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L_2 Theory for the Stochastic Ising Model

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0. Introduction

There are two main purposes of this paper. The first is an attempt to resolve, at least partially, one of the more disturbing aspects of stochastic Ising model theory; the second is to demonstrate that the stochastic theory can be used as a powerful tool in deriving results about the equilibrium theory. The rest of this introduction is devoted to explaining what it is that we have in mind by these statements, while, at the same time, introducing the notation and background with which we will work.

Let $E=(\{-1,1\})^{Z^d}$ be given the product topology and let $\mathscr{B}$ be the associated Borel field. Elements of E will be denoted by $\eta(=\{\eta_k\colon k\in Z^d\})$ and should be thought of as the state or configuration of an infinite system of "spins" placed on the lattice Z^d. Given a non-empty finite subset F of Z^d, let $\mathscr{B}^F=\mathscr{B}[\eta_k\colon k\in F]$ (i.e., the smallest σ-algebra of subsets of E with respect to which η_k is measurable for all $k\in F$) and $\tilde{\mathscr{B}}^F=\mathscr{B}[\eta_k\colon k\notin F]$. A *potential* is a set $\{J_F\colon F$ a finite subset of $Z^d\}\subseteq R$ satisfying

$$J_{F+k}=J_F, \qquad \text{for all } k\in Z^d \text{ and all } F,$$

$$\sum_{F\ni 0}|J_F|<\infty.$$

Given a potential $\{J_F\}$, we say that μ is a *Gibbs state* with potential $\{J_F\}$ if μ is a probability measure on E and, for every $k\in Z^d$,

$$(0.1)\quad \rho_k(\{\eta_k\}|\tilde{\eta}^k)\equiv\big(1+\exp\big[2\sum_{F\ni k}J_F\prod_{j\in F}\eta_j\big]\big)^{-1}$$

is the regular conditional probability distribution on $\mathscr{B}^{\{k\}}$ of μ given $\tilde{\mathscr{B}}^{\{k\}}$. The set $\mathscr{G}=\mathscr{G}(\{J_F\})$ of Gibbs states with potential $\{J_F\}$ is a non-empty, weakly compact, convex subset of the probability measures on E (cf. [1]).

The idea behind the introduction of stochastic Ising models is to realize Gibbs states as the stationary measures of an evolution on E. The sort of evolution that we are talking about consists of the individual spins flipping, no more than one at a time, at exponential holding times determined by rate constants $\{c_k(\eta): k\in Z^d\}\subseteq\mathscr{C}^+(E)$ (the non-negative, continuous functions on E). The use of such stochastic process in this connection was initiated by Glauber [2]. In [5], we showed how such processes can be described in an abstract "martingale problem" formulation. Namely, let $\Omega=D([0,\infty),E)$ be the set of all right continuous functions $\omega:(0,\infty)\to E$ having left limits, and endow Ω with the Skorohod topology. Letting $\eta(t,\omega)$ denote the position of ω at time $t\geqq 0$, we define $\mathscr{M}_t=\mathscr{B}[\eta(s): 0\leqq s\leqq t]$ and $\mathscr{M}=\mathscr{B}[\eta(s): s\geqq 0]$. $\mathscr{M}$ is then also the Borel field over Ω). Define $\Delta_k f=f_{,k}$ for $k\in Z^d$ and $f: E\to R$ by $\Delta_k f(\eta)=f({}^k\eta)-f(\eta)$ where

$$({}^k\eta)_j=\begin{cases}\eta_j & \text{if } j\neq k\\ -\eta_k & \text{if } j=k,\end{cases}$$

and set

$$\mathscr{L}f=\sum_k c_k f_{,k}$$

for $f\in\mathscr{D}\equiv\{f\in\mathscr{C}(E): f_{,k}\equiv 0$ for all but a finite number of $k\in Z^d\}$. A probability measure P_η on $\langle\Omega,\mathscr{M}\rangle$ is said to solve the *martingale problem for $\mathscr{L}$ starting from* $\eta\in E$ if $P_\eta(\eta(0)=\eta)=1$ and $\left\langle f(\eta(t))-\int_0^t\mathscr{L}f(\eta(s))\,ds,\mathscr{M}_t,P_\eta\right\rangle$ is a martingale for all $f\in\mathscr{D}$. We showed in [5] that for any $\eta\in E$ and any choice of $\{c_k: k\in Z^d\}\subseteq\mathscr{C}^+(E)$, the martingale problem for $\mathscr{L}=\sum_k c_k\Delta_k$ starting at η has a solution. Moreover, under additional assumptions on the c_k's, we were able to show that there is exactly one such solution P_η for each $\eta\in E$; in which case the family $\{P_\eta: \eta\in E\}$ enjoys the following properties:

(i) $\{P_\eta: \eta\in E\}$ is a strong Markov, Feller continuous family,

(ii) if $\{c_k^{(n)}: k\in Z^d\}\subseteq\mathscr{C}^+(E), n\geqq 1$, and $\{\eta^n\}_1^\infty\subseteq E$ satisfy $c_k^{(n)}\to c_k$ uniformly as $n\to\infty$ for each k and $\eta^n\to\eta$, then $P_{\eta^n}^{(n)}\to P_\eta$ weakly, where $P_{\eta^n}^{(n)}$ is any solution to the martingale problem for $\mathscr{L}^{(n)}=\sum c_k^{(n)}\Delta_k$ starting from η^n.

A discussion of the connection between the martingale problem and our intuitive description of the evolution is discussed in the introduction to [5].

Let $\{J_F\}$ be a potential and define $\rho_k(\eta_k|\tilde\eta^k)$ as in (0.1). Suppose $\{c_k: k\in Z^d\}\subseteq\mathscr{C}^+(E)$ are given so that:

$$(0.2)\quad c_k(\eta)\rho_k(\eta_k|\tilde\eta^k)=c_k({}^k\eta)\rho_k(-\eta_k|\tilde\eta^k),\qquad k\in Z^d\quad\text{and}\quad \eta\in E.$$

Equation (0.2) is called the *detailed balance equation.* If the martingale problem for $\mathscr{L}=\sum_k c_k\Delta_k$ is well-posed (i.e., it has exactly one solution for each $\eta\in E$), we will say that $\{P_\eta: \eta\in E\}$ is a *stochastic Ising model with potential* $\{J_F\}$. In Lemma (1.2) we show that every Gibbs state with potential $\{J_F\}$ is a stationary measure for every stochastic Ising model with potential $\{J_F\}$. This is the essential connection between stochastic Ising models and Gibbs states.

We now come to the first problem mentioned at the beginning of this section. Namely, if there is one stochastic Ising model with potential $\{J_F\}$, then there are infinitely many. So far as we know, the physics does not dictate a canonical choice of the c_k's, although it is customary to take $c_k(\eta)=\rho_k(-\eta_k|\tilde\eta^k)$ and in some cases the choice is narrowed by special considerations (cf. [4]). It is the aim of Section (1) to show that, at least in some sense, the conclusions that can be drawn about one choice of the stochastic Ising model are valid for them all. The most serious flaw in our results is that we have been unable to show that ergodicity for one stochastic Ising model implies ergodicity for them all, although Theorem (1.12) is a step in that direction.

