nu=1}^{n} X_\nu, \qquad n = 1, 2, \ldots, \tag{1}$$

as a (generalized) random walk in the Euclidean space R_d. In most cases, we shall be concerned only with linear and planar random walks.

If $EX_1 = 0$, $[EX_1 = 0$, and $E\,|X_1|^2 < \infty]$, then by CHUNG and FUCHS [3] every possible state of the linear [planar] random walk S is recurrent. If $EX_1 \neq 0$, then it is well known that S does not possess any recurrent state. Furthermore, an investigation of this case by CHUNG and DERMAN [2] showed: if S is a linear random walk having all positive integers as possible states and if $0 < EX_1 < \infty$, then any infinite set of positive integers is visited by S infinitely often with probability 1.

In this paper we want to make another study of the case where $EX_1 \neq 0$, $d = 1, 2$. Centering the X_n at expectations leads to a new random walk $S^* = \{S_n^*\}$, defined by

$$S_n^* = \sum_{\nu=1}^{n} (X_\nu - EX_1), \tag{2}$$

for which all possible states are recurrent, but which in general no longer proceeds in the lattice. This suggests the following question: Does there always exist a sequence of constant vectors c_n with integer components, such that the random walk $V = \{V_n\}$, given by

$$V_n = \sum_{\nu=1}^{n} (X_\nu + c_\nu), \tag{3}$$

possesses recurrent states? The substitution of S by V may be interpreted as a *control* of S: A particle which performs the random walk S undergoes during the n-th step an additional translation c_n. Our aim is to determine c_n in such a way, that the particle is prevented from drifting away. It is important that c_n is a constant vector and hence does not depend on the position of the particle at time n. This means that V is a spatially homogeneous and temporarily inhomogeneous random walk. Recurrence problems for spatially inhomogeneous and temporarily homogeneous random walks have been studied by several authors (see e.g. [5]).

We derive a necessary criterion and a sufficient criterion for the existence of recurrent states in V. In particular, we show that any linear [planar] random walk S, for which X_1 has a genuine one-dimensional [two-dimensional] distribution, may be controlled in infinitely many ways such that all lattice points become recurrent states.

The proof of the main result (theorem 2, sufficient criterion) is based on three facts: (i) the cited result of CHUNG and FUCHS; (ii) a generalization of a theorem of CHUNG on Markov chains with stationary transition probabilities to Markov chains with non-stationary transition probabilities; (iii) an analysis of the set of possible states of the random walk S (lemmas 2—4).

2. Definitions

Let $U = \{U_n\}$ be an arbitrary sequence of d-dimensional random vectors $U_n = (U_{n1}, \ldots, U_{nd})$. We use the following *definitions* (cf. [3]):

(a) The point $a \in R_d$ is a *possible value* of U_n, if $P(|U_n - a| < \varepsilon) > 0$ for every $\varepsilon > 0$.

(b) The point $a \in R_d$ is a *possible state* of U, if for every $\varepsilon > 0$ there exists an $n = n(\varepsilon)$ such that $P(|U_n - a| < \varepsilon) > 0$.

(c) The point $a \in R_d$ is a *recurrent state* of U, if $P(|U_n - a| < \varepsilon \text{ i.o.}) = 1$ for every $\varepsilon > 0$; or equivalently, if a is a limit point of the sequence $\{U_n\}$ with probability 1. In the case of lattice random vectors these definitions coincide with the usual ones.

It is sometimes convenient to describe V in such a way that only the non-vanishing ones of the c_ν — denote them by $c_{n_1}, c_{n_2}, \ldots,$ — are used. (In the sequel we consider only those random walks V, for which infinitely many of the c_ν do not vanish, since this is the only interesting case.) Put $n_0 = 0$, $c_0 = 0$, $S_0 \equiv 0$; and for $k = 0, 1, \ldots$ put $C_k = \sum_{j=0}^{k} c_{n_j}$. Then V is given by

$$V_{n_k+m} = S_{n_k+m} + C_k, \quad \begin{array}{l} k = 0, 1, \ldots, \\ m = 0, 1, \ldots, n_{k+1} - n_k - 1. \end{array} \tag{4}$$

We call the numbers n_k *control times* and the vectors c_{n_k} *control values*.

3. Necessary conditions for recurrent states

Theorem 1. *Assume that the random walk* $V = \{V_n\}$, $V_n = \sum_{\nu=1}^{n} (X_\nu + c_\nu)$, *possesses a recurrent state.*

(i) *If* $E|X_1|^r < \infty$ *for some* r, $1 \leqq r < 2$, *then*

$$\varliminf_{n} \frac{1}{n^{1/r}} \left| n E X_1 + \sum_{1}^{n} c_\nu \right| = 0.$$

In particular, in any case $\varliminf_{n} \left| E X_1 + \frac{1}{n} \sum_{1}^{n} c_\nu \right| = 0$.

