Z. Wahrscheinlichkeitstheorie verw. Geb. 28, 89–98 (1974)


# Thinning and Rare Events in Point Processes

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In 1956 Rényi [10] proved a first theorem on the thinning of point processes: In a renewal process on the non-negative line (= time) independently retain each point with probability $p$ and cancel it with probability $1-p$. Change the time scale by a factor $p^{-1}$ and let $p \to 0$. Then the process approaches a Poisson stream. Råde [9] treated more general deletion procedures of this type: Given that a point is retained, then the next point not cancelled is its $k$-th successor with probability $p_k$, $k=1, 2, \ldots$. In this formulation Rényi thinning amounts to the particular choice $p_k=(1-p)^{k-1} p$, geometric cancelling.

Both authors rely upon analytic tools, requiring that the thinning mechanism is independent of the renewal process to be thinned. Like Lindvall in [7], we use a probabilistic approach. This one shows, that the independence is, in fact, not necessary. Therefore we can study fairly complicated thinnings like those associated with the incidence of rare patterns in the underlying process. We can also handle a somewhat richer class of processes to be thinned, including except renewal processes for example stationary ergodic ones.

For such processes the original Rényi thinning has been studied by Nawrotzki [8] and Belyaev [1] (the latter's relevant Theorem 1 is correct in spite of the mistaken Lemma 1).

Some applications are made, to renewal processes, to thinning procedures that treat different points independently, and to the process of large claims in risk theory.

## 1. Preliminaries

Denote by $\mathscr{B}(R)$ the Borel algebra on the real numbers, $R$, and $\mathscr{N}$ the class of integer valued measures on $\mathscr{B}(R)$ which are finite on compact sets. Endow $\mathscr{N}$ with the $\sigma$-algebra $\mathscr{B}(\mathscr{N})$ generated by the sets

$$\{\mu \in \mathscr{N};\, \mu A = j\}, \quad A \in \mathscr{B}(R), \quad j=0, 1, \ldots.$$

A *point process* is a measurable map from some probability space (supposed to be fixed in the sequal—we shall use $P$ for its probability measure) into $(\mathscr{N}, \mathscr{B}(\mathscr{N}))$.

The natural topology for $\mathscr{N}$ is the vague topology: $\mu_n \xrightarrow{v} \mu$ (read vaguely) if

$$\int_R f\, d\mu_n \to \int_R f\, d\mu$$

for all continuous functions $f: R \to R$ with compact support. Equivalently $\mu_n \xrightarrow{v} \mu$ if and only if $\mu_n A \to \mu A$ for all bounded measurable $A$, whose boundaries have $\mu$-measure zero. $\mathscr{B}(\mathscr{N})$ is actually the Borel algebra of the vague topology [6, Prop. 1.1].

On $\mathcal{N}$ with its vague topology weak convergence in the usual sense can be studied: Let $\xi_n$ and $\xi$ be point processes, $\xi_n \xrightarrow{d} \xi$ (read weakly) if and only if for all bounded continuous $\varphi$: $\mathcal{N} \to R$ the expectations satisfy $E[\varphi(\xi_n)] \to E[\varphi(\xi)]$. This holds if and only if for all continuous $f$: $R \to R$ with compact support the random variables $\int f\,d\xi_n$, $\int f\,d\xi$ satisfy $\int f\,d\xi_n \xrightarrow{d} \int f\,d\xi$ (for random variables $\xrightarrow{d}$ is, of course, convergence in distribution).

A *point of occurrence* of $\xi$ is a point $x$ such that $\xi\{x\} \geqq 1$. Since $\xi$ is finite on bounded sets there cannot, for any outcome, be more than countably many points of occurrence. Another necessary and sufficient condition for $\xi_n \xrightarrow{d} \xi$ is that

$$(\xi_n A_1, \ldots, \xi_n A_k) \xrightarrow{d} (\xi A_1, \ldots, \xi A_k)$$

for all bounded measurable $A_j$, $1 \leqq j \leqq k$, such that $\xi$ with probability one has no occurrences in the boundary of any $A_j$. The convergence $\xi_n \xrightarrow{d} \xi$ often implies weak convergence in the Skorohod $J_1$-sense [6, Sect. 3].

