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On the Convergence of Convolutions of Distributions with Regularly Varying Tails

THOMAS HÖGLUND

Summary. Let $x_1, \dots, x_n$ be a sequence of independent random variables with the common distribution F. Suppose $E\, x_k = 0$ and that F belongs to the domain of attraction of the normal distribution. Under conditions which do not involve the existence of any particular moment we show that

$$P\{x_1 + \dots + x_n \leqq x\, a_n\} = \Phi(x) - \frac{n}{a_n^3} \int_{-a_n}^{a_n} |y|^3 F(dy) \big(\omega\, \Phi(x) + o(1)\big)$$

uniformly in x, provided the norming constants $a_1, a_2, \dots$ are properly chosen. Here Φ is the standard normal distribution and ω a certain operator (depending on F).

The local counterparts are also treated.

1. Introduction and Results

Suppose $x_1, \dots, x_n$ are independent, identically distributed random variables with zero expectation and with a distribution F that belongs to the domain of attraction of the normal distribution. Then there is a sequence of numbers $a_1, a_2, \dots \to \infty$ satisfying

$$\frac{n}{a_n^2} \int_{-a_n}^{a_n} x^2 F(dx) \to 1 \qquad \text{as } n \to \infty. \tag{1}$$

Obviously $a_n \sim \sqrt{n}\, \sigma$ if the variance exists and equals σ^2. Let F_n denote the distribution of the normed sum $a_n^{-1}(x_1 + \dots + x_n)$. If we require that the third moment μ_3 exists and that F is not a lattice distribution, then (Esséen [2] p. 49)

$$F_n(x) = \Phi(x) - \frac{\mu_3}{6\sigma^3 \sqrt{n}} (x^2 - 1)\, \varphi(x) + o\left(\frac{1}{\sqrt{n}}\right)$$

uniformly in x. Here Φ stands for the normal distribution with zero expectation and unit variance and φ for its density.

The aim of the present paper is to prove an analogous result when the third moment does not exist. (Concerning estimates of the Berry-Esséen type in this case, see Feller [5], Esséen [3] and the references given there.) Instead we have to impose other conditions. To formulate these we shall say that a positive function R defined on $(0, \infty)$ varies regularly with exponent $\alpha\,(-\infty < \alpha < \infty)$ if, for each $x > 0$, $\frac{R(t\,x)}{R(t)} \to x^\alpha$ as $t \to \infty$. R varies slowly if R varies regularly with exponent zero.

Our conditions will be: for some slowly varying function L, as $x\to\infty$,

$$\int_{-x}^{x} |y|^3\, F(dy) \sim x^{1-\delta} L(x), \qquad 0\leqq\delta\leqq 1, \tag{2}$$

and

$$\frac{\int_0^x y^3\, F(dy)}{\int_{-x}^{x} |y|^3\, F(dy)} \to p, \qquad \frac{\int_{-x}^{0} |y|^3\, F(dy)}{\int_{-x}^{x} |y|^3\, F(dy)} \to q. \tag{3}$$

Except when $\delta=1$, (2) and (3) are equivalent to

$$1-F(x)+F(-x) \sim \frac{1-\delta}{2+\delta}\, x^{-2-\delta} L(x) \tag{2'}$$

and

$$\frac{1-F(x)}{1-F(x)+F(-x)} \to p, \qquad \frac{F(-x)}{1-F(x)+F(-x)} \to q. \tag{3'}$$

The verification of this equivalence is immediate, given Lemma 1 below. Conditions similar to those above have proved successful in connection with large deviation probabilities, see Heyde [8] and especially Heyde [6] where a result of McLaren concerning the convergences toward a normal distribution is cited.

Introduce the truncation function

$$\tau(x)=x \;\text{ if }\; |x|\leqq 1, \;=0\; \text{ if }\; |x|>1$$

and the functions

$$\Omega_\delta(x)=\begin{cases} p\, x^{1-\delta}, & x>0,\\ q\,|x|^{1-\delta}, & x<0,\end{cases}$$

where p and q are defined in (3). The results will be formulated in terms of the derivation operator D and the operators ω_δ and $\tilde{\omega}_\delta$ defined, for those u for which the definitions make sense, by

$$\begin{aligned} \omega_\delta\, u(x) &= \frac{1}{\delta(\delta+1)(\delta+2)} \int u(x-y)\, \Omega_\delta(dy), \qquad 0<\delta\leqq 1\\ \tilde{\omega}_\delta\, u(x) &= \int \frac{u(x-y)-u(x)+y\,Du(x)-\frac{1}{2}\tau(y)^2 D^2 u(x)}{y^3}\, \Omega_\delta(dy), \qquad 0\leqq\delta\leqq 1. \end{aligned} \tag{4}$$