The second section of the paper is concerned with proving mixing properties of Gibbs states from properties of corresponding stochastic Ising models. Our main result in this direction is that if the semi-group determined by stochastic Ising model with potential $\{J_F\}$ approaches a Gibbs state μ, with the same potential, in $L^2(\mu)$ at an exponential rate and if $J_F=0$ for all $F\ni 0$ not contained in a fixed finite set F_0, then there is an $\alpha>0$ such that if $\Lambda_n=\{j\colon |j|\leqq n\}$,

$$\liminf_{n\to\infty}\frac{-1}{|n|}\log\sup_{B\in\tilde{\mathscr{B}}^{\Lambda_n}}|\mu(A\cap B)-\mu(A)\mu(B)|\geqq\alpha$$

for all $A\in\mathscr{B}$ depending only on a finite number of coordinates. Conversely, we use this result to show that there is no exponential rate at which a stochastic Ising model approaches the Gibbs measure in the critical case of the 2-dimensional classical stationary Ising model.

Some of the theorems in this paper appear, at least implicitely, in other places (cf. [3] and [9]). However, it is our impression that those which have been proved elsewhere (e.g., (i)$\Leftrightarrow$(ii) in Theorem (1.12)) find themselves in a more natural context here and that some of those which have been stated before receive here for the first time, regorous statements and complete proofs (e.g., Theorem (1.4)).

1. L_2 (Gibbs States)

Let $\{c_k\colon k\in Z^d\}\subseteq c^+(E)$ be given and set

$$\mathscr{L}=\sum_k c_k\Delta_k,$$

defined on $\mathscr{D}$. The basic assumption which we will be making throughout this section is that the martingale problem for $\mathscr{L}$ is well-posed. We will denote by $\{P_\eta\colon \eta\in E\}$ the corresponding Markov family of probability measures on Ω and by $\{T_t, t\geqq 0\}$ the Feller semi-group on $\mathscr{C}(E)$ determined by $\{P_\eta\colon \eta\in E\}$.

Let $\mathscr{G}_{\mathscr{L}}$ stand for the set of probability measures μ on E such that:

(1.1) $\int\varphi\mathscr{L}\psi\,d\mu=\int\psi\mathscr{L}\varphi\,d\mu,\qquad \varphi,\psi\in\mathscr{D}.$

It is clear that either $\mathscr{G}_{\mathscr{L}}=\emptyset$ or it is a weakly compact convex subset of probability measures on E. We will be assuming that $\mathscr{G}_{\mathscr{L}}\neq\emptyset$.

(1.2) **Lemma** (cf. Remark (1.16)). *Given a probability measure μ on E and $k\in Z^d$, let $\mu_k(\{\cdot\}|\tilde{\eta}^k)$ denote the regular conditional probability distribution on $\mathscr{B}^{\{k\}}$ of μ*

given $\mathscr{B}^{\{k\}}$. *Then the following are equivalent:*

(i) $\mu \in \mathscr{G}_{\mathscr{L}}$,

(ii) $\mu_k(\{\eta_k\}|\tilde{\eta}^k)c_k(\eta)=\mu_k(\{-\eta_k\}|\tilde{\eta}^k)c_k({}^k\eta)$ (*a.s.* μ) *for all* $k \in Z^d$,

(iii) $\int \varphi \mathscr{L} \psi \, d\mu = -1/2 \sum_k \int c_k \varphi_{,k} \psi_{,k} \, d\mu$ *for* $\varphi, \psi \in \mathscr{D}$,

(iv) $\int \varphi T_t \psi \, d\mu = \int \psi T_t \varphi \, d\mu$ *for all* $\varphi, \psi \in \mathscr{C}(E)$.

Proof. We first show that (i), (ii), and (iii) are equivalent. To prove that (i) implies (ii), note that if $\varphi, \psi \in \mathscr{D}$ and $\varphi_{,k}\psi_{,k}=0$ for all k, then $\mathscr{L}(\varphi \cdot \psi)=\varphi \mathscr{L}\psi + \psi \mathscr{L}\varphi$. In particular, if $\mu \in \mathscr{G}_{\mathscr{L}}$, then for such $\varphi, \psi \in \mathscr{D}$ we have

$$0=\int \mathscr{L}(\varphi \cdot \psi)\, d\mu = 2\int \varphi \mathscr{L}\psi \, d\mu$$

Take $\psi(\eta)=\eta_k$ and $\varphi \in \mathscr{D}$ such that $\varphi_{,k}=0$. Then $\mu \in \mathscr{G}_{\mathscr{L}}$ implies

$$\begin{aligned} 0&=\int \varphi \mathscr{L} \psi \, d\mu \\ &= -2\int \varphi(\eta)[\mu_k(\{1\}|\tilde{\eta}^k)c_k([1,\tilde{\eta}^k])-\mu_k(\{-1\}|\tilde{\eta}^k)c_k([-1,\tilde{\eta}^k])]\,\mu(d\eta), \end{aligned}$$

and (ii) follows immediately from this. The passage from (ii) to (iii) is easy, and clearly (iii) implies (i).

Finally, we must show that (i) is equivalent to (iv). Assume (i) and let $\mathscr{L}^{(n)} = \sum_{|k| \leq n} c_k \Delta_k$. Then, since (i) is equivalent to (ii), $\mu \in \mathscr{G}_{\mathscr{L}^{(n)}}$. Since $\mathscr{L}^{(n)}$ is bounded on $\mathscr{C}(E)$, the associated semi-group $\{T_t^{(n)}: t \geqq 0\}$ admits the representation

$$T_t^{(n)}\varphi = \sum_{k \geqq 0} \frac{t^k}{k!}(\mathscr{L}^{(n)})^k \varphi, \qquad \varphi \in \mathscr{C}(E),$$

and therefore

$$\int \varphi T_t^{(n)} \psi \, d\mu = \int \psi T_t^{(n)} \varphi \, d\mu$$

for all $\varphi, \psi \in \mathscr{C}(E)$. Because the solution of the martingale problem for $\mathscr{L}$ is unique, $T_t^{(n)}\varphi(\eta) \to T_t \varphi(\eta)$, uniformly in η for $\varphi \in \mathscr{C}(E)$ and $t \geqq 0$, as $n \to \infty$. This shows that (i) implies (iv). To see that (iv) implies (i), note that

$$T_t \varphi - \varphi = \int_0^t T_s \mathscr{L} \varphi \, ds, \qquad \varphi \in \mathscr{D},$$

and therefore if (iv) holds:

$$0=\int \psi T_t \varphi \, d\mu - \int \varphi T_t \psi \, d\mu = \int_0^t ds \int (\psi T_s \mathscr{L}\varphi - \varphi T_s \mathscr{L}\psi)\, d\mu$$

for $\varphi, \psi \in \mathscr{D}$. Dividing by t and letting $t \to 0$, one sees that

$$\int \psi \mathscr{L} \varphi \, d\mu = \int \varphi \mathscr{L} \psi \, d\mu$$

(1.3) **Lemma.** *If* $\mu \in \mathscr{G}_{\mathscr{L}}$, *then*

$$\int (T_t \varphi)^2 \, d\mu \leqq \int \varphi^2 \, d\mu, \qquad \varphi \in \mathscr{C}(E).$$