(ii) *If* $E|X_1|^r < \infty$ *for some* r, $1 \leqq r \leqq 2$, $b_n \uparrow \infty$ *and* $\sum_{1}^{\infty} b_n^{-r} < \infty$, *then*

$$\varliminf_{n} \frac{1}{b_n} \left| n E X_1 + \sum_{1}^{n} c_\nu \right| = 0.$$

(iii) *If* $\lim_{k\to\infty} \frac{n_{k+1}}{n_k} = 1$, *then*

$$\underline{\lim}_{k} \left| E X_1 + \frac{1}{n_k} \sum_{j=1}^{k} c_{n_j} \right| = 0 .$$

Proof. Since $\max_{\nu=1,\dots,d} |X_{1\nu}| \leqq |X_1|$, the conditions on the moments of $|X_1|$ imply the corresponding conditions for all components of X_1. In (i), by KOLMOGOROV-MARCINKIEWICZ (see e.g. [6], p. 243), we have

$$\frac{1}{n^{1/r}} \sum_{1}^{n} (X_\nu - E X_\nu) \equiv \frac{1}{n^{1/r}} (V_n - E V_n) \to 0 \text{ a. s.}$$

In (ii), by LOÈVE [6], p. 241, we have $\frac{1}{b_n} (V_n - E V_n) \to 0$ a.s. Putting $b_n = n^{1/r}$ in case (i), the definition of almost sure convergence yields in both cases

$$P(|V_n - E V_n| \geqq \varepsilon b_n \text{ i.o.}) = 0 \text{ for every } \varepsilon > 0 . \tag{5}$$

If a is now a recurrent state of V, then by definition

$$P(|V_n - E V_n| \geqq |E V_n - a| - 1 \text{ i.o.}) \geqq P(|V_n - a| < 1 \text{ i.o.}) = 1 . \tag{6}$$

The comparison of (5) and (6) implies that for infinitely many n, $|E V_n - a| - 1 < \varepsilon b_n$; therefore $|E V_n| \leqq |E V_n - a| + |a| < \varepsilon b_n + |a| + 1$ for infinitely many n. As $b_n \to \infty$, $\underline{\lim}_{n} \frac{|E V_n|}{b_n} \leqq 2\varepsilon$. This being valid for every $\varepsilon > 0$, we get

$$\underline{\lim}_{n} \frac{|E V_n|}{b_n} = 0 . \tag{7}$$

From (7), (i) and (ii) follow immediately. To prove (iii), we note that (i), $r = 1$, applies. Hence there exists a sequence of positive integers $l_s \uparrow \infty$ such that

$$\frac{1}{l_s} \sum_{\nu=1}^{l_s} c_\nu \to -E X_1 .$$

One can choose l_s in such a way that each interval $(l_{s-1}, l_s]$ contains at least one of the n_k. Denote by n_{k_s} the largest of the n_k lying in that interval. From

$$1 \leqq \frac{l_s}{n_{k_s}} < \frac{n_{k_s+1}}{n_{k_s}} \to 1 \quad (s \to \infty)$$

and $\frac{1}{n_{k_s}} \sum_{j=1}^{k_s} c_{n_j} = \frac{l_s}{n_{k_s}} \frac{1}{l_s} \sum_{1}^{l_s} c_\nu$, we get $\frac{1}{n_{k_s}} \sum_{j=1}^{k_s} c_{n_j} \to -E X_1$, which proves (iii).

Remarks. 1. The conditions in theorem 1 are in general not sufficient for the existence of recurrent states in V. Take, for example, $X_1 = c$ (c some integer), and $c_n = [\sqrt{n}] - [\sqrt{n-1}] - c$, where $[b]$ denotes the greatest integer $\leqq$ b . Then

$$\underline{\lim}_{n} \left| E X_1 + \frac{1}{n} \sum_{1}^{n} c_\nu \right| = \underline{\lim}_{n} \frac{[\sqrt{n}]}{n} = 0, \text{ but } V_n = [\sqrt{n}] \to \infty \text{ a.s.}$$

2. In general, the existence of a recurrent state does not tell us anything about $\overline{\lim}_{n} \left| E X_1 + \frac{1}{n} \sum_{1}^{n} c_\nu \right|$. Take for example $X_1 = c$ (c some integer). Then, for any b,

$0 \leqq b \leqq \infty$, there exists a sequence of integers c_n such that V has a recurrent state and

$$\overline{\lim_n} \left| EX_1 + \frac{1}{n} \sum_1^n c_\nu \right| = b .$$

3. In the proof of theorem 1 we did not need the assumption that X_1 has integer-valued components. Furthermore, part (ii) does not require that the X_n are identically distributed. This remark and a modification of the conditions on the moments of the X_n (see LOÈVE [6], p. 241) lead to the following recurrence criterion for sums of independent random vectors.

