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Explosiveness of Age-Dependent Branching Processes

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1. Introduction

Consider a continuous-time Markov branching process with state-space the non-negative integers and family-size generating function $h(s)$. It is well-known (see Harris [3]) that the process is explosive if and only if $h'(1)=\infty$ and

$$\int_{1-\varepsilon}^{1} \frac{ds}{s-h(s)}<\infty \tag{1}$$

for suitably small $\varepsilon>0$.

Ikeda, Nagasawa and Watanabe [4] define a Markov branching process on a more general state-space, for which Savits [5] obtains some results in special cases; these results are essentially analogous to the above.

In this paper we consider the problem of explosiveness of the age-dependent Bellman-Harris process (see [3]) with life-length distribution function $G(t)$ and family-size generating function $h(s)$. It is seen that if G is suitably well-behaved at the origin, the condition (1) is again necessary and sufficient; examples are also given where necessity and sufficiency are violated due to the form of G.

2. The Age-Dependent Branching Process

In the Bellman-Harris process, individuals have random life-lengths with distribution function $G(t)$ $(t\geqq 0)$; at death, an individual produces a family of random size with probability generating function $h(s)$; and all life-lengths and family-sizes are independent of each other. We will assume that the process starts at time $t=0$ with one individual of age zero. We will also assume that $G(0)=0$; this involves no loss of generality, since it may be shown that if $G(0)>0$, the process is equivalent to one characterised by G^* and h^*, where

$$G^*(t)=\frac{G(t)-G(0)}{1-G(0)}$$

and $h^*(s)$ is, for each s, the unique solution of

$$h^*(s)=h\{[1-G(0)]\,s+G(0)\,h^*(s)\}$$

and obviously, $G^*(0)=0$.

We shall usually only be concerned with the process $\{N_t\}$, where N_t is the number of individuals alive at time t, and the sample-paths of $\{N_t\}$ are assumed to be right-continuous. The p.g.f. $F(s,t)=E\,s^{N_t}$ is determined by the integral equation

$$F(s,t)=s[1-G(t)]+\int_{[0,t]} h[F(s,t-u)]\,dG(u) \tag{2}$$

and the p.g.f.'s of the higher-order joint distributions satisfy similar equations.

For $0 \leqq s < 1$, $F(s,t)$ is the unique solution of (2) and may be expressed as $\lim_{k\to\infty} F_k(s,t)$ where the $F_k(s,t)$ are defined recursively by $F_0 \equiv 0$ and

$$F_{k+1}(s,t) = s[1-G(t)] + \int_{[0,t]} h[F_k(s,t-u)]\, dG(u). \tag{3}$$

3. Explosiveness

The process is called *explosive* if there is a positive probability that N_t becomes infinite for finite t; otherwise it is *conservative*. Since

$$P[N_t=\infty] = 1 - \sum_{n=0}^{\infty} P[N_t = n] = 1 - F(1,t)$$

it follows that the process is explosive if and only if $F(1,t) < 1$ for some t.

Harris [3] proves that if $G(0)=0$, $h'(1)<\infty$ is a sufficient condition for the process to be conservative, and so in looking for criteria for explosiveness or otherwise, the only interesting case is when $h'(1)=\infty$. In general, $F(1,t) \equiv \phi(t)$ (say) satisfies

$$\phi(t) = 1 - G(t) + \int_{[0,t]} h[\phi(t-u)]\, dG(u). \tag{4}$$

$\phi(t) \equiv 1$ is always a solution of (4), but may not be the only one. $\phi(t)$ may be written as $\lim_{k\to\infty} \phi_k(t)$ where the ϕ_k are defined recursively by $\phi_0 \equiv 0$ and

$$\phi_{k+1}(t) = 1 - G(t) + \int_{[0,t]} h[\phi_k(t-u)]\, dG(u). \tag{5}$$

The following theorem tells us which solution of (4) is the "right" one.