The points of occurrence can be enumerated in the order they appear:

$$X_j = \inf\{x \geqq 0;\ \xi[0, x] \geqq j+1\}, \quad j \geqq 0,$$

$$X_j = \sup\{x < 0;\ \xi[x, 0) \geqq -j\}, \quad j < 0.$$

Then

$$\cdots \leqq X_{-2} \leqq X_{-1} < 0 \leqq X_0 \leqq X_1 \leqq \cdots$$

and only finitely many $X_j$ can coincide. As usual $\inf \emptyset$ is interpreted as $+\infty$ and $\sup \emptyset$ as $-\infty$.

We shall make use of two simple lemmata. A point $x \in R$ is an *atom* of $\xi$ if it is with positive probability a point of occurrence, i.e. $P(\xi\{x\} \geqq 1) > 0$. The process $\xi$ is *completely random* if $\xi A_1, \ldots, \xi A_k$ are independent as soon as $A_1, \ldots, A_k$ are disjoint.

**Lemma 1.** *No point process can have more than a countable number of atoms.*

*Proof.* Let $j$, $k$ be positive integers. An argument parallelling [2, p. 124] shows that

$$P(\xi\{x\} \geqq 1) \geqq 1/j$$

is possible only for finitely many $x \in [-k, k]$. Take the union over all $j$ and $k$.

**Lemma 2.** *If $\xi_n \xrightarrow{d} \xi$ and the $\xi_n$ are completely random, then so is $\xi$.*

*Proof.* Write $\mathcal{A}(\xi)$ for the class of bounded $A \in \mathcal{B}(R)$ with $P(\xi(\partial A) = 0) = 1$, "$\partial$" denoting "boundary of". $\mathcal{A}(\xi)$ is a ring and by Lemma 1 it contains a class of intervals whose right as well as left end points are dense. Hence it generates $\mathcal{B}(R)$.

Consider the set

$$\mathcal{M}_1 = \left\{ B \in \mathcal{B}(R);\ \xi\left(B \setminus \bigcup_{j=2}^{k} A_j\right),\ \xi A_2, \ldots, \xi A_k \right.$$
$$\left. \text{are independent for all disjoint } A_2, \ldots, A_k \in \mathcal{A}(\xi) \right\}.$$

A passage to the limit $\xi_n \xrightarrow{d} \xi$ reveals that $\mathcal{A}(\xi) \subset \mathcal{M}_1$ and by the usual monotone class argument [5, Sect. 6., Th. 3, p. 27] $\mathcal{M}_1 = \mathcal{B}(R)$. The rest goes inductively:

Define for $1<j<k$

$$\mathcal{M}_j=\Big\{B\in\mathcal{B}(R);\ \xi\Big(B\setminus\bigcup_{i=j+1}^{k}A_j\Big),\ \xi(B_1\setminus B),\dots,\xi(B_{j-1}\setminus B),\ \xi(A_{j+1}),\dots,\xi(A_k)$$
$$\text{are independent for all disjoint } B_1,\dots,B_{j-1},\ A_{j+1},\dots,A_k$$
$$\text{such that the } A\text{-sets} \in\mathcal{A}(\xi) \text{ and the } B\text{-sets} \in\mathcal{B}(R) \text{ are bounded}\Big\}.$$

If $\mathcal{M}_1=\dots=\mathcal{M}_j=\mathcal{B}(R)$, monotonicity yields that $\mathcal{M}_{j+1}=\mathcal{B}(R)$ and, finally,

$$\mathcal{B}(R)=\mathcal{M}_k=\{B\in\mathcal{B}(R);\ \xi(B),\xi(B_1\setminus B),\dots,\xi(B_{k-1}\setminus B)$$
$$\text{are independent for all disjoint bounded } B_1,\dots,B_{k-1}\in\mathcal{B}(R)\}.$$

The fact that also the numbers of points in disjoint possibly nonbounded sets are independent follows by approximation.