Corollary. *Let $Y_1, Y_2, \ldots$ be independent d-dimensional random vectors. Assume there are two sequences of real numbers $b_n \uparrow \infty$ and r_n, $1 \leqq r_n \leqq 2$, such that* $\sum_{n=1}^{\infty} \frac{E|Y_n|^{r_n}}{b_n^{r_n}} < \infty$. *Then, if the sequence $U_n = \sum_{\nu=1}^{n} Y_\nu$ possesses a recurrent state,*

$$\underline{\lim_n} \frac{|EU_n|}{b_n} = 0 .$$

4. A lemma on Markov chains

The following lemma is a generalization of a theorem of CHUNG (see e.g. [2], theorem 1) to the case where the Markov chain does not have stationary transition probabilities.

Lemma 1. *Let $U = \{U_n, n \geqq 0\}$ be a Markov chain, not necessarily with stationary transition probabilities. Let Z be the minimal state space of U. For $B \subset Z$, $i \in Z$ and $m \geqq 0$ define, if $P(U_m = i) > 0$,*

$$f_{i,B}(m) = P \ (U_n \in B \text{ for some } n > m \,|\, U_m = i).$$

Then the following statement holds: If $P(U_n \in A \text{ i.o.}) = 1$ and $\inf_{i \in A, m \geqq 0} f_{i,B}(m) > 0$, then $P(U_n \in B \text{ i.o.}) = 1$★.

As the *proof* is just a modification and extension of the proof of theorem 5 in [1], p. 19, we indicate only the necessary alterations.

Put $\alpha = \inf_{i \in A, m \geqq 0} f_{i,B}(m)$. Then for $0 < N' < N$

$$P \ (U_n \in A \text{ for some } n \geqq N;\ U_n \notin B \text{ for } n \geqq N')$$

$$= \sum_{i \in A} \sum_{m=N}^{\infty} P \ (U_n \notin A \text{ for } N \leqq n < m;\ U_m = i;\ U_n \notin B \text{ for } n \geqq N') \leqq$$

$$\leqq \sum_{i \in A} \sum_{m=N}^{\infty} P(U_n \notin A \text{ for } N \leqq n < m;\ U_m = i;\ U_n \notin B \text{ for } N' \leqq n \leqq N;$$
$$U_n \notin B \text{ for } n > m)$$

★ For inf and $\sum$ only those m are considered, for which the corresponding conditional probabilities are defined.

$$= \sum_{i \in A} \sum_{m=N}^{\infty} P\,(U_n \notin A \text{ for } N \leqq n < m;\ U_m = i;\ U_n \notin B \text{ for } N' \leqq n \leqq N)$$
$$\times P\,(U_n \notin B \text{ for } n > m \,|\, U_m = i)$$
$$\leqq (1-\alpha)\, P\,(U_n \in A \text{ for some } n \geqq N;\ U_n \notin B \text{ for } N' \leqq n \leqq N).$$

The rest follows then by letting $N \to \infty$ and $N' \to \infty$.

5. Analysis of the set of possible values of S_n

If one tries to apply the preceding lemma to our recurrence problem, it turns out, that the verification of the condition $\inf\limits_{i \in A, m \geqq 0} f_{i,B}(m) > 0$ requires some information about the manner, in which the probability mass of S_n spreads over the line (or the plane) as $n \to \infty$. In this section we state only the pertinent results. The proofs for lemmas 2—4, which require several facts from the theory of diophantine equations, are part of a forthcoming paper [4] which deals with other problems on the set of possible values of S_n. In order to avoid repetition, we shall describe the properties of the sets in question for d-dimensional random walks, though we shall make use of it only for $d = 1$ and $d = 2$.

Denote by L_0 the set of all points in the Euclidean space R_d with integer coordinates. For any $d + 1$ points $b_i = (b_{i1}, \dots, b_{id}) \in L_0$, $i = 1, 2, \dots, d + 1$, we put

$$\Delta \equiv \Delta(b_1, \dots, b_d) = \begin{vmatrix} b_{11} & b_{12} & \dots & b_{1d} \\ b_{21} & b_{22} & \dots & b_{2d} \\ \dots & \dots & \dots & \dots \\ b_{d1} & b_{d2} & \dots & b_{dd} \end{vmatrix}$$

and

$$D \equiv D(b_1, \dots, b_{d+1}) = \begin{vmatrix} b_{11} & b_{12} & \dots & b_{1d} & 1 \\ b_{21} & b_{22} & \dots & b_{2d} & 1 \\ \dots & \dots & \dots & \dots & \dots \\ b_{d+1,1} & b_{d+1,2} & \dots & b_{d+1,d} & 1 \end{vmatrix}.$$

If $b_1, b_2, \dots, b_s$, $1 \leqq s \leqq d$, are linearly independent d-dimensional (constant) vectors, any set of the form $L \equiv L(b_0; b_1, \dots, b_s) = \{b : b = b_0 + \sum_1^s y_i b_i,\ b_0 \in L_0,$ y_i arbitrary integers$\}$ is called a s-dimensional sublattice of L_0. A lattice has many different representations. It is not difficult to see that for $s = d$ the positive integer $m_L = |\Delta(b_1, \dots, b_d)|$ is invariant under the class of all representations of the lattice $L(b_0; b_1, \dots, b_d)$. Hence m_L is a geometrical property of the lattice, namely the content of a mesh of the lattice.