Proof. Taking $\psi \equiv 1$ in part (iv) of Lemma (1.2) we see that μ is a stationary measure for the family $\{P_\eta: \eta \in E\}$. Thus the result follows immediately since

$$(T_t \varphi(\eta))^2 \leqq T_t(\varphi^2)(\eta).$$

(1.4) **Theorem.** *If $\mu \in \mathcal{G}_{\mathcal{L}}$, then there is a unique strongly continuous semi-group $\{T_t^\mu: t \geqq 0\}$ of self-adjoint contradictions on $L_2(\mu)$ such that $T_t^\mu \varphi = T_t \varphi$, $\varphi \in \mathcal{D}$. In particular, if $\mathcal{L}^\mu$ is the generator of $\{T_t^\mu: t \geqq 0\}$, then $\mathcal{L}^\mu$ is a self-adjoint extension of $\mathcal{L}$. Finally, if $\{E_\lambda^\mu: \lambda \geqq 0\}$ is the resolution of the identity in $L_2(\mu)$ such that*

$$T_t^\mu = \int_0^\infty e^{-\lambda t}\, dE_\lambda^\mu,$$

then

$$\text{(1.5)} \quad \int_0^\infty \lambda\, d(E_\lambda^\mu \varphi, \varphi) = 1/2 \sum_k \int c_k \varphi_{,k}^2\, d\mu$$

for all $\varphi \in L_2(\mu)$; and therefore if $\varphi, \psi \in L_2(\mu)$ satisfy

$$\sum_k \int c_k(\varphi_{,k}^2 + \psi_{,k}^2)\, d\mu < \infty,$$

then

$$\text{(1.6)} \quad \int_0^\infty \lambda\, d(E_\lambda \varphi, \psi) = 1/2 \sum_k c_k\, \varphi_{,k}\, \psi_{,k}\, d\mu.$$

Proof. Everything except the last part of the theorem is immediate from Lemmas (1.2) and (1.3). Also (1.6) follows from (1.5) by the usual polarization trick.

To prove (1.5) set $\mathcal{L}^{(n)} = \sum_{|k| \leqq n} c_k \Delta_k$ and let $\{T_t^{(n)}: t \geqq 0\}$ and $\{E_\lambda^{(n)}: \lambda \geqq 0\}$ be defined accordingly (we have dropped the superscript μ here to simplify the notation). Then it is obvious that

$$1/2 \sum_{|k| \leqq n} \int c_k\, \varphi_{,k}^2\, d\mu = \int_0^\infty \lambda\, d(E_\lambda^{(n)} \varphi, \varphi)$$

for all $\varphi \in L_2(\mu)$, since this equality holds for $\varphi \in \mathcal{D}$ and both sides are continuous in $L_2(\mu)$.

Thus for $t > 0$:

$$\begin{aligned} 1/2 \sum_k \int c_k\, \varphi_{,k}^2\, d\mu &\geqq 1/2 \sum_{|k| \leqq n} \int c_k\, \varphi_{,k}^2\, d\mu \\ &= \int_0^\infty \lambda\, d(E_\lambda^{(n)} \varphi, \varphi) \geqq \int_0^\infty \lambda\, e^{-2\lambda t}\, d(E_\lambda^{(n)} \varphi, \varphi) \\ &= 1/2 \sum_{|k| \leqq n} \int c_k (T_t^{(n)} \varphi)_{,k}^2\, d\mu \end{aligned}$$

for all $\varphi \in L_2(\mu)$; and so by Fatou's Lemma:

$$\begin{aligned} 1/2 \sum_k \int c_k\, \varphi_{,k}^2\, d\mu &\geqq \limsup_{n \to \infty} \int_0^\infty \lambda\, e^{-2\lambda t}\, d(E_\lambda^{(n)} \varphi, \varphi) \\ &\geqq 1/2 \sum_k \int c_k (T_t \varphi)_{,k}^2\, d\mu \end{aligned}$$

for $t>0$ and $\varphi\in L_2(\mu)$. But if $t>0$ and $\varphi\in L_2(\mu)$, choose $\{\varphi_m\}_1^\infty\subseteq\mathscr{D}$ such that $\varphi_m\to\varphi$ in $L_2(\mu)$. Then

$$\begin{aligned}&\left|\int_0^\infty \lambda e^{-\lambda t}\,d(E_\lambda\varphi,\varphi)-\int_0^\infty \lambda e^{-\lambda t}\,d(E_\lambda^{(n)}\varphi,\varphi)\right|\\
&\quad\leqq\left|\int_0^\infty \lambda e^{-\lambda t}\,d(E_\lambda\varphi,\varphi)-\int_0^\infty \lambda e^{-\lambda t}\,d(E_\lambda\varphi_m,\varphi_m)\right|\\
&\qquad+\left|\int_0^\infty \lambda e^{-\lambda t}\,d(E_\lambda\varphi_m,\varphi_m)-\int_0^\infty e^{-\lambda t}\,d(E_\lambda^{(n)}\varphi_m,\varphi_m)\right|\\
&\qquad+\left|\int_0^\infty \lambda e^{-\lambda t}\,d(E_\lambda^{(n)}\varphi_m,\varphi_m)-\int_0^\infty \lambda e^{-\lambda t}\,d(E_\lambda^{(n)}\varphi,\varphi)\right|\\
&\quad\leqq\frac{4}{t}\,\|\varphi-\varphi_m\|_{L_2(\mu)}\sup_m\|\varphi_m\|_{L_2(\mu)}+|(T_t\varphi_m,\mathscr{L}\varphi_m)-(T_t^{(n)}\varphi_m,\mathscr{L}^{(n)}\varphi_m)|.\end{aligned}$$

The first term tends to zero uniformly in n as $m\to\infty$ and the second term tends to zero for each m as $n\to\infty$. Hence

$$\int_0^\infty \lambda e^{-2\lambda t}\,d(E_\lambda^{(n)}\varphi,\varphi)\to\int_0^\infty \lambda e^{-2\lambda t}\,d(E_\lambda\varphi,\varphi)$$

as $n\to\infty$ for each $t>0$ and $\varphi\in L_2(\mu)$. We therefore have

$$\begin{aligned}1/2\sum_k\int c_k\varphi_{,k}^2\,d\mu&\geqq\int_0^\infty \lambda e^{-2\lambda t}\,d(E_\lambda\varphi,\varphi)\\
&\geqq 1/2\sum_k\int c_k(T_t\varphi)_{,k}^2\,d\mu\end{aligned}$$

for all $t>0$ and $\varphi\in L_2(\mu)$. Letting $t\downarrow 0$, applying the monotone convergence theorem to the middle term and Fatou's lemma to the term on the far right we get (1.5).