## 2. The Thinning Theorem

Let $\xi$ be any point process and $\{Y_{nj}\}_{j\in Z}$, $Z$ the integers, for each $n=1,2,\dots$ a sequence of random variables taking only the values zero or one. Define, for $A\in\mathcal{B}(R)$, $\xi_n A$ to be the number of points $X_j\in A$ such that $Y_{nj}=1$ and similarly $\eta_n A$ to be the number of indices $j\in A$ such that $Y_{nj}=1$. These point processes $\eta_n$ we shall refer to as thinning.

Define also, for any $a>0$ the operator $C_a$ taking a measure $\mu$ into $C_a\mu$, where $(C_a\mu)A=\mu(aA)=\mu\{ax;\ x\in A\}$.

**Theorem 1.** *Assume that*

$$\lim_{j\to\pm\infty} X_j/j=m>0 \quad a.s.$$

*or, what is the same,*

$$\lim_{x\to\infty}\xi[0,x]/x=-\lim_{x\to-\infty}\xi[-x,0]/x=m^{-1}.$$

*Then, for $a_n\to\infty$,*

$$C_{a_n}\xi_n\xrightarrow{d}C_{m^{-1}}\eta$$

*if and only if,*

$$C_{a_n}\eta_n\xrightarrow{d}\eta.$$

*Note.* As will be evident from the proof it is enough that there be some enumeration of $\xi$'s points of occurrence, not necessarily the one of their order of appearance, such that the first condition holds. With this formulation the theorem is valid in $R^k$, though the limit process will be supported by the line through the origin with direction $m$ (which of course is a vector in $R^k$). (To obtain such a case let $\langle Y_n\rangle_{n\in Z}$ be a sequence of random vectors in $R^k$ for which the strong law of large numbers holds, $\lim_{j\to\pm\infty}(Y_0+\dots+Y_j)/j=m$ a.s. Define a point process by putting unit mass at each $Y_0+\dots+Y_j$, $j\geqq 0$, and each $Y_{-1}+\dots+Y_j$, $j\leqq -1$.)

It should also be clear that $R$ could be replaced by $R_+=[0,\infty)$. Then we need of course no requirement on $\lim_{j\to-\infty}X_j/j$.

*Proof.* The proof will be given for the implication

$$C_{a_n}\eta_n \xrightarrow{d} \eta \Rightarrow C_{a_n}\xi_n \xrightarrow{d} C_{m^{-1}}\eta$$

but the same arguments would yield the converse.

Let $f\colon R \to R$ be any continuous function with compact support. We wish to show that

$$\int f\, d\, C_{a_n}\xi_n \xrightarrow{d} \int f\, d\, C_{m^{-1}}\eta .$$

However,

$$\int f\, d\, C_{a_n}\xi_n = \int f(a_n^{-1}x)\,\xi_n(dx) = \sum_{j\in Z} f(a_n^{-1}X_j)\, Y_{nj}$$

$$= \sum_j [f(a_n^{-1}X_j) - f(a_n^{-1}mj)]\, Y_{nj} + \sum_j f(a_n^{-1}mj)\, Y_{nj}.$$

The last sum is nothing but

$$\int f(a_n^{-1}mj)\,\eta_n(dx) = \int f(mx)\, C_{a_n}\eta_n(dx) \xrightarrow{d} \int f\, d\, C_{m^{-1}}\eta$$

by assumption.