Theorem 1. *$\phi(t)$ is the smallest non-negative solution of (4), in the sense that if $\psi(t)$ is any non-negative solution, then $\phi(t) \leqq \psi(t)$ for all $t \geqq 0$.*

Proof. We prove by induction on k that $\phi_k(t) \leqq \psi(t)$ for all k and t. It is true for $k=0$; now suppose it is true for some k. Then for all $t \geqq 0$,

$$\begin{aligned}\phi_{k+1}(t) &= 1 - G(t) + \int_{[0,t]} h[\phi_k(t-u)]\, dG(u) \\ &\leqq 1 - G(t) + \int_{[0,t]} h[\psi(t-u)]\, dG(u) \qquad \text{(since } h \text{ non-decreasing)} \\ &= \psi(t).\end{aligned}$$

Hence the induction follows. So $\phi(t) = \lim_{k\to\infty} \phi_k(t) \leqq \psi(t)$.

Corollary 1.1. *The process characterised by G and h (henceforth called "the process (G,h)") is explosive iff (4) has a solution with $0 \leqq \phi(t) \leqq 1$ and $\phi(t) \not\equiv 1$.*

Corollary 1.2. *The process (G,h) is explosive iff there exist $T>0$ and a function $\psi(t)$ on $[0,T]$ such that $0 \leqq \psi(t) \leqq 1$, $\psi(t) \not\equiv 1$ and*

$$\psi(t) \geqq 1 - G(t) + \int_{[0,t]} h[\psi(t-u)]\, dG(u) \qquad \textit{for } 0 \leqq t \leqq T. \tag{6}$$

Proof. Necessity is obvious, as $\phi(t)$ itself must satisfy the above conditions. For sufficiency, suppose ψ exists. Define ϕ_0 by

$$\phi_0(t)=\psi(t) \quad (0\leqq t\leqq T)$$
$$=1 \quad (t>T)$$

and $\{\phi_k\}$ recursively by (5). Then $\phi_0(t)\geqq\phi_1(t)$ for $0\leqq t\leqq T$ by the given condition on ψ; and also trivially for $t>T$. It follows by induction that $\phi_k(t)$ is decreasing in k for each fixed t, and so $\phi(t)=\lim_{k\to\infty}\phi_k(t)$ exists; also, by bounded convergence it satisfies (4). $\phi(t)\leqq\psi(t)<1$ for some $t\in[0,T]$ and so by Corollary 1.1, the process is explosive.

Further information about ϕ is given by the following two theorems, which will be of use later.

Theorem 2. *Each ϕ_k is a non-increasing function of t; and hence also ϕ is.*

Proof. The Eqs. (5) may be written

$$1-\phi_{k+1}(t)=\int_{[0,t]}\{1-h[\phi_k(t-u)]\}\,dG(u).$$

We prove the result by induction on k. $\phi_0(t)\equiv 0$; suppose ϕ_k is non-increasing. Then if $0\leqq t<t'$,

$$\begin{aligned}\phi_{k+1}(t)-\phi_{k+1}(t')&=\{1-\phi_{k+1}(t')\}-\{1-\phi_{k+1}(t)\}\\&=\int_{[0,t]}\{h[\phi_k(t-u)]-h[\phi_k(t'-u)]\}\,dG(u)\\&\quad+\int_{(t,t']}\{1-h[\phi_k(t'-u)]\}\,dG(u)\geqq 0\end{aligned}$$

since h non-decreasing, ϕ_k non-increasing. Hence the result.

Theorem 3. *If $\phi(t)\not\equiv 1$ then $\phi(t)<1$ for all $t>0$.*

Proof. Suppose on the contrary there exists $t>0$ for which $\phi(t)=1$. Then since ϕ is non-increasing, there must be a $t_0>0$ such that

$$\phi(t)=1 \quad \text{for } t<t_0$$
$$\phi(t)<1 \quad \text{for } t>t_0.$$

Then for $t\geqq t_0$,

$$\begin{aligned}\phi(t)&=1-G(t)+\int_{[0,t-t_0]}h[\phi(t-u)]\,dG(u)+\int_{(t-t_0,t]}dG(u)\\&=1-G(t-t_0)+\int_{[0,t-t_0]}h[\phi(t-u)]\,dG(u).\end{aligned}$$