Let M_n and M be the set of possible values of S_n and possible states of S respectively. Then $M = \bigcup\limits_{n=1}^{\infty} M_n$, and M_n and M are uniquely determined by M_1, the set of possible values of X_1, for

$$M_n = \{z : z = \sum_1^n a_\nu,\ a_\nu \in M_1\}.$$

For this and the following sections we make the assumption, that X_1 has a genuine d-dimensional distribution, i.e., that M_1 contains $d+1$ points for which the determinant D is not equal to zero. Then there exist the greatest common divisor* (g.c.d.) g of the determinants Δ and the g.c.d. h of the determinants D, all determinants taken for the points in M_1. Obviously g divides h.

Lemma 2. *There exists a unique smallest d-dimensional lattice L_S containing M. The mesh-content of L_S is equal to g. In particular, L_S is a proper sublattice of L_0 if and only if $g > 1$.*

Lemma 3. *If $g = 1$, then there exists a finite subset M_1' of M_1 with the following property: to every positive integer l there exist a positive integer $n(l)$ and a d-dimensional cube $W(l)$ (the position of which may depend on l) with sides of length l, such that*

$$L_0 \cap W(l) \subset \bigcup_{t=1}^{h} \{b : b = \sum_{i=1}^{n(l)+t} a_i, a_i \in M_1'\} \subset \bigcup_{t=1}^{h} M_{n(l)+t}. \tag{8}$$

Lemma 4. *If $g = 1$, then L_0 is the disjoint union of h congruent lattices $L_1, \ldots, L_h$, which have mesh-content h. If $n \equiv t \pmod h$, then $M_n \subset L_t$, $t = 1, 2, \ldots, h$.*

The lattice L_S in Lemma 2 will be called the *minimal lattice* of S. Lemma 3 is of special importance for our problem. The relation (8), in which the second inclusion needs no proof, tells us e.g. in the case $h = 1$, that the possible values of S_n 'cluster' in the following sense: as $n \to \infty$, the lattice points of larger and larger cubes can be reached in n steps, even if one uses in the random walk only some finite set of possible values of X_1.

6. Sufficient conditions for recurrent states

Theorem 2. *Let S be the linear or planar random walk (1), and assume $E|X_1|^2 < \infty$ in the planar case. If $\{n_k\}$ is any sequence of control times, then all points of the minimal lattice L_S are recurrent states of the random walk V, defined by*

$$V_{n_k+m} = S_{n_k+m} + C_k, \tag{4}$$

if the following conditions are satisfied:

(i) $C_k = -n_k EX_1 + 0(1), \ (k \to \infty)$;

(ii) $\sup_k (n_{k+1} - n_k) < \infty$;

(iii) *there exists an integer $K > 0$ such that for $s = 1, 2, \ldots$*

$$\max_{k=s,s+1,\ldots,s+K} (n_{k+1} - n_k) \geqq \frac{h}{g}.$$

Remarks. 1. Condition (i) is plausible, for it says that one selects C_k in such a way that $|EV_{n_k}|$ cannot go to infinity. 2. Condition (iii)** cannot be totally

* We use the usual convention, that if a and b are non-negative integers, then g.c.d. $(a, -b)$ = g.c.d. $(a, b) \geqq 0$ and g.c.d. $(a, 0)$ = g.c.d. $(a) = a$.

** It is easy to see that theorem 2 remains valid if condition (iii) is only satisfied for all but a finite number of the positive integers s.

dispensed with, as is seen by the following example. Take $M_1 = \{1, 4, 7\}$, and assign probabilities to the points of M_1 such that $EX_1 = 2.5$. Then take $n_k = 2k$, $C_k = -n_k EX_1 = -5k$. Conditions (i) and (ii) are satisfied, but none of the points $2 + 3\lambda$, where λ is an arbitrary integer, are in M; hence none of them is recurrent, while all of them belong to $L_S = L_0$.