For the first sum let $b_n \uparrow \infty$ but so slowly that $b_n/a_n \to 0$. Set

$$A_n = \{|X_j/j - m| < m/2 \text{ for } |j| \geqq b_n\}.$$

Obviously $PA_n \uparrow 1$. Then, $1_A$ denoting the indicator function,

$$\Big|\sum_j [f(a_n^{-1}X_j) - f(a_n^{-1}mj)]\, Y_{nj}\Big| \leqq \sum_{|j|\leqq b_n} |f(a_n^{-1}X_j) - f(a_n^{-1}mj)|\, Y_{nj}$$

$$+ 1_{A_n} \sum_{|j|>b_n} |f(a_n^{-1}X_j) - f(a_n^{-1}mj)|\, Y_{nj} + (1 - 1_{A_n}) \sum_{|j|>b_n} |f(a_n^{-1}X_j) - f(a_n^{-1}mj)|\, Y_{nj}.$$

The last sum tends to zero a.s. since $1_{A_n}$ with probability one actually equals one ultimately. Introduce the modulus of continuity

$$\delta(\varepsilon) = \sup_{|x_1 - x_2| \leqq \varepsilon} |f(x_1) - f(x_2)|$$

and the a.s. finite random variable $X = \sup_j |X_j/j - m|$.

The first term is not greater than

$$\sum_{|j|\leqq b_n} \delta(|a_n^{-1}X_j - a_n^{-1}mj|)\, Y_{nj} \leqq \max_{|j|\leqq b_n} \delta(a_n^{-1} j\, |X_j/j - m|)\, \eta_n[-b_n, b_n]$$

$$\leqq \delta(a_n^{-1} b_n \cdot X) \cdot \eta_n[-b_n, b_n] \xrightarrow{d} 0$$

since $\{\eta_n[-b_n, b_n]\}$ is tight and $f$ is uniformly continuous. For the intermediate sum let the support of $f$ be contained in $[-a, a]$. Since $a_n^{-1}|X_j| \geqq a_n^{-1} mj/2$ on $A_n$, this sum is not greater than, writing $b = 2a/m$,

$$1_{A_n} \cdot \sum_{|b_n| < |j| \leqq a_n b} |f(X_j \cdot j^{-1} \cdot j\, a_n^{-1}) - f(m \cdot j\, a_n^{-1})|\, Y_{nj}$$

$$\leqq \sup_{\substack{|x|\leqq b \\ |j| > b_n}} |f(X_j \cdot j^{-1} \cdot x) - f(mx)|\, C_{a_n}\eta_n[-b, b].$$

The supremum tends to zero a.s. and as $C_{a_n}\eta_n \xrightarrow{d} \eta$ the proof is complete.

## 3. Independent Deletions

For any Borel measure $\lambda$, which is finite on bounded sets, the Poisson process with intensity $\lambda$, $\Pi_\lambda$, is defined by the requirements that $\Pi_\lambda A_1, \ldots, \Pi_\lambda A_n$ be independent for disjoint sets $A_1, \ldots, A_n$ and

$$P(\Pi_\lambda A = k) = e^{-\lambda A} \cdot (\lambda A)^k / k!,$$

$k = 0, 1, \ldots$, [4]. In particular, if $\lambda$ is a multiple of Lebesgue measure $L$ we talk of stationary Poisson processes.

It is of particular interest to know when the thinned process is approximately Poisson. By Theorem 1 this is the case if and only if the thinning processes are themselves Poisson in the limit. In this section we shall see that Poisson limits always arise if the deletion mechanism treats different points independently, in the next that rare configurations in renewal processes are approximately Poisson.