Now for $\tau\geqq 0$, define $\phi^*(\tau)=\phi(\tau+t_0)$. Then for $\tau\geqq 0$,

$$\begin{aligned}\phi^*(\tau)=\phi(\tau+t_0)&=1-G(\tau)+\int_{[0,\tau]}h[\phi(\tau+t_0-u)]\,dG(u)\\&=1-G(\tau)+\int_{[0,\tau]}h[\phi^*(\tau-u)]\,dG(u).\end{aligned}$$

Hence ϕ^* is a solution of (4), but is strictly smaller than ϕ at at least one point, e.g. $\tau=\frac{1}{2}t_0$, contradicting Theorem 1. Hence the result is proved.

4. Comparison Theorems

In this section we show that explosiveness is essentially determined by the steepness of $G(t)$ near $t=0$ and the steepness of $h(s)$ near $s=1$, and obtain ways of comparing different processes.

Theorem 4. [*Comparison of G's.*] *Let* G, G^* *be distribution functions with* $G(0)=G^*(0)=0$. *If there is a* $T>0$ *such that* $G(t)\leqq G^*(t)$ *for* $0\leqq t\leqq T$, *and the process* (G, h) *is explosive, then the process* (G^*, h) *is explosive.*

Proof. Let $\phi(t)$ correspond to the (G, h) process. Then $\phi(t)<1$ for $t>0$ (Theorem 3). For any $t\leqq T$, we have

$$\begin{aligned}\phi(t)&=1-G(t)+\int_{[0,t]} h[\phi(t-u)]\,dG(u)\\&=1-G(t)+\int_{[0,t]} \chi_t(u)\,dG(u)\end{aligned}$$

where $\chi_t(u)\equiv h[\phi(t-u)]$. Since h is non-decreasing, and ϕ non-increasing (Theorem 2), χ_t is a non-decreasing function of u. An integration by parts gives

$$\int_{[0,t]} \chi_t(u)\,dG(u)+\int_{[0,t]} G(u)\,d\chi_t(u)=G(t).$$

Hence

$$\begin{aligned}\phi(t)&=1-\int_{[0,t]} G(u)\,d\chi_t(u)\\&\geqq 1-\int_{[0,t]} G^*(u)\,d\chi_t(u)\\&=1-G^*(t)+\int_{[0,t]} h[\phi(t-u)]\,dG^*(u).\end{aligned}$$

Hence ϕ satisfies the condition (6) for the process (G^*, h) and so that process is explosive.

Corollary 4.1. [*Comparison with Markov Process.*] *If there exist* $T>0$ *and constants* $\beta>\alpha>0$ *such that* $\alpha t\leqq G(t)\leqq \beta t$ *for* $0\leqq t\leqq T$, *then the process* (G, h) *is explosive iff* $h'(1)=\infty$ *and* $\int_{1-\varepsilon}^{1}\frac{ds}{s-h(s)}<\infty$ *for some* $\varepsilon>0$.

This follows as a consequence of two-way comparison with a Markov process for which $G(t)=1-e^{-\lambda t}\sim\lambda t$ as $t\to 0$ for some $\lambda>0$.

Theorem 5. [*Comparison of h's.*] *If* h, h^* *are p.g.f.'s such that* $h^*(s)\leqq h(s)$ *for* $\theta\leqq s\leqq 1$, *some* $\theta<1$, G *is a distribution function with* $G(0)=0$, *and the process* (G, h) *is explosive, then the process* (G, h^*) *is explosive.*

Proof. Let $\phi(t)$ correspond to the (G, h) process. Then $\phi(t)<1$ for $t\geqq 0$; also, from the equation defining ϕ, $\phi(t)\geqq 1-G(t)$ and so $\phi(t)\uparrow 1$ as $t\downarrow 0$. Hence there exists $T>0$ such that $\phi(t)\geqq\theta$ for $0\leqq t\leqq T$. For these values of t,

$$\begin{aligned}\phi(t)&=1-G(t)+\int_{[0,t]} h[\phi(t-u)]\,dG(u)\\&\geqq 1-G(t)+\int_{[0,t]} h^*[\phi(t-u)]\,dG(u)\end{aligned}$$

and ϕ satisfies the criterion (6) for the (G, h^*) process. Hence that process is explosive.