For later use we here mention that $\tilde{\omega}_\delta \Phi$ has the Fourier-Stieltjes transform $e^{-t^2/2} h(t)$, where

$$h(t)=\int \frac{e^{itx}-1-itx-\frac{1}{2}(it\,\tau(x))^2}{x^3}\, \Omega_\delta(dx). \tag{5}$$

This is seen if we use the representation (4) of $\tilde{\omega}_\delta \Phi$, then apply Fubinis theorem and remember that

$$\int e^{itx} D^k \Phi(dx) = (-it)^k e^{-t^2/2}, \qquad k=0,1,\ldots.$$

The following theorem is closely related to the theorem of Bergström, [1] and also to that of Heyde, [7] in which he shows that (with $a_n=\sigma\sqrt{n}$ and without any extra assumptions such as ours (2) and (3)).

$$\sum_{n=1}^{\infty} n^{-1+\delta/2} \sup_x |F_n(x) - \Phi(x)| < \infty$$

if and only if

$$E|x_k|^{2+\delta} < \infty, \quad 0<\delta<1, \quad E|x_k| \log(1+|x_k|) < \infty, \quad \delta=0.$$

Theorem 1. *Suppose F satisfies* (2) *and* (3) *and has an infinite absolute third moment.*

(i) *If* $\delta>0$ *we choose* $a_n = \sigma\sqrt{n}$, *where* $\sigma^2 = \int x^2 F(dx)$; *then*

$$F_n(x) = \left(1 - \frac{L(\sqrt{n})}{\sigma^{2+\delta} n^{\delta/2}} \omega_\delta D^3\right) \Phi(x) + o\left(\frac{L(\sqrt{n})}{n^{\delta/2}}\right) \tag{6}$$

uniformly in x.

(ii) *If* $\delta=0$ *then it is possible to choose the sequence* $a_1, a_2, \ldots$ *such that*

$$\frac{n}{a_n^2} \int_{-a_n}^{a_n} y^2 F(dy) = 1 + o\left(\frac{n L(a_n)}{a_n^2}\right). \tag{7}$$

With such a choice of $a_1, a_2, \ldots$,

$$F_n(x) = \left(1 + \frac{n L(a_n)}{a_n^2} \tilde{\omega}_0\right) \Phi(x) + o\left(\frac{n L(a_n)}{a_n^2}\right)$$

uniformly in x.

Remark. A choice of $a_1, a_2, \ldots$ according to (7) gives the fastest possible rate of convergence if $\delta=0$; if $\delta>0$ we have nothing to gain by another choice than $a_n = \sigma\sqrt{n}$. In order to show this, it will in the proof of Theorem 1 be shown that $F_n(x) = (1 + c_n \tilde{\omega}_\delta + \rho_n \frac{1}{2} D^2) \Phi(x) + o(c_n) + o(\rho_n)$ uniformly in x, where

$$c_n = \frac{n}{a_n^3} \int_{-a_n}^{a_n} |y|^3 F(dy) \sim \frac{n L(a_n)}{a_n^{2+\delta}}, \qquad \rho_n = \frac{n}{a_n^2} \int_{-a_n}^{a_n} y^2 F(dy) - 1. \tag{8}$$

We first show that the choice of a_n does not influence the order of magnitude of c_n. Suppose $a_1^{(i)}, a_2^{(i)}, \ldots i=1,2$, are two sequences satisfying (1), then

$$a_n^{(1)} \sim a_n^{(2)}. \tag{9}$$

Let $c_n^{(i)}$ and $\rho_n^{(i)}$ stand for the c_n and ρ_n corresponding to $a_n^{(i)}$, $i=1,2$. Now $c_n^{(i)} = n\, a_n^{(i)^{-3}} R(a_n^{(i)})$, where R is a nondecreasing, regularly varying function; because of (9) we therefore have $c_n^{(1)} \sim c_n^{(2)}$.