If for each $n$ the random variables $Y_{nj}, j \in Z$, are independent (but not necessarily independent of $\xi$) we talk of independent deletions; if they also have the same distribution, the deletions will be called i.i.d. The following generalization of Rényi's theorem holds:

**Theorem 2.** *Let $\xi$ be as in Theorem* 1 *and $a_n \to \infty$. Then, with independent deletions $Y_{nj}$ such that for any constant $c$ $\lim_{n\to\infty} \max_{|j| \leq c a_n} P(Y_{nj} = 1) = 0$, $C_{a_n} \xi_n$ can only have Poisson limits. Further $C_{a_n} \xi_n \xrightarrow{d} \Pi_\lambda$ if and only if for all bounded intervals $A$, whose boundaries have $C_m \lambda$-measure zero,*

$$(*) \qquad \lim_{n\to\infty} \sum_{j \in a_n A} P(Y_{nj} = 1) = (C_m \lambda) A.$$

*If the deletions are i.i.d., $P(Y_{nj} = 1) = p_n > 0$, then to obtain a limit $a_n$ must be of the same order of magnitude as $p_n^{-1}$ and*

$$C_{a_n} \xi_n \xrightarrow{d} \Pi_{\alpha/m\,L}$$

*if and only if $a_n p_n \to \alpha$.*

*Proof.* It is here that Lemmata 1 and 2 are needed. Assume that $C_{a_n} \xi_n \xrightarrow{d} \xi$. Then by Theorem 1 $C_{a_n} \eta_n \xrightarrow{d} C_m \xi$. Since the $Y_{nj}$ are independent, the $C_{a_n} \eta_n$ are completely random, forcing $C_m \xi$ to be the same. For $A \in \mathscr{A}(C_m \xi)$ as introduced in the proof of Lemma 2

$$\sum_{j \in a_n A} Y_{nj} = (C_{a_n} \eta_n) A \xrightarrow{d} (C_m \xi) A.$$

By classical theory it follows that

$$\sum_{j \in a_n A} P(Y_{nj} = 1)$$

must converge, as $n \to \infty$, to some number $\nu A$ and that $C_m \xi A$ must have a Poisson distribution with parameter $\nu A$. The reader can verify this from any general central limit theorem, like [4, Theorem 2, XVII, 7, p. 552], obtaining first the convergence of the sum of probabilities from the necessary conditions for existence of a limit distribution, then the Poissonness from the shape of the canonical measure.

It follows that

$$\nu A = -\log P(C_m \xi A = 0)$$

for $A \in \mathscr{A}(C_m \xi)$. Since $C_m \xi$ is completely random, $\nu$ is additive. Further if $A_n \uparrow A$, then $C_m \xi A_n \uparrow C_m \xi A$ and thus $P(C_m \xi A_n = 0) \to P(C_m \xi A = 0)$. Hence $\nu$ is countably additive on $\mathscr{A}(C_m \xi)$. It follows that $\nu$ has a unique extension to a measure on the $\sigma$-algebra generated by this class, which is nothing but all of $\mathscr{B}(R)$ (as was pointed out in the proof of Lemma 2). We denote the extension by $\nu$, too, and note that it is finite on bounded sets.

If $B$ is any bounded interval, there is by Lemma 1 a sequence of intervals $A_k \in \mathscr{A}(C_m \xi)$ decreasing to $B$. Thus,

$$P(C_m \xi B = j) = \lim_{k \to \infty} P(C_m \xi A_k = j) = \lim_{k \to \infty} e^{-\nu A_k} \cdot (\nu A_k)^j / j! = e^{-\nu B} \cdot (\nu B)^j / j!.$$

This and the complete randomness enhance that $C_m \xi = \Pi_\nu$ and therefore $\xi = \Pi_\lambda$ with $\lambda = C_m^{-1} \nu$.

This proved the first assertion of the theorem but also the only-if part of the second one: if there is convergence, $(*)$ must hold for all $A \in \mathscr{A}(C_m \xi)$. But these are exactly the sets with $\nu = C_m \lambda$-null boundary.