(1.7) **Corollary.** *If $\mu\in\mathscr{G}_{\mathscr{L}}$ and*

$$\alpha_0\equiv\inf\{\sum_k\int c_k\varphi_{,k}^2\,d\mu/2\,\|\varphi-E_0^\mu\varphi\|_{L_2(\mu)}^2\colon \varphi\in L_2(\mu)\text{ and }\varphi\neq E_0^\mu\varphi\},$$

then $E_\lambda^\mu-E_0^\mu=0$ for $0\leqq\lambda<\alpha_0$ and $E_\lambda^\mu-E_0^\mu\neq 0$ for $\lambda>\alpha_0$. In particular, α_0 is the largest $\alpha\geqq 0$ such that $\|T_t\varphi-E_0^\mu\|_{L_2(\mu)}\leqq e^{-\alpha t}\|\varphi\|_{L_2(\mu)}$ for all $t\geqq 0$ and $\varphi\in L_2(\mu)$. Finally, the null space, $\mathscr{N}(\mathscr{L}^\mu)$, of $\mathscr{L}^\mu$ coincides with the space of $\varphi\in L_2(\mu)$ such that $\sum_k\int c_k\varphi_{,k}^2\,d\mu\equiv 0$.

(1.8) **Corollary.** *Let $\{c_k^{(i)}\colon k\in Z^d\}\subseteq\mathscr{C}^+(E)$, $i=1,2$, be given and set $\mathscr{L}^{(i)}=\sum_k c_k^{(i)}\Delta_k$. Assume that the martingale problem for $\mathscr{L}^{(i)}$ is well-posed and that $\{c_k^{(1)}>0\}=\{c_k^{(2)}>0\}$ for all $k\in Z^d$. If $\mu\in\mathscr{G}_{\mathscr{L}^{(1)}}\cap\mathscr{G}_{\mathscr{L}^{(2)}}$, then $\mathscr{N}((\mathscr{L}^{(1)})^\mu)=\mathscr{N}((\mathscr{L}^{(2)})^\mu)$ and therefore $E_0^{(1)\mu}=E_0^{(2)\mu}$. Finally, if in addition to $\{c_k^{(1)}>0\}=\{c_k^{(2)}>0\}$ we have $c_k^{(2)}\geqq\gamma c_k^{(1)}$ $k\in Z^d$, for some $\gamma>0$, then not only does $E_0^{(1)\mu}=\pi=E_0^{(2)\mu}$, but also*

$$\limsup_{t\to\infty}-\frac{1}{t}\ln\|T_t^{(1)}\varphi-\pi\varphi\|_{L_2(\mu)}\geqq\alpha$$

for all $\varphi \in \mathscr{D}$ *implies*

$$\|T_t^{(2)}\varphi - \pi\varphi\|_{L_2(\mu)} \leqq e^{-\gamma\alpha t}\|\varphi\|_{L_2(\mu)}$$

for all $t \geqq 0$ *and* $\varphi \in L_2(\mu)$.

Proof. The only part that isn't immediate from Corollary (1.7) is the last. But

$$\limsup_{t\to\infty} -\frac{1}{t}\ln\|T_t^{(1)}\varphi - \pi\varphi\|_{L_2(\mu)} \geqq \alpha, \qquad \varphi \in \mathscr{D},$$

implies $(E_\lambda^{(1)\mu} - \pi)\varphi = 0$ for $\varphi \in \mathscr{D}$ and $\lambda < \alpha$. Hence $E_\lambda^{(1)\mu} - \pi = 0$ for $\lambda < \alpha$, and so, by Corollary (1.7)

$$\inf\{\sum_k \int c_k^{(1)}\varphi_{,k}^2\, d\mu/2\,\|\varphi - \pi\varphi\|_{L_2(\mu)}: \varphi \in L_2(\mu) \text{ and } \varphi \neq \pi\varphi\} \geqq \alpha.$$

But this means that

$$\inf\{\sum_k \int c_k^{(2)}\varphi_{,k}^2\, d\mu/2\,\|\varphi - \pi\varphi\|_{L_2(\mu)}: \varphi \in L_2(\mu) \text{ and } \varphi \neq \pi\varphi\} \geqq \gamma\alpha,$$

and therefore $\|T_t^{(2)\mu}\varphi - \pi\varphi\|_{L_2(\mu)} \leqq e^{-\gamma\alpha t}\|\varphi\|_{L_2(\mu)}$, $t \geqq 0$ and $\varphi \in L_2(\mu)$.

(1.9) **Lemma.** *Assume that* $c_k > 0$ *for all* $k \in Z^d$. *Given* $\mu \in \mathscr{G}_{\mathscr{L}}$, *set* $\Lambda_\mu = \{\varphi \in L_2(\mu)$: $\int \varphi_{,k}^2\, d\mu = 0$ *for all* $k \in Z^d\}$. *Then* $\Lambda_\mu = \{\varphi \in L_2(\mu)$: φ *is* $\mathscr{T}$*-measurable*$\}$, *where* $\mathscr{T}$ *is the tail field of subsets of* E $\left(\text{i.e., } \mathscr{T} = \bigcap_0^\infty \tilde{\mathscr{B}}^{F_N} \text{ and } F_N = \{k \in Z^d: |k| \leqq N\}\right)$.

Proof. From Lemma (1.2) and the positivity of the c_k's, we see that $\mu_k(\{\pm 1\}\,|\,\tilde{\eta}^k)$ is uniformly positive for each $k \in Z^d$. It is easy to see from this that if F is a non-empty finite subset of Z^d and $\mu_F(\cdot\,|\,\tilde{\eta}^F)$ is regular conditional probability distribution of μ on $\mathscr{B}^F$ given $\tilde{\mathscr{B}}^F$, then $\mu_F(\{\alpha\}\,|\,\tilde{\eta}^F)$ may be assumed to be uniformly positive for $\alpha \in \{-1, 1\}^F$. In particular, there is an $a_F < \infty$ such that

$$\mu(\{\eta: \eta_k = \alpha_k,\, k \in F\}\,|\,\tilde{\mathscr{B}}^F) \leqq a_F\, \mu(\{\eta: \eta_k = 1,\, k \in F\}\,|\,\tilde{\mathscr{B}}^F)$$

(a.s. μ) for all $\alpha \in \{-1, 1\}^F$.

Clearly $\{\varphi \in L_2(\mu)$: φ is $\mathscr{T}$-measurable$\} \subset \Lambda_\mu$. To prove the opposite inclusion, define $\pi_F: E \to E$ for finite non-empty $F \subseteq Z^d$ so that

$$(\pi_F\,\eta)_k = \begin{cases} 1 & \text{if } k \in F \\ \eta_k & \text{if } k \notin F. \end{cases}$$

We must show that if $\varphi \in \Lambda_\mu$, then $\varphi = \varphi \circ \pi_F$ for all finite non-empty $F \subseteq Z^d$. To this end, first note that there is a $c_F < \infty$ such that

$$\mu(\pi_F^{-1}(A)) \leqq c_F\, \mu(A)$$

for all $A \in \mathscr{B}$. Indeed, it is enough to prove this for $A = A_F \cap A^F$, where $A_F = \{\eta: \eta_k = 1 \text{ for } k \in F\}$ and $A^F \in \tilde{\mathscr{B}}^F$. But

$$\begin{aligned}\mu(\pi_F^{-1}(A_F \cap A^F)) &= \sum_{\alpha \in \{-1,1\}^F} \mu(\{\eta_k = \alpha_k,\, k \in F\} \cap A^F)\\ &= \sum_{\alpha} \int_{A^F} \mu(\{\eta_k = \alpha_k = k \in F \mid \tilde{\mathscr{B}}^F)\, d\mu\\ &\leqq 2^{|F|}\, a_F \int_{A^F} \mu(\{\eta_k = 1 : k \in F\} \mid \tilde{\mathscr{B}}^F)\, d\mu\\ &= c_F\, \mu(A_F \cap A^F).\end{aligned}$$