Thus we obtain the fastest possible rate of convergence if we choose a_n such that $\rho_n = O(c_n)$; this is certainly satisfied if we choose a_n according to (7). If the variance σ^2 exists and $a_n = \sigma\sqrt{n}$, then

$$-\rho_n = \frac{1}{\sigma^2} \int_{|y|>\sigma\sqrt{n}} y^2 F(dy).$$

It follows from Lemma 1 that in this case

$$-\frac{\rho_n}{c_n} \to \begin{cases} \dfrac{1-\delta}{\delta} & \text{if } \delta>0, \\ \infty & \text{if } \delta=0. \end{cases} \tag{10}$$

This shows that $\rho_n = O(c_n)$ if $\delta > 0$ and $a_n = \sigma\sqrt{n}$; however, if $\delta = 0$, then a choice according to (7) is preferable to the choice $a_n = \sigma\sqrt{n}$.

Theorems 2 and 3 are the local counterparts of Theorem 1, for absolutely continuous and lattice distributions respectively.

Theorem 2. *Suppose F satisfies the conditions of Theorem 1, is absolutely continuous, and has characteristic function $f \in L^r$ for some $1 \leq r < \infty$. Denote the density of F_n by v_n.*

(i) *If $\delta > 0$ we choose $a_n = \sigma\sqrt{n}$, where $\sigma^2 = \int x^2 F(dx)$; then*

$$v_n(x) = \left(1 - \frac{L(\sqrt{n})}{\sigma^{2+\delta} n^{\delta/2}} \omega_\delta D^3\right) \varphi(x) + o\left(\frac{L(\sqrt{n})}{n^{\delta/2}}\right)$$

uniformly in x.

(ii) *If $\delta = 0$ then it is possible to choose the sequence $a_1, a_2, \ldots$ such that* (7) *is valid. With such a choice of $a_1, a_2, \ldots$*

$$v_n(x) = \left(1 + \frac{n L(a_n)}{a_n^2} \tilde{\omega}_0\right) \varphi(x) + o\left(\frac{n L(a_n)}{a_n^2}\right)$$

uniformly in x.

If F is a lattice distribution with span h then the variables x_j are restricted to values of the form $b, b \pm h, b \pm 2h, \ldots$ and h is the largest positive number with this property. The atoms of the distribution of $a_n^{-1}(x_1 + \cdots + x_n)$ are among the points of form $x = a_n^{-1}(nb + kh)$, where $k = 0, \pm 1, \pm 2, \ldots$. For such x we define

$$p_n(x) = P\left(a_n^{-1}(x_1 + \cdots + x_n) = x\right).$$

Theorem 3. *Suppose F is a lattice distribution with span h and assume that F satisfies the conditions of Theorem 1. Let x be of form $a_n^{-1}(nb + kh)$, k integer.*

(i) *If $\delta > 0$ we choose $a_n = \sigma\sqrt{n}$, where $\sigma^2 = \int x^2 F(dx)$, then*

$$p_n(x) = \left(1 - \frac{L(\sqrt{n})}{\sigma^{2+\delta} n^{\delta/2}} \omega_\delta D^3\right) \varphi(x) + o\left(\frac{L(\sqrt{n})}{n^{\delta/2}}\right)$$

uniformly in x.

(ii) *If $\delta = 0$ then it is possible to choose the sequence $a_1, a_2, \ldots$ such that* (7) *is valid. With such a choice of $a_1, a_2, \ldots$*

$$p_n(x) = \left(1 + \frac{n L(a_n)}{a_n^2} \tilde{\omega}_0\right) \varphi(x) + o\left(\frac{n L(a_n)}{a_n^2}\right)$$

uniformly in x.

2. Three Lemmas

The following lemma can be found in problem 30, p. 279 of [4].

Lemma 1. *Let F be a probability distribution concentrated on $(0, \infty)$. Put*

$$U_\zeta(x) = \int_0^x y^\zeta F(dy), \qquad V_\sigma(x) = \int_x^\infty y^{-\sigma} F(dy).$$

If F possesses moments of order $<\alpha$, but not of order $>\alpha>0$, then either U_ζ and V_σ vary regularly for all $\zeta>\alpha$ and all $\sigma>-\alpha$ or for no such value. In the first case the exponents are $\zeta-\alpha$ and $\sigma-\alpha$, and this case arises if and only if

$$\frac{x^{\sigma+\zeta}V_\sigma(x)}{U_\zeta(x)}\to\frac{\zeta-\alpha}{\sigma+\alpha} \tag{11}$$

($0<\alpha<\zeta$). *If U_ζ varies slowly,* (11) *holds with $\alpha=\zeta$; if V_σ varies slowly the left side tends to* ∞.

In applications of Lemma 1 to distributions not concentrated on $(0,\infty)$ we treat the two tails separately.