However, it also follows that under $(*)$ the only possible limits are completely random processes with the number of points in intervals of the type appearing in $(*)$ Poisson distributed with parameter $C_m \lambda A$. But as above this implies that if $\eta$ is a limit point of $\{C_{a_n} \eta_n\}$, then $\eta = \Pi_{C_m \lambda}$. To complete the proof of the second assertion we need therefore only establish the tightness of this sequence and apply Theorem 1. By the note following Prop. 3.2 in [6] the tightness follows from the tightness of all sequences $\{C_{a_n} \eta_n B\}$ for bounded $B$. However $(*)$ implies the convergence of $C_{a_n} \eta_n A$ for bounded intervals $A$ with $C_m \lambda$-null boundary and hence the asked for tightness.

Finally, if $P(Y_{nj} = 1) = p_n$, then

$$(C_{a_n} \eta_n) A = \sum_{j \in a_n A} Y_{nj}$$

has a limit if and only if $a_n p_n$ has and if $a_n p_n \to \alpha$, then

$$C_{a_n} \eta_n \xrightarrow{d} \Pi_{\alpha L}$$

and therefore, by Theorem 1,

$$C_{a_n} \xi_n \xrightarrow{d} \Pi_{\alpha/m L}.$$

*Note.* Define $\eta_{nj}$ to be the point process with mass $Y_{nj}$ at the point $j/a_n$. Then

$$C_{a_n} \eta_n = \sum_j \eta_{nj}$$

and part of Theorem 2 can be obtained from the asymptotic Poissonness of superpositions of point processes, cf. [6, Sect. 7].

Theorem 2 has a natural application to insurance mathematics. With each point of occurrence $X_j$ associate a claim $Z_j \geqq 0$, $P(Z_j \leqq x) = G(x) < 1$ for all $x$. The

$Z_j$'s are supposed independent of one another. Let $b_n \to \infty$ and

$$Y_{nj}=1 \quad \text{if } Z_j > b_n,$$
$$Y_{nj}=0 \quad \text{if } Z_j \leqq b_n.$$

Then we are in the case of i.i.d. deletions and with $a_n=[1-G(b_n)]^{-1}$ the large claims process, consisting of the occurrences where claims larger than $b_n$ were raised, is, with a contraction of the scale by $a_n$, approximately a stationary Poisson process with intensity $L/m$. Note that we have assumed neither that $\xi$ is renewal nor that $\{Z_j\}$ and $\xi$ are independent.

## 4. Rare Configurations in Renewal Processes

Let us now consider renewal processes on $[0, \infty)$, i.e. point processes such that the waiting times $U_j = X_j - X_{j-1}, j \geqq 1$, $U_0 = X_0$ are i.i.d. random variables. We shall assume that $0 < E[U_j] = m < \infty$. Then Theorem 1 applies by the law of large numbers.

Weak convergence of renewal processes reduces to weak convergence of the distributions of the corresponding waiting times. Explicitly, if $F_n$ is the distribution of the waiting times of $\xi_n$ then $\xi_n \xrightarrow{d} \xi$ if and only if $F_n \xrightarrow{w} F$, where $F$ is the distribution of $U_j$ in $\xi$. We give a proof of the *if*-part. Thus, assume that $F_n \xrightarrow{w} F$. Then the product measures $F_n^\infty, F^\infty$ on $R_+^\infty$ satisfy $F_n^\infty \xrightarrow{w} F^\infty$ [1, p. 14]. Set

$$A = \left\{ u = \langle u_j \rangle_1^\infty \in R_+^\infty;\ \sum_1^k u_j \to \infty \ \text{ as } k \to \infty \right\}.$$

By the law of large numbers (provided $F(0) < 1$), $F^\infty(A) = 1$. For $u \in A$ let $h(u) \in \mathcal{N}$ give mass one exactly to the points $u_0, u_0 + u_1, \ldots$. For $u \notin A$ define $h(u)$ to have no mass. The map $h$: $R_+^\infty \to \mathcal{N}$ is continuous on $A$ and by the continuous mapping theorem

$$\xi_n = h(U_0^{(n)}, U_1^{(n)}, \ldots) \xrightarrow{d} h(U_0, U_1, \ldots) = \xi$$

as $n \to \infty$.