5. Existence Theorems

In cases where comparison with the Markov process is not possible, we would expect the necessary and sufficient condition (1) no longer to be applicable, and explosiveness to depend on the forms of both G and h. The following theorems, and the examples given in the next section, emphasise the extent to which this is so. We start with a lemma, which makes use of the inverse function h^{-1} of h, which is defined on $[h(0), 1]$ since h is strictly increasing on $[0, 1]$.

Lemma A. *If there exists $T>0$ such that*

$$1-h^{-1}(1-t) \leqq \int_{[0,t]} G(u)\,du \qquad \text{for } 0 \leqq t \leqq T \tag{7}$$

then the process (G, h) is explosive.

Proof. Define $\psi(t)=h^{-1}(1-t)$ for $0 \leqq t \leqq T$. Then for these values of t,

$$\begin{aligned}
&1-G(t)+\int_{[0,t]} h[\psi(t-u)]\,dG(u)\\
&\quad =1-G(t)+\int_{[0,t]} [1-t+u]\,dG(u)\\
&\quad =1-\int_{[0,t]} (t-u)\,dG(u)=1-\int_{[0,t]} G(u)\,du \qquad \text{(integration by parts)}\\
&\quad \leqq h^{-1}(1-t)=\psi(t).
\end{aligned}$$

Hence ψ satisfies the conditions of Corollary 1.2 and the process is explosive.

This lemma immediately yields the first existence theorem:

Theorem 6. *Given h such that $h'(1)=\infty$, there is a G with $G(0)=0$ such that the process (G, h) is explosive.*

Proof. Put $G(t)=\dfrac{1}{h'[h^{-1}(1-t)]}$ for t in a neighbourhood of 0; then (7) holds with equality.

The following theorem shows that the "reverse" can be done but the proof gives little indication of the nature of the h constructed.

Theorem 7. *Given a distribution function G with $G(0)=0$, and $G(t)>0$ for all $t>0$, there is a p.g.f. h such that the process (G, h) is explosive.*

Proof. Since $G(0)=0$ and $G(t)>0$ for $t>0$, the function

$$\Gamma(t)=\int_{[0,t]} G(u)\,du \qquad (t \geqq 0)$$

is continuous and strictly increasing, and $\Gamma(0)=0$. So it has an inverse which is continuous and $\Gamma^{-1}(0)=0$. The condition of Lemma A may be written

$$h^{-1}(1-t) \geqq 1-\Gamma(t) \qquad \text{for } t \text{ in a neighbourhood of } 0$$

or equivalently,

$$b(x) \equiv 1-\Gamma^{-1}(1-x) \geqq h(x) \qquad \text{for } x \text{ in a neighbourhood of } 1.$$

Such an h may be found as a consequence of the following lemma.

Lemma B. *If b is a continuous function with $b(1)=1$, there exist $\theta<1$ and a p.g.f. h such that $h(x)\leqq b(x)$ for $\theta\leqq x\leqq 1$.*

Proof (after Besicovitch [1]). Let $\{p_i\}$ be a sequence of numbers with each $p_i>0$ and $\sum_{i=1}^{\infty} p_i=1$. Choose θ so that $b(x)>p_1$ for $\theta\leqq x\leqq 1$. We find recursively a sequence $\{n_1, n_2, \ldots\}$ of integers such that

$$\pi_k(x)=\sum_{i=1}^{k} p_i x^{n_i}<b(x) \quad \text{for } \theta\leqq x\leqq 1.$$