We also state a lemma which is contained in Theorem 2a, p. 32 of [2].

Lemma 2. *Let F be a probability distribution with characteristic function f. Let G be a real function of bounded variation over the whole real axis, such that $G(-\infty)=0$, $G(+\infty)=1$, while the derivative G' exists everywhere and satisfies $|G'|<K$ for some constant K. Write*

$$g(t)=\int e^{itx}G(dx).$$

Suppose that for some positive constants T and η we have

$$\int_{-T}^{T}\left|\frac{f(t)-g(t)}{t}\right|dt=\eta.$$

Then there are positive constants A and B independent of T and η such that for all real x

$$|F(x)-G(x)|<A\eta+\frac{B}{T}.$$

Finally we need a bound for characteristic functions of distributions in the domain of attraction of the normal distribution.

Lemma 3. *Let F be a probability distribution with characteristic function f. Suppose $\int_{-x}^{x}y^2F(dy)$ is slowly varying and the sequence $a_1,a_2,\ldots$ satisfies* (1). *Then there are positive constants ε and c such that*

$$|f^n(t/a_n)|\leqq e^{-c|t|} \tag{12}$$

for $1<|t|<\varepsilon a_n$ and all sufficiently large n.

Proof. We may without loss of generality assume that $\int xF(dx)=0$. Introduce the symmetrized distribution ${}^0F(x)=\int F(x+y)F(dy)$; then also

$$M(x)=\int_{-x}^{x}y^2\,{}^0F(dy)$$

is slowly varying, and

$$\frac{n}{a_n^2}M(a_n)\to 2. \tag{13}$$

This becomes obvious if we use the fact that a distribution G with zero expectation belongs to the domain of attraction of the normal distribution if and only if $\int_{-x}^{x} y^2 G(dy)$ varies slowly. The variance of the corresponding normal distribution then equals the limit $\lim_n \frac{n}{a_n^2} \int_{-a_n}^{a_n} y^2 G(dy)$ if we use a_n as norming constants. (See [3] p. 544.) Further ${}^0F$ has characteristic function $|f(t)|^2$ and because ${}^0F$ is symmetric

$$1-|f(t)|^2=\int(1-\cos tx)\,{}^0F(dx).$$

Now $\frac{1-\cos y}{y^2}$ is non negative for each y and larger than $\frac{2}{5}$ for $|y|\leqq 1$, hence

$$1-|f(t)|^2 \geqq \int_{|x|\leqq 1/|t|} \cdot \frac{1-\cos tx}{(tx)^2}(tx)^2\,{}^0F(dx) \geqq \tfrac{2}{5} t^2 M\left(\frac{1}{|t|}\right).$$

From this and the inequality $1-x\leqq e^{-x}$ valid for all $x\geqq 0$ we obtain

$$|f(t)|^2 \leqq e^{-\frac{2}{5}t^2 M(1/|t|)},$$

i.e.,

$$|f(t)| \leqq e^{-\frac{1}{5}t^2 M(1/|t|)}.$$

Thus it suffices to show that there are positive numbers ε and c such that

$$|t|\frac{n}{a_n^2} M\left(\frac{a_n}{|t|}\right) > c \tag{14}$$

for $1<|t|<\varepsilon a_n$ and all sufficiently large n. From the representation

$$M(x)=b(x)\exp\left(\int_1^x \eta(y)\frac{dy}{y}\right)$$

where $b(x)\to b>0$ and $\eta(x)\to 0$ as $x\to\infty$ (see [4], p. 274) it now follows that

$$\frac{M\left(\frac{a_n}{|t|}\right)}{M(a_n)} \geqq \frac{b\left(\frac{a_n}{|t|}\right)}{b(a_n)} \exp(-\sup_{y>a_n/|t|} |\eta(y)| \log|t|) \geqq \tfrac{1}{2}|t|^{-1}$$

for $1\leqq|t|\leqq\varepsilon a_n$ and all sufficiently large n, if $\varepsilon>0$ is chosen so small that $|\eta(y)|<1$ and $b(y)>\frac{2}{3}b$ for all $y>1/\varepsilon$.

Because of (13), (14) follows.