Next note that

$$(\varphi \circ \pi_F)_{,k} = \begin{cases} 0 & \text{if } k \in F\\ \varphi_{,k} \circ \pi & \text{if } k \notin F.\end{cases}$$

Thus if $\varphi \in \Lambda_\mu$, then $\varphi \circ \pi_F \in \Lambda_\mu$. Observe that if $k \notin F$ and $F' = F \cup \{k\}$, then

$$\varphi \circ \pi_{F'} = \varphi \circ \pi_F + \frac{1 - \eta_k}{2} (\varphi \circ \pi_F)_{,k}.$$

Hence if $\varphi \in \Lambda_\mu$, then $\varphi \circ \pi_{F'} = \varphi \circ \pi_F$ (a.s. μ). Working by induction, one now sees that $\varphi = \varphi \circ \pi_F$ (a.s. μ) for all finite non-empty F.

(1.10) **Lemma.** *If $\mu \in \mathscr{G}_{\mathscr{L}}$ and $f \geqq 0$ is a bounded $\mathscr{T}$-measurable function satisfying $\int f\, d\mu = 1$, then the measure ν defined by $\dfrac{d\nu}{d\mu} = f$ is again an element of $\mathscr{G}_{\mathscr{L}}$.*

Proof. Let $\varphi, \psi \in \mathscr{D}$ be given and note that

$$\sum_k \int c_k((f\varphi)_{,k}^2 + \psi_{,k}^2)\, d\mu < \infty,$$

since $(f\varphi)_{,k} = f\varphi_{,k}$. Hence

$$\begin{aligned}\int \varphi \mathscr{L} \psi\, d\nu &= -\int_0^\infty \lambda\, d(E_\lambda^\mu(f\varphi), \psi)\\ &= -1/2 \sum_k \int c_k (f\varphi)_{,k}\, \psi_{,k}\, d\mu\\ &= -1/2 \sum_k \int c_k\, \varphi_{,k}\, \psi_{,k}\, d\nu,\end{aligned}$$

and we can apply Lemma (1.2).

(1.11) **Lemma.** *Let $\mu \in \mathscr{G}_{\mathscr{L}}$ and assume that*

$$\{\varphi \in L_2(\mu): \varphi = T_t^\mu \varphi, t \geqq 0\} = \{\varphi \in L_2(\mu): \varphi = \int \varphi\, d\mu \text{ (a.s. } \mu)\}.$$

Set $P_\mu = \int P_\eta\, \mu(d\eta)$. Then $\langle \Omega, \mathscr{M}, \theta_t, P_\mu \rangle$ is an ergodic dynamical system, where $\theta_t: \Omega \to \Omega$ is the time shift.

Proof. Clearly $P_\mu = P_\mu \circ \theta_t^{-1}$. Thus, all we have to show is that if F is a non-negative, bounded $\mathscr{M}$-measurable function satisfying $F = F \circ \theta_t$ (a.s. P_μ) for $t \geqq 0$, then

$F = E^{P_\mu}[F]$ (a.s. P_μ). Given such an F, set $f(\eta) = E^{P_\eta}[F]$. Then

$$T_t f(\eta) = E^{P_\eta}[E^{P_{\eta(t)}}[F]] = E^{P_\eta}[F \circ \theta_t] \overset{(\text{a.s. } \mu)}{=\!=\!=} E^{P_\eta}[F] = f(\eta)$$

and so $f = \int \gamma \, d\mu$ (a.s. μ). Now let $A \in \mathscr{M}_s$ be given and define, for $\Gamma \in \mathscr{B}$,

$$\nu(\Gamma) = E^{P_\mu}[I_A \, I_\Gamma(\eta(s))].$$

Clearly $\nu \ll \mu$, since $\nu(\Gamma) \leqq E^{P_\mu}[I_\Gamma(\eta(s))] = \mu(\Gamma)$. Thus

$$\begin{aligned} E^{P_\mu}[I_A F] &= E^{P_\mu}[I_A F \circ \theta_s] = E^{P_\mu}[I_A f(\eta(s))] \\ &= \int f \, d\nu = \int f \, d\mu \, \nu(E) = E^{P_\mu}[F] \, P_\mu \cdot (A), \end{aligned}$$

since $f = \int f \, d\mu$ (a.s. ν). Because $s \geqq 0$ and $A \in \mathscr{M}_s$ were arbitrary, this proves that $F = E^{P_\mu}[F]$ (a.s. P_μ).

(1.12) **Theorem.** *Assume that $c_k > 0$ for all $k \in Z^d$. Given $\mu \in \mathscr{G}_{\mathscr{L}}$, $E_0^\mu \varphi = E^\mu[\varphi \mid \mathscr{T}]$, where $\mathscr{T}$ is the tail field; and therefore $T_t^\mu \to E^\mu[\cdot \mid \mathscr{T}]$ strongly in $L_2(\mu)$ as $t \to \infty$. Moreover, the following are equivalent:*

(i) *μ is an extreme point of $\mathscr{G}_{\mathscr{L}}$,*

(ii) *$\mathscr{T}$ is μ-trivial,*

(iii) *$E_0^\mu \varphi = \int \varphi \, d\mu$, $\varphi \in L_2(\mu)$,*

(iv) *$T_t^\mu \varphi \to \int \varphi \, d\mu$ in $L_2(\mu)$ as $t \to \infty$ for $\varphi \in L_2(\mu)$,*

(v) *if $P_\mu = \int P_\eta \, \mu(d\eta)$ and $\theta_t \colon \Omega \to \Omega$ is the time shift, then $\langle \Omega, \mathscr{M}, \theta_t, P_\mu \rangle$ is an ergodic dynamical system.*

Proof. The first assertion is an immediate consequence of Lemma (1.9). Using this in conjunction with Lemma (1.10) and the obvious argument by contradiction, one sees that (i) implies (ii). Clearly, (ii) implies (iii). From the spectral representation of T_t^μ it is obvious that $T_t^\mu \to E_0^\mu$ strongly in $L_2(\mu)$ as $t \to \infty$, and therefore (iii) implies (iv). If (iv) holds, then it is clear that $\varphi = T_t^\mu \varphi$, $t \geqq 0$ implies $\varphi = \int \varphi \, d\mu$ (a.s. μ), and so (iv) implies (v) by Lemma (1.11). Finally, if $\mu = \theta \mu_1 + (1-\theta) \mu_2$ where $0 < \theta < 1$ and $\mu_1, \mu_2 \in \mathscr{G}_{\mathscr{L}}$ are distinct, then $P_\mu = \theta P_{\mu_1} + (1-\theta) P_{\mu_2}$ and therefore cannot be ergodic. Thus (v) implies (i).