By the choice of θ, we may take $n_1=0$. Suppose now that $n_1, n_2, \ldots, n_k$ have been chosen. Then $\rho_k(x)=b(x)-\pi_k(x)$ is a strictly positive continuous function on $[\theta, 1]$ with $\rho_k(1)=\sum_{i=k+1}^{\infty} p_i$. Let $m=\min_{\theta\leqq x\leqq 1} \rho_k(x)>0$. Choose $\xi\in[\theta, 1)$ such that $\rho_k(x)>p_{k+1}$ for $\xi\leqq x\leqq 1$. Choose n_{k+1} such that $\xi^{n_{k+1}}<\frac{m}{p_{k+1}}$. Then for $\theta\leqq x\leqq\xi$,

$$p_{k+1} x^{n_{k+1}}\leqq p_{k+1}\xi^{n_{k+1}}<m\leqq\rho_k(x)$$

and for $\xi<x\leqq 1$,

$$p_{k+1} x^{n_{k+1}}\leqq p_{k+1}<\rho_k(x)$$

and so for all $x\in[\theta, 1]$,

$$b(x)-\pi_k(x)>p_{k+1} x^{n_{k+1}}, \quad \text{i.e. } b(x)>\sum_{i=1}^{k+1} p_i x^{n_i}.$$

Hence n_{k+1} has the required property. Thus $b(x)>\sum_{i=1}^{k} p_i x^{n_i}$ for $x\in[\theta, 1]$ for all k, and so in the limit as $k\to\infty$,

$$b(x)\geqq\sum_{i=1}^{\infty} p_i x^{n_i}=h(x) \quad \text{say,}$$

and h is a p.g.f. Hence the lemma is proved.

Finally in this section, we have a theorem ensuring the existence of a conservative process for given h.

Theorem 8. *Given a p.g.f. h such that $h'(1)=\infty$, there is a distribution function G with $G(t)>0$ for all $t>0$ such that the process (G, h) is conservative.*

Proof. Take a sequence $\{t_n\}_{n\geqq 0}$, decreasing to zero, for which $\sum_{n=0}^{\infty} t_n=\infty$.

Consider the *generation sizes* of a process (G, h); these are well-known to form a Galton-Watson process, the n-th generation having p.g.f. $h^{(n)}$, the n-fold functional iterate of h. Let A_n be the event that the n-th generation contains a member whose life-length is not greater than t_n. Because of the independence of life-lengths and family-sizes, we have

$$\begin{aligned} P(A_n)&=\sum_{j=1}^{\infty} h_j^{(n)}\{1-[1-G(t_n)]^j\} \\ &=1-h^{(n)}[1-G(t_n)]. \end{aligned}$$

Now if $\sum_{n=0}^{\infty} P(A_n) < \infty$, we have by the Borel-Cantelli Lemma that with probability one, A_n occurs only finitely often. And as $h^{(n)}(s) \uparrow 1$ as $s \uparrow 1$, it is possible to choose the values of G successively at the points t_n so that this condition holds.

But if with probability one *all* the members of *all but finitely many* generations live for at least the corresponding t_n, it follows since $\sum_{n=0}^{\infty} t_n = \infty$ that the process (G, h) cannot explode.

6. Examples

There follow some examples of processes where the explosiveness question has been decided; in all, the Markov comparison is not applicable, and in Examples 1 and 3 the necessity and sufficiency of the condition (1) are violated.

Example 1. The usual example given of a Markov process with $h'(1) = \infty$ but which is not explosive is $h(s) = s + (1-s)\log(1-s)$: a distribution in the domain of attraction of a stable law of order 1. Theorem 6 ensures that G may be found such that the process (G, h) is explosive; G must simply satisfy the condition (7), for which in this case it is sufficient that

$$\int_0^t G(u)\,du = \frac{t}{(-\log t)} \qquad \text{for small } t$$

and so

$$G(t) \sim \frac{1}{(-\log t)} \qquad \text{as } t \to 0.$$

As expected, this function is very "steep" at the origin.

Example 2. If h is the p.g.f. of a distribution in the domain of attraction of a stable law of order α, $0 < \alpha < 1$, then, using results of Feller [2], we have that

$$1 - h(s) = (1-s)^{\alpha} L\left(\frac{1}{1-s}\right) \qquad \text{as } s \to 1,$$

where L is a function varying slowly at ∞.