3. Proof of Theorem 1

Below we will be concerned with the sequence $U_1, U_2, \ldots$ of measures defined by

$$U_n(dx)=\frac{(a_n x)^3 F(a_n dx)}{\int_{-a_n}^{a_n} |y|^3 F(dy)}.$$

Let us point out that

$$U_n(I) \to \int_I \Omega_\delta(dx) \tag{15}$$

for each bounded interval I whose boundary has Ω_δ-measure zero, and that for each $\varepsilon > 0$ there is an $x > 0$ such that for all n

$$\int_{|y|>x} y^{-2} |U_n|(dy) < \varepsilon. \tag{16}$$

The first of these statements is a direct consequence of (2) and (3). If we treat each tail of U_n separately and apply Lemma 1 we get

$$\int_{|y|>x} y^{-2} |U_n|(dy) \to \frac{1-\delta}{1+\delta} x^{-1-\delta} \quad \text{as } n \to \infty$$

from which the second statement follows.

Denote the characteristic function of F by f. Using the identity

$$u^n - v^n = (u-v)(u^{n-1} + u^{n-2} v + \cdots + v^{n-1})$$

with $u = f(t/a_n)$ and $v = \exp\left(-\frac{1}{2}\frac{t^2}{n}\right)$ we obtain

$$f^n(t/a_n) - e^{-t^2/2} = d_n(t)\, S_n(t) \tag{17}$$

where

$$d_n(t) = n\left[f(t/a_n) - \exp\left(-\frac{1}{2}\frac{t^2}{n}\right)\right],$$

$$S_n(t) = \frac{1}{n}\sum_{j=0}^{n-1} f^j(t/a_n) \exp\left(-\frac{1}{2} t^2 \frac{n-j-1}{n}\right).$$

Remembering that F has zero expectation we get trivially

$$f(t/a_n) = \int \left[e^{itx} - 1 - itx - \tfrac{1}{2}(it\,\tau(x))^2\right] F(a_n\, dx) + 1 + \tfrac{1}{2}(it)^2 \int \tau(x)^2 F(a_n\, dx).$$

Hence

$$d_n(t) = c_n h_n(t) + \rho_n \tfrac{1}{2}(it)^2 + r_n(t) \tag{18}$$

where c_n and ρ_n are defined in (8), and

$$h_n(t) = \int \frac{e^{itx} - 1 - itx - \frac{1}{2}(it\,\tau(x))^2}{x^3} U_n(dx),$$

$$r_n(t) = -n\left[\exp\left(-\frac{1}{2}\frac{t^2}{n}\right) - 1 + \frac{1}{2}\frac{t^2}{n}\right].$$

From (15) and (16) we obtain $h_n(t) \to h(t)$, where h is defined in (5). The difference $S_n(t) - e^{-t^2/2}$ we estimate in the following way:

$$\left|\frac{1}{n}\sum_{j=0}^{n-1} u^j v^{n-1-j} - v^n\right| = \left|\frac{1}{n}\sum_{j=0}^{n-1} v^{n-j-1}(u^j - v^j) + v^{n-1} - v^n\right|$$
$$\leqq \frac{1}{n}\sum_{j=0}^{n-1} j\,|u-v| + |1-v| \leqq n\,|u-v| + |1-v|.$$

(Here we have used the fact that $|u|\leqq 1$, $|v|\leqq 1$ and that the inequality $|u^j-v^j|\leqq j|u-v|$ is valid for such u and v.) Now $f^n(t/a_n)\to e^{-t^2/2}$ implies $n(f(t/a_n)-1)\to -t^2/2$. Hence $S_n(t)\to e^{-t^2/2}$. The inequality $|e^{-x}-1+x|\leqq\frac{1}{2}x^2$ valid for $x\geqq 0$ gives

$$|r_n(t)|\leqq\frac{1}{n}t^4/8. \tag{19}$$

A glance at (17) and (18) now suggest introducing the functions g_n defined by

$$g_n(t)-e^{-t^2/2}=[c_n h(t)+\rho_n \tfrac{1}{2}(it)^2]\,e^{-t^2/2}; \tag{20}$$

then we have

$$\begin{aligned}f^n(t/a_n)-g_n(t)=r_n(t)\,S_n(t)+c_n\big[&(h_n(t)-h(t))\,e^{-t^2/2}\\&+h_n(t)(S_n(t)-e^{-t^2/2})\big]+\rho_n\tfrac{1}{2}(it)^2[S_n(t)-e^{-t^2/2}].\end{aligned}$$