The *proof* of theorem 2 is carried through in three steps. At first we compare the controlled random walk

$$V_{n_k+m} = S_{n_k+m} + C_k, \quad \begin{matrix} k = 0, 1, \ldots, \\ m = 0, 1, \ldots, n_{k+1} - n_k - 1, \end{matrix} \tag{4}$$

with the centered random walk

$$\begin{aligned} S_n^* &= S_n - nEX_1, \qquad n = 1, 2, \ldots, \\ S_0^* &\equiv 0. \end{aligned} \tag{2}$$

If we put $\eta(k) = C_k + n_k EX_1$, our assumptions imply the existence of constants T_1, T_2 such that $n_{k+1} - n_k \leqq T_1 < \infty$ and $|\eta(k)| \leqq T_2 < \infty$. Hence

$$\begin{aligned} |V_{n_k+m} - S_{n_k+m}^*| = |C_k + n_k EX_1 + mEX_1| \leqq |\eta(k)| + m\,|EX_1| < \\ < T_2 + T_1\,|EX_1| < \infty, \end{aligned}$$

therefore, for some constant T,

$$\sup_n |V_n - S_n^*| \leqq T < \infty. \tag{9}$$

Let b be any possible value of X_1. Then $P(S_1^* = b - EX_1) = P(X_1 = b) > 0$, which means that $b' = b - EX_1$ is a possible state of S^*. Now S_n^* is a sum of independent, identically distributed random vectors $X_\nu^* = X_\nu - EX_1$, for which $EX_1^* = 0$ and $E|X_1^*| < \infty$ in the linear case, and $EX_1^* = 0$, $E|X_1^*|^2 < \infty$ in the planar case. By the result of Chung and Fuchs [3], b' is then a recurrent state of S^*, which implies on account of (9) that

$$P(|V_n - b'| < 2T \text{ i.o.}) \geqq P(|S_n^* - b'| < T \text{ i.o.}) = 1. \tag{10}$$

In the second step we apply lemma 1 of section 4. We have now to distinguish between the linear and the planar random walk. The proof will be continued only for the first of these two cases, the other being completely analogous. We remark that it is sufficient to prove theorem 2 for the case $g = 1$, the general case can be reduced easily to the special case by replacing the lattice L_0 by the lattice L_S. If we take now in Lemma 1 for A the set of all lattice points in the interval $[b' - 2T, b' + 2T]$, and for B an arbitrary lattice point a', then equation (10) and the finiteness of A imply, that a' is a recurrent state of V, if for any $a \in A$

$$\inf_\nu f_{a,a'}(\nu) > 0. \tag{11}$$

The verification of (11) will be based on the fact that $f_{a,a'}(\nu)$ can be represented in the form

$$f_{a,a'}(\nu) = P(S_\varrho = a' - a + \varrho EX_1 - \zeta(\nu, \varrho) \quad \text{for some} \quad \varrho > 0), \tag{12}$$

where $\zeta(\nu, \varrho)$ is a bounded double-sequence. This is proved in the following way.

We have

$$
\begin{aligned}
f_{a,a'}(\nu) &= P(V_n = a' \quad \text{for some} \quad n > \nu \mid V_\nu = a) \\
&= P(S_n = a' - \sum_0^n c_i \quad \text{for some} \quad n > \nu \mid S_\nu = a - \sum_0^\nu c_i) \\
&= P(S_n - S_\nu = a' - a - \sum_{\nu+1}^n c_i \quad \text{for some} \quad n > \nu) \\
&= P(S_{n-\nu} = a' - a - \sum_{\nu+1}^n c_i \quad \text{for some} \quad n > \nu) \\
&= P(S_\varrho = a' - a - \sum_{i=0}^{\varrho-1} c_{\nu+1+i} \quad \text{for some} \quad \varrho > 0) \\
&= P(S_\varrho = a' - a + \varrho EX_1 - \zeta(\nu,\varrho) \quad \text{for some} \quad \varrho > 0),
\end{aligned}
$$

where $\zeta(\nu,\varrho) = \sum_{i=0}^{\varrho-1} (c_{\nu+1+i} + EX_1)$.

Now define $n_{k(\nu)}$ to be the largest of the numbers n_k which are $\leqq \nu$. From $\eta(k) = C_k + n_k EX_1$, we get $c_{n_k} = (n_{k-1} - n_k) EX_1 + \eta(k) - \eta(k-1)$, which yields

$$
\begin{aligned}
|\zeta(\nu,\varrho)| &= \Big| \sum_{\nu+1 \leqq n_k \leqq \nu+\varrho} c_{n_k} + \varrho EX_1 \Big| \\
&= \Big| EX_1 \sum_{\nu+1 \leqq n_k \leqq \nu+\varrho} (n_{k-1} - n_k) + \varrho EX_1 + \sum_{\nu+1 \leqq n_k \leqq \nu+\varrho} (\eta(k) - \eta(k-1)) \Big| \leqq \\
&\leqq |EX_1| \cdot |n_{k(\nu)} - n_{k(\nu+\varrho)} + \varrho| + |-\eta(k(\nu)) + \eta(k(\nu+\varrho))| \leqq \\
&\leqq |EX_1| \cdot |n_{k(\nu)} - n_{k(\nu+\varrho)} + \varrho| + 2T_2 .
\end{aligned}
$$