(1.13) *Remark.* This section has been written from the point of view that the operator $\mathscr{L}$ is the object of primary interest. In applications to the study of phase transitions, the central role is played by the 1-dimensional conditional distributions. To see the connection between these two points of view, let $\{\rho_k(\cdot \mid \tilde{\eta}^k) \colon k \in Z^d\}$ be a family of 1-dimensional conditional Gibbsian distributions and denote by $\mathscr{G}$ the set of all probability measures with those as 1-dimensional conditionals. Choose c_k's so that $c_k > 0$ and $c_k(\eta) \, \rho_k(\eta_k \mid \tilde{\eta}^k) = c_k({}^k\eta) \, \rho_k(-\eta_k \mid \tilde{\eta}^k)$. Then, by Lemma (1.2), $\mathscr{G} = \mathscr{G}_{\mathscr{L}}$, where $\mathscr{L} = \sum\limits_k c_k \Delta_k$.

(1.14) *Remark.* It may be useful to summarize the main results of this section in the language of analysis. For this purpose, assume there is a constant $\beta > 0$ such that $c_k \geqq \beta$ for all $k \in Z^d$. Then what we have shown is that $T_t^\mu \varphi \to \int \varphi \, d\mu$ in $L_2(\mu)$ as $t \to \infty$ for $\varphi \in L_2(\mu)$ is equivalent to the statement that if $\varphi \in L_2(\mu)$ and $\int \varphi_{,k}^2 \, d\mu = 0$ for all $k \in Z^d$ then $\varphi = \int \varphi \, d\mu$ (a.s. μ). Also we have shown that the existence of an $\alpha > 0$ such that $\| T_t^\mu - \mu(\varphi) \|_{L_2(\mu)} \leqq e^{-\alpha t} \, \|\varphi\|_{L_2(\mu)}$, $t \geqq 0$ and $\varphi \in L_2(\mu)$, is equivalent to the existence of an $\alpha''>0$ such that

$$\sum_k \int \varphi_{,k}^2\, d\mu \geqq \alpha'' \, \|\varphi-\mu(\varphi)\|_{L_2(\mu)}^2, \qquad \varphi\in L_2(\mu).$$

This last inequality is, in the present context, Poincaré's inequality (or in the modern terminology: a coercive inequality). Thus, we rephrase Corollary (1.7) by saying $\|T_t^\mu-\mu(\cdot)\|_{L^2(\mu)}\leqq e^{-\alpha t}$ for some $\alpha>0$ if and only if μ satisfies Poincaré's inequality. In this connection, it is amusing to check that when μ is the Haar measure on E, then μ satisfies Poincaré's inequality with $\alpha'=2$.

(1.15) *Remark.* An easy corollary of Theorem (1.12) is that if all the c_k's are positive and if $\mu_1, \mu_2\in\mathscr{G}_{\mathscr{L}}$, then $\mu_1=\mu_2$ if and only if $\mu_1|_{\mathscr{T}}=\mu_2|_{\mathscr{T}}$. In particular if μ_1 and μ_2 are extreme points of $\mathscr{G}_{\mathscr{L}}$, then they are either equal or singular.

(1.16) *Remark.* It is important to note that there is definitely something to be proved when passing from Equation (1.1) to part (iv) of Lemma (1.2). The point is that we have not assumed that the generator of $\{T_t\colon t\geqq 0\}$ is the minimal closure of $\mathscr{L}$ restricted to $\mathscr{D}$, in which case the result would have been immediate. The trick which we used here to circumvent this point seems to have very limited application. For instance, we are unable to prove, under the stated hypotheses about $\mathscr{L}$, that if $\int\mathscr{L}\varphi\, d\mu=0$ for $\varphi\in\mathscr{D}$, then $\int T_t\varphi\, d\mu=\int\varphi\, d\mu$, $\varphi\in\mathscr{C}(E)$. The analogous problem arises in the theory of elliptic differential equations, where it is well known not to be a simple one.

2. Cluster Properties of Gibbs States

In the previous section we saw that if μ is a Gibbs state with trivial tail field and $\mu\in\mathscr{G}_{\mathscr{L}}$ then $T_t^\mu f\to\int f\, d\mu$ in $L_2(\mu)$. Trivial tail field also implies that for $A\in\mathscr{B}$

(2.1) $$\lim_{\Lambda\uparrow Z^d}\ \sup_{B\in\bar{\mathscr{B}}^\Lambda}|\mu(A\cap B)-\mu(A)\,\mu(B)|=0$$

In this section we want to improve (2.1) by giving an exponential rate at which the left side converges to zero. We do this under the assumption that T_t^μ converges to E_0^μ exponentially fast and that the flip rates, c_k, have finite range. By finite range we mean that for some $M<\infty$

(2.2) $$c_{k,j}\equiv 0 \qquad \text{if } |k-j|\geqq M.$$

Note that if

(2.3) $$\sup_k \sup_\eta |c_k(\eta)|=\sup_k \|c_k\|<\infty$$

and the c_k's have finite range, then there is a $C<\infty$ such that

(2.4) $$\sup_k \Big[\|c_k\|+\sum_j \|c_{k,j}\|\Big]\leqq C.$$

Let $\mathscr{C}^1(E)=\{f\in\mathscr{C}(E)\colon \sum_k \|f_{,k}\|<\infty\}$. If (2.4) holds, the operator $\mathscr{L}=\sum_k c_k\,\Delta_k$ can be extended to $\mathscr{C}^1(E)$ in the obvious way and in this case it is known (see [5] or [10]) that if $f\in\mathscr{C}^1(E)$ then $T_t f\in\mathscr{C}^1(E)$ for all $t\geqq 0$.

(2.5) **Lemma.** *Assume that* (2.2) *and* (2.3) *hold and let* C *be as in* (2.4). *Let* $f \in \mathscr{C}^1(E)$ *be such that* $\Delta_j f \equiv 0$ *for* $L \leqq |j| \leqq L + 2MN$, *where* M *is as in* (2.2), $L \in R^+$ *and* $N \in Z^+$. *Then*

$$(2.6) \qquad \sum_{L+(N-1)M<|k|<L+(N+1)M} \sup_{0 \leqq s \leqq t} \|\Delta_k T_s f\| \leqq 2(L+M+2MN)^d \frac{(Ct)^N}{N!} \|f\|.$$

Proof. For $f \in \mathscr{C}^1(E)$ let $u(t, \eta) = T_t f(\eta)$. By (4.11) in [5], we know that $u(t, \cdot) \in \mathscr{C}^1(E)$ and that

$$\frac{\partial u_{,k}(t, \eta)}{\partial t} = \mathscr{L} u_{,k}(t, \eta) + \sum_j c_{j,k}(\eta)\, u_{,j}(t, {}^k\eta).$$

Hence by Lemma (4.1) of [5]

$$X(s) \equiv u_{,k}(t-(t \wedge s), \eta(t \wedge s)) + \sum_j \int_0^{t \wedge s} c_{j,k}(\eta(\sigma))\, u_{,j}(t-\sigma, {}^k\eta(\sigma))\, d\sigma$$

is a P_η-martingale, and therefore

$$E^{P_\eta}[f_{,k}(\eta(t))] - u_{,k}(t, \eta) = -E^{P_\eta}\left[\sum_j \int_0^t c_{j,k}(\eta(\sigma))\, u_{,j}(t-\sigma, {}^k\eta(\sigma)\, d\sigma\right].$$