Hence $(1-s)^{\alpha+\varepsilon} \leqq 1 - h(s) \leqq (1-s)^{\alpha-\varepsilon}$ $(\varepsilon > 0)$ for small s and by Theorem 5 (comparison of h's) there is no loss of generality in representing this class of p.g.f.'s by

$$h(s) = 1 - (1-s)^{\alpha} \qquad \text{for } 0 < \alpha < 1.$$

It is well-known that for this h, the *Markov* process is explosive. Here it is shown that even if $G(t) = \exp\left\{-\frac{k}{t^{\beta}}\right\}$ $(k > 0, \beta > 0)$, an extremely "flat" function at the origin, the process (G, h) is explosive—which in the absence of Theorem 8 might lead one to suppose that all processes are explosive for this particular h.

The condition of Corollary 1.2 becomes, in this case, writing $\eta(t) = 1 - \psi(t)$: we require $\eta(t)$ such that $\eta(t) > 0$ for some $t > 0$ and

$$\eta(t) \leqq \int_{[0,t]} \{\eta(t-u)\}^{\alpha}\, dG(u) \qquad \text{for } 0 \leqq t \leqq T. \tag{8}$$

We will show that $\eta(t)=\exp\left\{-\frac{c}{t^\beta}\right\}$ is a suitable choice for sufficiently large c. If $\eta(t)$ takes this form,

$$\text{L.H.S. of (8)}=\exp\left\{-\frac{c}{t^\beta}\right\}$$

and

$$\begin{aligned}\text{R.H.S. of (8)}&=\int_0^t \exp\left\{-\frac{c\alpha}{(t-u)^\beta}-\frac{k}{u^\beta}\right\}\frac{\beta k}{u^{\beta+1}}\,du\\ &=\int_0^1 \frac{\beta k}{t^\beta\theta^{\beta+1}}\exp\left\{-\frac{1}{t^\beta}\left[\frac{c\alpha}{(1-\theta)^\beta}+\frac{k}{\theta^\beta}\right]\right\}d\theta \qquad \text{where } u=t\theta\\ &\geqq \frac{\beta k L}{t^\beta}\exp\left\{-\frac{M}{t^\beta}\right\}\end{aligned}$$

where M is a constant to be chosen and L is the length of the interval on which

$$\frac{c\alpha}{(1-\theta)^\beta}+\frac{k}{\theta^\beta}\leqq M.$$

So provided we choose M so that L is positive *and* $M<c$, we have

$$\text{L.H.S.}\geqq\frac{\beta k L}{t^\beta}\exp\left\{-\frac{M}{t^\beta}\right\}\geqq\exp\left\{-\frac{c}{t^\beta}\right\}=\text{R.H.S.} \qquad \text{for small enough } t.$$

But clearly L is positive if and only if

$$\begin{aligned}M&>\text{minimum value of }\frac{c\alpha}{(1-\theta)^\beta}+\frac{k}{\theta^\beta} \qquad \text{on } [0,1]\\ &=\left\{(c\alpha)^{\frac{1}{1+\beta}}+k^{\frac{1}{1+\beta}}\right\}^{1+\beta} \qquad \text{by elementary calculus.}\end{aligned}$$

Hence M must be chosen so that

$$\left\{(c\alpha)^{\frac{1}{1+\beta}}+k^{\frac{1}{1+\beta}}\right\}^{1+\beta}<M<c$$

and this is possible provided $\left\{(c\alpha)^{\frac{1}{1+\beta}}+k^{\frac{1}{1+\beta}}\right\}^{1+\beta}<c$, which is true for

$$c>\frac{k^{\frac{1}{1+\beta}}}{1-\alpha^{\frac{1}{1+\beta}}}.$$

So c and hence $\eta(t)$ can be found, and the process is explosive.

Example 3. For the same choice of h as in the above example, the proof of Theorem 8 gives us a way of making G "flat" enough to obtain a conservative process. For in this case,

$$h^{(n)}(s)=1-(1-s)^{\alpha^n}.$$

So taking $t_n = \frac{1}{n}$ for each n, and $G(t) = \exp\left\{-\exp\frac{k}{t}\right\}$ for small t ($k>0$), we have

$$\sum_n \{1 - h^{(n)}[1 - G(t_n)]\} = \sum_n \exp\{-(\alpha\, e^k)^n\} < \infty \qquad \text{provided } \alpha\, e^k > 1.$$

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