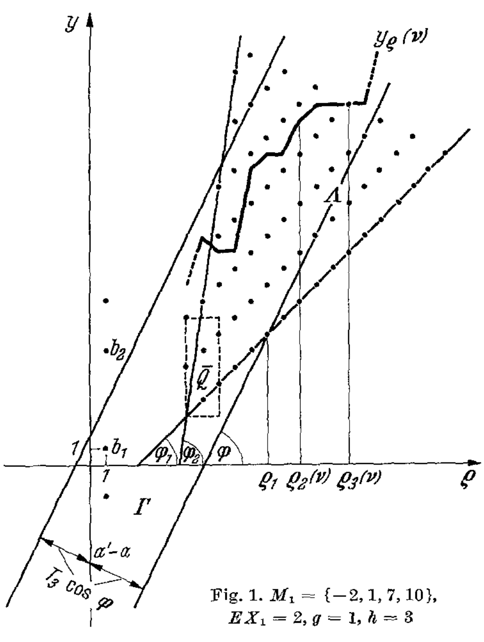

Fig. 1. $M_1 = \{-2, 1, 7, 10\}$, $EX_1 = 2, g = 1, h = 3$

Since $n_{k+1} - n_k \leqq T_1 < \infty$, the definition of $k(\nu)$ implies

$$
(13) \qquad \nu - T_1 \leqq n_{k(\nu)} \leqq \nu, \quad \nu = 1, 2, \ldots .
$$

Applying (13) twice, we get

$$
-T_1 \leqq n_{k(\nu)} - n_{k(\nu+\varrho)} + \varrho \leqq T_1
$$

and finally

$$
(14) \qquad |\zeta(\nu,\varrho)| \leqq T_1 |EX_1| + 2T_2 = T_3 < \infty \quad \text{for all } \nu \text{ and } \varrho .
$$

This proves that $f_{a,a'}(\nu)$ has the asserted representation.

The third step of the proof consists in the verification of relation (11). The argument will become clearer by representing the linear random walk $\{S_\varrho\}$ as a sequence of (random) points (ϱ, S_ϱ) in a two-dimensional (ϱ, y)-space $\bar{R}$ (see Fig. 1).

For any set $B \subset M_1$ we define $B_n = \{z : z = \sum_1^n b_i, b_i \in B\}$. For any set $A_n \subset L_0$ we define $\bar{A}_n = \{(\varrho, x) : \varrho = n, x \in A_n\}$ and $\bar{A} = \bigcup_1^\infty \bar{A}_n$. Thus, $\bar{L}_0$ is the two-dimensional representation of the minimal lattice of S.

Since X_1 has a genuine one-dimensional distribution (i.e. X_1 does not degenerate into one point) and since EX_1 lies in the convex hull of M_1, there are points $b_1, b_2 \in M_1$ such that $b_1 < EX_1 < b_2$. If we put $EX_1 = \tan \varphi$, $b_1 = \tan \varphi_1$, $b_2 = \tan \varphi_2$, then $\varphi_1 < \varphi < \varphi_2$, if one takes the appropriate determination for arc tan. By lemma 3 there is a number ϱ_0, a finite subset $M_1' \subset M_1$ and in $\bar{R}$ some closed rectangle $\bar{Q}$, whose sides are parallel to the axes and of length $h - 1$ and $b_2 - b_1$*, such that $\bar{L}_0 \cap \bar{Q} \subset \bigcup_{t=1}^h \bar{M}_{\varrho_0+t}$. This relation implies, since S_n is a sum of independent, identically distributed random variables, that there exists a sector Λ in $\bar{R}$ with the following properties: (a) the legs of Λ form with the ϱ-axis angles of size φ_1 and φ_2; (b) $\Lambda \cap \bar{L}_0 \subset \bigcup_1^\infty \bar{M}_n''$, where $M_1'' = M_1' \cup \{b_1, b_2\}$; this means, that all points of the minimal lattice $\bar{L}_0$, which lie in Λ, can be reached by S, using only the points b_1, b_2, and the points in M_1'.

In view of (12), we consider in $\bar{R}$ for every fixed ν the sequence $\{(\varrho, y_\varrho(\nu))\}$, where

$$y_\varrho(\nu) = a' - a + \varrho EX_1 - \zeta(\nu, \varrho). \tag{15}$$

By (14), the sequence $\{(\varrho, y_\varrho(\nu))\}$ lies inside the parallel strip Γ which is symmetric to the line $y = a' - a + \varrho EX_1$ and which has width $2T_3 \cos \varphi$. Hence there exists a number $\varrho_1 = \varrho_1(a, a')$, independent of ν, such that $(\varrho, y_\varrho(\nu)) \in \Lambda$ for $\varrho > \varrho_1$ and for all ν. Denote by $\varrho_2(\nu)$ the smallest of the numbers $\varrho > \varrho_1$ for which $y_\varrho(\nu) \neq y_{\varrho_1}(\nu)$. Since $y_\varrho(\nu) = a' - a - \sum_{i=0}^{\varrho-1} c_{\nu+1+i}$, the condition (ii) implies $\varrho_2(\nu) \leqq \varrho_1 + T_1$ for all ν. Furthermore, it follows from conditions (ii) and (iii) that $y_\varrho(\nu)$, considered as a function of ϱ, is constant on a subinterval of length h of the ϱ-interval $[\varrho_2(\nu), \varrho_2(\nu) + (K+1)T_1]$. From lemma 4 we know that there is a sequence of congruent lattices L_ϱ, $\varrho = 1, 2, \ldots$, such that $\bigcup_{t=1}^h L_{\varrho+t} = L_0$ for all ϱ. Since L_0 is the projection of $\bar{L}_0$ onto the y-axis, the preceding considerations show that the sequence $\{(\varrho, y_\varrho(\nu))\}$ — considered as a set of points in $\bar{R}$ — intersects $\Lambda \cap \bar{L}_0 \subset \bigcup_1^\infty \bar{M}_n''$ in some point, the first coordinate of which is some number