In particular if $f_{,k} \equiv 0$, then

$$\sup_{0 \leqq s \leqq t} \|u_{,k}(s, \cdot)\| \leqq \int_0^t \sum_j \|c_{j,k}\|\, \|u_{,j}(s, \cdot)\|\, ds.$$

Thus if $L \leqq a < b \leqq L + 2MN$

$$\begin{aligned}(2.7) \qquad \sum_{a<|k|<b} \sup_{0 \leqq s \leqq t} \|u_{,k}(s, \cdot)\| &\leqq \int_0^t \sum_{a<|k|<b} \sum_j \|c_{j,k}\|\, \|u_{,j}(s, \cdot)\|\, ds \\ &\leqq \int_0^t \sum_{a-M<|j|b+M} \sum_k \|c_{j,k}\|\, \|u_{,j}(s, \cdot)\|\, ds \\ &\leqq C \int_0^t \sum_{a-M<|k|<b+M} \sup_{0 \leqq s \leqq \tau} \|u_{,k}(s, \cdot)\|\, d\tau.\end{aligned}$$

Now let

$$\Phi_m(t) = \sum_{L+(N-m)M<|k|<L+(N+m)M} \sup_{0 \leqq s \leqq t} \|u_{,k}(s, \cdot)\|.$$

The left side of (2.6) is just $\Phi_1(t)$, and by (2.7) we have for $1 \leqq m \leqq N$

$$(2.8) \qquad \Phi_m(t) \leqq C \int_0^t \Phi_{m+1}(s)\, ds.$$

Also

(2.9) $\Phi_{N+1}(t) \leqq 2\|f\| \sum_{L-M<|k|<L+2MN+M} 1 \leqq 2[L+M+2MN]^d \|f\|.$

The result follows easily from (2.8) and (2.9) by induction.

(2.10) **Lemma.** *Let S and T be disjoint, non-empty, complementary subsets of Z^d. Suppose $\{c_k = k \in Z^d\} \subseteq \mathscr{C}^+(E)$ has the property that $c_{k,j} \equiv 0$ for $k \in S$ $(k \in T)$ and $j \in T (k \in S)$. If the martingale problem for $\mathscr{L} = \sum_k c_k \Delta_k$ starting from η has exactly one solution, then $P_\eta(A \cap B) = P_\eta(A) P_\eta(B)$ for $A \in \mathscr{M}^S \equiv \mathscr{B}[\eta_k(\cdot): k \in S]$ $B \in \mathscr{M}^T \equiv \mathscr{B}[\eta_k(\cdot): k \in T]$, where P_η is the unique solution.*

Proof. Let $\mathscr{L}^s = \sum_{k \in S} c_k \Delta_k$ and $\mathscr{L}^T = \sum_{k \in T} c_k \Delta_k$. Given $\eta \in E$, let $\eta^S = \{\eta_k : k \in S\}$, $\eta^T = \{\eta_k : k \in T\}$, $P^S_{\eta^S}$ on $\Omega_S = D([0, \infty], \{-1, 1\}^S)$ be a solution to the martingale problem for $\mathscr{L}^S$ starting from η^S, and $P^T_{\eta^T}$ on $\Omega_T = D([0, \infty], \{-1, 1\}^T)$ be a solution to the martingale problem for $\mathscr{L}^T$ starting from η^T. Let $\varphi: \{-1, 1\}^S \times \{-1, 1\}^T \to E$ be the natural isomorphism between these spaces and let $\Phi: \Omega_S \times \Omega_T \to \Omega$ be the obvious lifting of φ (i.e., $\eta(\cdot, \Phi(\omega_S, \omega_T)) = \varphi(\eta^S(\cdot, \omega_S), \eta^T(\cdot, \omega_T))$. Set $P = (P^S_{\eta^S} \times P^T_{\eta^T}) \circ \Phi^{-1}$. Then it is elementary to check that P solves the martingale problem for $\mathscr{L}$ starting from η. (c.f. the equivalence of (i) and (iv) of Theorem (1.1) in [5]). Since, if $A \in \mathscr{M}^S$ and $B \in \mathscr{M}^T$, then $A \cap B = \Phi(A^S \times B^T)$, where A^S and B^T are measurable subsets of Ω_S and Ω_T, respectively, this completes the proof.

We learned the trick used in the proof of the next theorem from Sullivan [11].

(2.11) **Theorem.** *Assume that* (2.2) *and* (2.3) *hold and let C be as in* (2.4). *Let $f, g \in \mathscr{C}(E)$ be such that $\Delta_j f \equiv 0$ if $|j| \geqq L$ and $\Delta_j g \equiv 0$ if $|j| \leqq L + 2MN$. Then*

$$\sup_{0 \leqq s \leqq t} \|T_s(f \cdot g) - T_s f \cdot T_s g\| \leqq 6(L+M+2MN)^d \frac{(Ct)^{N+1}}{(N+1)!} \|f\| \|g\|. \tag{2.12}$$

Proof. Let L, N, f, and g be fixed. Let $\mathscr{L}'$ be the operator

$$\mathscr{L}' = \sum_{|k| \leqq L+(N-1)M} c_k \Delta_k + \sum_{|k| \geqq L+(N+1)M} c_k \Delta_k. \tag{2.13}$$

Since the c_k's satisfy (2.4), the martingale problem for $\mathscr{L}'$ is well-posed and hence by Lemma (2.10), if $\{U_t : t \geqq 0\}$ is the semi-group associated with $\mathscr{L}'$, we have

$$U_t(f \cdot g) = (U_t f) \cdot (U_t g). \tag{2.14}$$

Let $\varphi \in \mathscr{C}(E)$ be such that $\varphi_{,k} \equiv 0$ for $L \leqq |k| \leqq L + 2MN$. We show that

$$\|T_t \varphi - U_t \varphi\| \leqq 2(L+M+2MN)^d \frac{(Ct)^{N+1}}{(N+1)!} \|\varphi\|. \tag{2.15}$$

The theorem then follows by taking φ to be $f \cdot g$, f, and g in the inequality

$$\begin{aligned} &\|T_t(f \cdot g) - T_t f \cdot T_t g\| \\ &\quad \leqq \|T_t(f \cdot g) - U_t(f \cdot g)\| + \|U_t f\| \|U_t g - T_t g\| \\ &\qquad + \|T_t g\| \|U_t f - T_t f\|, \end{aligned}$$

which follows from (2.14).