Let us now turn to a fairly general thinning problem for renewal processes. As before $\xi$ is the underlying process with points $X_j$ and waiting times $U_j$, $0 < E[U_j] = m < \infty$. For a given natural number $k$ we retain the point $X_j$ if the $k$ preceding waiting times satisfy some specified property: let

$$V_{n0} = \min\{ j \geqq k-1;\ (U_{j-k+1}, \ldots, U_j) \in A_n \}$$

and

$$V_{ni} = \min\{ j \geqq V_{n,i-1} + k;\ (U_{j-k+1}, \ldots, U_j) \in A_n \}$$

for $i \geqq 1$. Let $Y_{nj} = 1$ if and only if $j \in \{V_{ni};\ 0 \leqq i < \infty\}$.

$\langle Y_{nj} \rangle_{j=0}^\infty$ is then a renewal process with waiting times distributed like $V_{n0}$. We assume that $0 < P((U_{j-k+1}, \ldots, U_j) \in A_n) = p_n \to 0$ as $n \to \infty$. Note that $\eta_n, \eta_n A = \sum_{j \in A} Y_{nj}$, is again a renewal process.

**Lemma 3.** *Assume that*

$$c_n = \max_{1 \leqq i \leqq k-1} P\big((U_0, \ldots, U_{k-1}) \in A_n, (U_i, \ldots, U_{k+i-1}) \in A_n\big)$$

*satisfies* $c_n/p_n \to 0$ *as* $n \to \infty$. *Then*

$$C_{p_n^{-1}} \eta_n \xrightarrow{d} \Pi_L.$$

*Proof.* We shall show that

$$\lim_{n\to\infty} P(Y_{nj}=0,\ 0 \leqq j \leqq x/p_n) = e^{-x}$$

for all $x>0$. Let the integers $l_n \uparrow \infty$ but so that $2^{l_n} p_n \to 0$ and $2^{l_n} c_n/p_n \to 0$ as $n \to \infty$. Let $B_{nj}$ be the full space when $j<k-1$ and

$$B_{nj} = \{(U_{j-k+1}, \dots, U_j) \notin A_n\}, \qquad j \geqq k-1,$$

$$C_{nj} = \bigcap_{i=(j-1)l_n}^{(j-1)l_n+k-1} B_{ni}, \qquad j \geqq 1,$$

$$D_{nj} = \bigcap_{i=(j-1)l_n+k}^{jl_n-1} B_{ni}, \qquad j \geqq 1,$$

for $l_n > k$, we obtain, writing $a_n = [([x/p_n]+1)/l_n]$,

$$P\left(\bigcap_{j=0}^{[x/p_n]} B_{nj}\right) = P\left(\bigcap_{j=1}^{a_n} (C_{nj} \cap D_{nj}) \cap \bigcap_{i=l_n a_n}^{[x/p_n]} B_{ni}\right).$$

Obviously the events $D_{nj}$, $j=1,2,\dots$, are independent with the same probability, and by the formula for the probability of the union of several events [3, p. 89] $P(D_{nj}) = 1-(l_n-k)p_n + r_n$ since, writing prime for complement, $P(B'_{nj}) = p_n, j \geqq k-1$. Here $r_n$ is the remaining part of the inclusion-exclusion formula. As, for any $(j-1)l_n + k \leqq j_1 < j_2 < \dots < j_r < jl_n$

$$P(B'_{nj_1} \cap \dots \cap B'_{nj_r}) \leqq \max_{k \leqq i_1 < i_2 < l_n} P(B'_{ni_1} \cap B'_{ni_2}) \leqq \max(c_n, p_n^2)$$