$$\varrho_3(\nu) \leqq \varrho_2(\nu) + (K+1)T_1 \leqq \varrho_1 + (K+2)T_1 = T_4 < \infty.$$

The constant T_4 depends on a and a', but not on ν.

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* For $h = 1$, $\bar{Q}$ degenerates to a segment of length $b_2 - b_1$.

From the last statement and equations (12) and (15) we get

$$\inf_{\nu} f_{a,a'}(\nu) = \inf_{\nu} P(S_\varrho = y_\varrho(\nu) \quad \text{for some} \quad \varrho > 0)$$
$$\geqq \inf_{\nu} P(S_{\varrho_3(\nu)} = y_{\varrho_3(\nu)}(\nu)) \geqq [\min_{z_i \in M_1''} P(X_1 = z_i)]^{T_4} > 0 .$$

This proves (11) and thereby theorem 2 for the linear case. If S is a planar random walk, all arguments go through as in the linear case. The only changes are, that $\overline{R}$ is a (ϱ, x_1, x_2)-space, $\overline{Q}$ is a three-dimensional rectangle, Λ is a (generalized) cone, Γ is a cylinder.

7. Examples and remarks

We call any control of S, for which all points of the lattice L_0 are recurrent states of V, a *recurrence-control* of S.

a) From theorem 2 one gets the following

Corollary. *Any linear random walk S has infinitely many recurrence-controls. Any planar random walk S, for which X_1 has a genuine two-dimensional distribution with $E|X_1|^2 < \infty$, has infinitely many recurrence-controls.*

Proof. If S is linear and X_1 degenerates to one point, then the corollary is trivial. (The situation is different in the degenerate planar case. Theorem 2 is certainly not applicable, since in this case a recurrence-control cannot satisfy condition (i).) Now assume, that S is linear [planar] and that X_1 has a genuine one-dimensional [two-dimensional] distribution. If $g = 1$, then the corollary is true on account of theorem 2, for there are infinitely many pairs $(\{n_k\}, \{c_{n_k}\})$ which satisfy the conditions (i)—(iii). For $g > 1$, we give only an outline of the proof. One can find a random walk V with the following properties.

(i) Every point of L_0 is a possible state of V.
(ii) V satisfies the conditions of theorem 2.
(iii) If one shifts the coordinate-system such that the origin coincides with an arbitrary point b of a mesh of L_S, then V can be regarded in this system as a control V_b of a random walk S_b which starts in b and otherwise is identical with S.

Now theorem 2 applies to V_b.

In the following examples we assume for simplicity that X_1 has a genuine one- or two-dimensional distribution and that $g = 1$.

b) For the simplest linear random walk, where

$$P(X_1 = 1) = p, \quad P(X_1 = -1) = q, \quad 0 < q < p < 1,$$

theorem 2 can be proved for $c_k \leqq 0$ without the number-theoretic apparatus of section 5 and without lemma 1. Since $n(\pm 1 + q - p)$ are possible values of S_n^*, the argumentation following equation (9) shows that for arbitrary large m,

$$P(V_n < -m \text{ i.o}) = P(V_n > m \text{ i.o.}) = 1 .$$

In a transition from the set $(-\infty, -m]$ to $[m, \infty)$ all steps in the direction of the positive x-axis have length one; hence V passes through all points $-m, -m+1, \ldots, m$, which are therefore recurrent.

c) There exists in any case a recurrence-control with equidistant control times. Take e.g. in the linear case $n_k = \bar{h} k$, where $\bar{h} \geqq \max\{h, |EX_1|^{-1}\}$, and $C_k = -[\bar{h} k E X_1]^\star$, where $[b]$ indicates the greatest integer $\leqq b$.

d) For any linear random walk S for which $EX_1 \neq 0$ there exists a recurrence-control with constant control values. Put $n_k = [tk]$, t real and $\geqq 1$, and take $c_{n_k} = c$, where c is some integer such that $\operatorname{sgn} c = -\operatorname{sgn} EX_1$, $|c| \geqq h\,|EX_1|$. Then V has all or no integers as recurrent states according to whether

$$t = -\frac{c}{EX_1} \text{ or } t \neq -\frac{c}{EX_1}.$$

The second assertion follows easily from theorem 1 (iii); the first one is a consequence of theorem 2, whereby only the checking of condition (iii) needs some consideration which we omit. Looking at the special random walk of example (b), we see that $c_{n_k} = -2$, $n_k = \left[\frac{2k}{p-q}\right]$ is a recurrence-control.