To prove (2.15) we first take $\varphi \in \mathscr{C}^1(E)$ and note that

$$T_0\,\varphi - U_0\,\varphi = 0$$

and

$$\frac{\partial}{\partial t}(T_t\,\varphi) - U_t\,\varphi) = \mathscr{L}\,T_t\,\varphi - \mathscr{L}'\,U_t\,\varphi$$
$$= \mathscr{L}'(T_t\,\varphi) - U_t\,\varphi) + (\mathscr{L} - \mathscr{L}')\,T_t\,\varphi.$$

Solving this equation we get

$$T_t\,\varphi - U_t\,\varphi = \int_0^t U_{t-s}(\mathscr{L} - \mathscr{L}')\,T_s\,\varphi\,ds.$$

Thus

$$\|T_t\,\varphi - U_t\,\varphi\| \leqq \int_0^t \|(\mathscr{L} - \mathscr{L}')\,T_s\,\varphi\|\,ds$$
$$\leqq \int_0^t \sum_{L+(N-1)M<|k|<L+(N+1)M} \|c_k\|\,\|\Delta_k\,T_s\,\varphi\|\,ds$$
$$\leqq 2(L+M+2MN)^d\,\frac{(C\,t)^{N+1}}{(N+1)!}\,\|\varphi\|.$$

The last inequality follows from Lemma (2.5). The proof of (2.15) for continuous φ follows by taking a limit.

(2.16) **Theorem.** *Assume that* (2.2) *and* (2.3) *hold and let C be as in* (2.4). *Let μ be a stationary distribution for $\{P_\eta\colon \eta \in E\}$ such that for some $\alpha > 0$ and $A < \infty$*

$$\sup_{t \geqq 0} e^{\alpha t}\,\|T_t\,\varphi - \int \varphi\,d\mu\|_{L_1(\mu)} \leqq A\,\|\varphi\|$$

for all $\varphi \in \mathscr{C}(E)$. Let $\varphi \in \mathscr{D}$ and $\{\Lambda_n\}$ be an increasing sequence of finite sets with $\bigcup_n \Lambda_n = Z^d$. Denote the distance from the origin to Λ_n^c by r_n. Then

$$\text{(2.17)}\quad \liminf_{n\to\infty}\,\inf\{-r_n^{-1}\ln|\int \varphi\,\psi\,d\mu - \int \varphi\,d\mu \int \psi\,d\mu|\colon \|\psi\|_{L_\infty(\mu)} \leqq 1$$
$$\textit{and } \psi \textit{ is } \tilde{\mathscr{B}}^{\Lambda_n}\textit{-measurable}\} \geqq \frac{\alpha\gamma}{2MC},$$

where $0 < \gamma < 1$ solves $\dfrac{-\alpha\gamma}{C} = 1 + \ln\gamma$.

Proof. First note that it suffices to prove (2.17) when the infumium is restricted to continuous functions. In the string of inequalities below the infumium over ψ is over continuous, $\tilde{\mathscr{B}}^{\Lambda_n}$-measurable, function ψ with $\|\psi\| \leqq 1$.

Let L be large enough so that φ is $\mathscr{B}[\eta_j: |j|<L]$-measurable. Let N_n be the greatest integer in $(r_n-L)/2M$. Then $r_n/(N_n+1)\to 2M$, and we have

$$\begin{aligned}
&\sup_\psi |\int \varphi\psi\, d\mu - \int \varphi\, d\mu \int \psi\, d\mu| \\
&\quad \leqq \sup_\psi |\int (T_t(\varphi\cdot\psi) - T_t\varphi\cdot T_t\psi)\, d\mu| \\
&\qquad + \sup_\psi |\int T_t\varphi\cdot T_t\psi\, d\mu - \int\varphi\, d\mu\int\psi\, d\mu| \\
&\quad \leqq 6(L+M+2MN_n)^d \frac{(C\,t)^{N_n+1}}{(N_n+1)!}\|\varphi\| \\
&\qquad + \sup_\psi \|\varphi\|\; \|T_t\psi - \int\psi\, d\mu\|_{L_1(\mu)} + \|T_t\varphi - \int\varphi\, d\mu\|_{L_1(\mu)} \\
&\quad \leqq \left[6(L+M+2MN_n)^d \frac{(C\,t)^{N_n+1}}{(N_n+1)!} + 2e^{-\alpha t} A\right]\|\varphi\|.
\end{aligned}$$

One now sets $t=\gamma(N_n+1)/C$ and applies Stirlings' formula to complete the proof.

(2.18) *Remark.* If the conclusion of Theorem (2.16) holds then there is an $\alpha>0$ such that for all finite A, $B\subset Z^d$

$$\text{(2.19)}\qquad \lim_{|r|\to\infty} e^{\alpha|r|}\,|\int \prod_{j\in A}\eta_j \prod_{k\in B+r}\eta_k\, d\mu - \int\prod_{j\in A}\eta_j\, d\mu \int \prod_{k\in B+r}\eta_k\, d\mu| = 0.$$

There are several interesting physical consequences of (2.19). Rather than try to explain them here we refer the reader to [7].

(2.20) *Remark.* Let μ be a Gibbs state with finite range potential $\{J_R\}$ and assume that μ has trivial tail field. Suppose there is an operator $\mathscr{L}$ with finite range flip rates such that $\mu\in\mathscr{G}_{\mathscr{L}}$ and that $\mathscr{L}^\mu$ has a gap in its spectrum immediately below 0. (This is really a property of μ and does not depend on the choice of $\mathscr{L}$, see Corollary (1.18).) Then from section one and Theorem (2.16) we see that (2.19) holds. Another condition which implies (2.19) is that the transfer matrix for the potential $\{J_R\}$ have a gap in its spectrum (see [8]). It would be interesting to know if the similarity in these two statements is more than just a coincidence.

(2.21) *Remark.* We can use Theorem (2.16) to give a natural example where $T_t^\mu f\to\int f\, d\mu$ for all $f\in L_2(\mu)$ and yet the convergence may be made arbitrairly slow by taking f properly. The example is the two dimensional stochastic Ising model at the critical temperature. Thus $d=2$ and

$$J_R=\begin{cases} -\sinh^{-1}(1) & \text{if } R=\{j,k\} \quad \text{with} \quad |j-k|=1 \\ 0 & \text{otherwise.}\end{cases}$$

Let μ be the Gibbs state with one point conditionals given by (0.1) with this potential. It is known that there is only one such μ. Also if we take $c_k(\eta)=\rho_k(-\eta_k\,|\,\tilde{\eta}^k)$, then the interaction is attractive in the sense of [4], and hence as $t\to\infty$

$$\|T_t f - \int f\, d\mu\| \to 0 \qquad \text{for all } f\in\mathscr{C}(E).$$

Thus $T_t^\mu\varphi\to\int\varphi\, d\mu$ in $L_2(\mu)$ for all $\varphi\in L_2(\mu)$. Also $\mathscr{L}$ has finite range flip rates; however, (2.19) does *not* hold (see [6]). Hence by Theorem (2.16) the $L_2(\mu)$ convergence of $T_t^\mu\varphi$ to $\int\varphi\, d\mu$ must not be uniformly exponentially fast. Therefore if $\{E_\lambda^\mu: \lambda \geqq 0\}$ is as in Theorem (1.4) we have $E_\lambda^\mu - E_0^\mu \neq 0$ for all $\lambda > 0$, and it follows that by choosing $\varphi \in L_2(\mu)$ properly the convergence can be made as slow as desired. This also shows that the convergence of $T_t f$ to $\int f d\mu$ in the uniform norm cannot be exponentially fast uniformly for $f \in \mathscr{D}$.

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