it follows that $r_n \leqq 2^{l_n} \cdot \max(c_n, p_n^2)$.
Hence,

$$P\left(\bigcap_{j=1}^{a_n} D_{nj}\right) = (1-(l_n-k)p_n + r_n)^{a_n} \to e^{-x}$$

as $n \to \infty$. But

$$P\left(\bigcap_{j=1}^{a_n} C_{nj} \cap \bigcap_{i=l_n a_n}^{[x/p_n]} B_{ni}\right) \geqq 1 - \sum_{j=1}^{a_n} \sum_{i=(j-1)l_n}^{(j-1)l_n+k-1} P(B'_{ni}) - \sum_{i=l_n a_n}^{[x/p_n]} P(B'_{ni})$$
$$\geqq 1-(a_n k + [x/p_n] - l_n a_n + 1)p_n \to 1$$

as $n \to \infty$, since $P(B'_{ni})=0$ for $0 \leqq i < k-1$, and $P(B'_{ni}) = p_n$ for larger $i$, we have proved that

$$P(Y_{nj}=0, 0 \leqq j \leqq x/p_n) = P\left(\bigcap_{j=0}^{[x/p_n]} B_{nj}\right) \to e^{-x}$$

as $n \to \infty$. By the remarks preceding the lemma this means that $C_{p_n^{-1}} \eta_n \xrightarrow{d} \Pi_L$.

**Theorem 3.** *Let $\xi$ be a renewal process on $R_+$ with finite expected time m between renewals. Let $U_j$, $j=0,1,\ldots$, be these times between renewals and let $A_n \subset R^k$ be as in Lemma 3. If $\xi_n$ is obtained from $\xi$ by retaining only those points of occurrence $X_j, j \geqq k-1$, where $(U_{j-k+1},\ldots,U_j)\in A_n$ and none of the points $X_{j-k+1},\ldots,X_{j-1}$ were retained, then*

$$C_{p_n^{-1}}\xi_n \xrightarrow{d} \Pi_{L/m}.$$

*This is true also without the last requirement that all $k-1$ points, preceding a retained one, be cancelled.*

*Proof.* By Theorem 1 and the lemma we need only check the last assertion. Let $\eta_n$ and $Y_{nj}$ be as in the lemma and

$$Y'_{nj}=1_{\{(U_{j-k+1},\ldots,U_j)\in A_n\}}, \qquad \eta'_n B=\sum_{j\in B} Y'_{nj}.$$

Evidently, $\eta'_n \geqq \eta_n$. However, for every $x>0$,

$$E[\eta'_n[0,x/p_n]]=([x/p_n]-k)\,p_n \to x$$

as $n\to\infty$, which is the same as

$$\lim_{n\to\infty} E[\eta_n[0,x/p_n]].$$

Hence

$$0\leqq E[\eta'_n[0,x/p_n]]-E[\eta_n[0,x/p_n]]\to 0$$

proving that $\eta'_n-\eta_n$ tends to the zero point process.

As an application we investigate when renewal points cluster. Assume that the $U_j$'s have distribution $F$ such that $F(0)=0$, $F(x)>0$ for $x>0$. Let $\varepsilon_n\to 0$. To get $\xi_n$ we keep those points $X_j$ for which $X_j-X_{j-k}\leqq\varepsilon_n$. Here $p_n=P(X_j-X_{j-k}\leqq\varepsilon_n)=F^{*k}(\varepsilon_n)\to 0$ and for $0\leqq i\leqq k-1$

$$P(X_{k-1}\leqq\varepsilon_n,\ X_{k+i}-X_i\leqq\varepsilon_n)\leqq P(X_{k-1}\leqq\varepsilon_n, X_k-X_{k-1}\leqq\varepsilon_n)=p_nF(\varepsilon_n).$$

Therefore Theorem 3 yields

$$C_{p_n^{-1}}\xi_n \xrightarrow{d} \Pi_{L/m}$$

provided $E[U_j]=m$.

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*(Received August 23, 1972; in revised form January 17, 1973)*