e) If S is a planar random walk for which $EX_1 \equiv (\alpha, \beta) \neq 0$, there does not necessarily exist a recurrence-control with constant control values. More precisely *there exists a recurrence-control with constant control values if and only if either $\beta = 0$ or $\frac{\alpha}{\beta}$ is rational.*

Proof. If $\alpha\beta = 0$, then one can take a control very similar to that of example (d). If $\alpha\beta \neq 0$, but $\frac{\alpha}{\beta}$ rational, then we have $\frac{\alpha}{\beta} = \frac{r}{s}$ for some integers r and $s > 0$. Take then $c_{n_k} = (-\lambda\,|r| \operatorname{sgn}\alpha, -s\lambda \operatorname{sgn}\beta)$, where λ is some integer >0 such that $\lambda\,|r| \geqq (h+1)\,|\alpha|$, and take furthermore $n_k = \left[\frac{\lambda\,|r|\,k}{|\alpha|}\right]$. Then theorem 2 is applicable. Now suppose that $\beta \neq 0$, $\frac{\alpha}{\beta}$ irrational, and that n_k and $c_{n_k} \equiv c \equiv (c_1, c_2) \neq 0$ are arbitrary. Then $EV_n = nEX_1 + t_n c$, where t_n is some real number. If n is fixed, then $y = nEX_1 + tc$, $-\infty < t < \infty$, represents a straight line in R_2, whose distance from the origin is given by $d_n = n\,|EX_1|\,|\sin(\varphi - \psi)|$, with $\tan\varphi = \frac{\alpha}{\beta}$, $\tan\psi = \frac{c_1}{c_2}$. Our assumptions imply $\sin(\varphi - \psi) \neq 0$, hence

$$\varliminf_{n} \frac{|EV_n|}{n} \geqq \varliminf_{n} \frac{d_n}{n} > 0.$$

This means that on account of theorem 1 (i) the random walk V cannot possess recurrent states.

f) If S is a linear or planar random walk for which $EX_1 \neq 0$, then there exists a recurrence-control with constant control times and constant control values if and only if the components of EX_1 are rational.

We prove the statement for the planar case and put $EX_1 = (\alpha_1, \alpha_2)$. Assume there is a recurrence-control of the form $n_k = bk$, $c_{n_k} = (c_1, c_2)$, where $b > 0$ and c_1, c_2 are integers. Since $n_{k+1}/n_k \to 1$ $(k \to \infty)$, theorem 1 (iii) implies

$$EX_1 = \left(\frac{c_1}{b}, \frac{c_2}{b}\right),$$

⋆ If $EX_1 = 0$, we have to modify n_k and C_k slightly.

so that the components of EX_1 are rational. On the other hand, if

$$EX_1 = \left(\frac{r_1}{s_1}, \frac{r_2}{s_2}\right),$$

where $s_1 > 0$, $s_2 > 0$, r_1, r_2 are some integers, then $n_k = hs_1s_2k$, $c_{n_k} = h(r_1s_2, r_2s_1)$ is a recurrence-control according to theorem 2.

g) Finally we consider the classical planar Pólya random walk with drift:

$$P(X_1 = (1,0)) = p_1,\ P(X_1 = (-1,0)) = q_1,\ P(X_1 = (0,1)) = p_2,$$
$$P(X_1 = (0,-1)) = q_2.$$

In order to exclude trivial cases, we assume $p_1q_1p_2q_2 > 0$ and $EX_{11}EX_{12} \neq 0$. If the ratio of the components of EX_1 is irrational, then there exists no recurrence-control with constant control values, but one might try to find a control which is 'symmetric' with respect to the two components. For this purpose we propose $n_k = a[t_1t_2k]$, $C_k = -a\ ([t_2k] \operatorname{sgn} EX_{11}, [t_1k] \operatorname{sgn} EX_{12})$, where a is an integer > 0 and t_1, t_2 are real numbers $\geqq 1$. Then by theorems 2 and 1 (iii) all or no lattice points of the plane are recurrent states of V according to whether

$$(t_1, t_2) = (|p_1 - q_1|^{-1}, |p_2 - q_2|^{-1}) \quad \text{or} \quad (t_1, t_2) \neq (|p_1 - q_1|^{-1}, |p_2 - q_2|^{-1}).$$

Note added in proof: After completion of the manuscript, the author noticed that a special case of lemma 3 occurs in D. MEISLER, O. PARASIUK, E. RVACHEVA: On the multidimensional local limit theorem of probability. [Russian.] Ukrain. Math. J. 1, 9–20 (1949).

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