Furthermore we have in connection with the definition (5) of h shown that g_n is the Fourier-Stieltjes transform of

$$G_n(x)=(1+c_n\,\tilde{\omega}_\delta+\rho_n\tfrac{1}{2}D^2)\,\Phi(x).$$

Given $\varepsilon>0$ we put $T=(\varepsilon c_n)^{-1}B$ in Lemma 2 and obtain

$$|F_n(x)-G_n(x)|\leqq A(I_1+I_2+I_3)+\varepsilon c_n$$

where

$$I_1=\int_{-T}^{T}|r_n(t)\,t^{-1}S_n(t)|\,dt,$$

$$I_2=c_n\int_{-T}^{T}\left(\left|\frac{h_n(t)-h(t)}{t}\right|e^{-t^2/2}+|h_n(t)\,t^{-1}(S_n(t)-e^{-t^2/2})|\right)dt,$$

$$I_3=\tfrac{1}{2}\rho_n\int_{-T}^{T}|t(S_n(t)-e^{-t^2/2})|\,dt.$$

Since

$$a_n c_n=\frac{n}{a_n^2}\int_{-a_n}^{a_n}y^2\,F(dy)\cdot b_n\sim b_n,$$

where

$$b_n=\int_{-a_n}^{a_n}|y|^3\,F(dy)\Big/\int_{-a_n}^{a_n}y^2\,F(dy),$$

the assumption of an infinite absolute third moment implies $T=o(a_n)$. Thus according to Lemma 3 there is a positive constant c such that

$$|f^n(t/a_n)|\leqq e^{-c|t|} \quad \text{for } 1<|t|<T$$

and all sufficiently large n. Hence also

$$\left|f^j(t/a_n)\exp\left(-\frac{1}{2}t^2\frac{n-1-j}{n}\right)\right|<e^{-c_0|t|},\quad 0\leqq j<n \tag{21}$$

for some $c_0>0$ if $1<|t|<T$ and n is large enough (because (21) is trivial when $j<n/2$). Because of (21), for the above-mentioned t and n,

$$|S_n(t)|<e^{-c_0|t|}. \tag{22}$$

From (19) and (22) we conclude $I_1 = O(1/n)$. The inequalities

$$|e^{itx} - 1 - itx - \tfrac{1}{2}(it\,\tau(x))^2| \leqq \begin{cases} \tfrac{1}{6}|tx|^3 & \text{if } |x| \leqq 1, \\ 2|tx| & \text{if } |x| > 1, \end{cases}$$

give

$$|h_n(t)| \leqq \tfrac{1}{6}|t|^3 \int_{|x|\leqq 1} |U_n|(dx) + 2|t| \int_{|x|>1} \frac{1}{x^2} |U_n|(dx).$$

This together with (15) and (16) shows that

$$|h_n(t)| \leqq (\text{constant not depending on } n)\, |t|\,(1+t^2). \tag{23}$$

The functions under the integral signs in I_2 and I_3 tend to zero and they are according to (22) and (23) dominated by integrable functions. Hence, by Lebesgues dominated convergence theorem, $I_2 = o(c_n)$, $I_3 = o(\rho_n)$.

Thus we have shown that

$$F_n(x) = (1 + c_n\,\tilde{\omega}_\delta + \tfrac{1}{2}\rho_n D^2)\,\Phi(x) + o(c_n) + o(\rho_n) \tag{24}$$

uniformly in x.

If $\delta > 0$ and $a_n = \sigma\sqrt{n}$ it follows from (10) that

$$\rho_n = -c_n \int_{|x|>1} \frac{1}{x} \Omega_\delta(dx) + o(c_n).$$

Hence we can get rid of the truncation in this case:

$$\begin{aligned}(c_n\,\tilde{\omega}_\delta + \rho_n \tfrac{1}{2}D^2)\,\Phi(x) \\ = c_n \int \frac{\Phi(x-y) - \Phi(x) + y\,D\,\Phi(x) - \tfrac{1}{2}y^2 D^2\,\Phi(x)}{y^3}\,\Omega_\delta(dy) + o(c_n),\end{aligned} \tag{25}$$

uniformly in x. Three partial integrations to the right in (25) show that (24) is equivalent to (6) if $\delta > 0$.

It remains to show that a choice of the sequence $a_1, a_2, \ldots$ according to (7) is possible. Put $V(x) = \int_{|y|\leqq x} y^2\,F(dy)$ and

$$a_n = \sup\left\{x > 0 \,\middle|\, \frac{n}{x^2}\,V(x) \geqq 1\right\}$$

for n large enough. Then, since V is nondecreasing, $n\,a_n^{-2}\,V(a_n) = 1$.

4. Proof of Theorem 2 and 3

From the inversion formula for Fourier transforms we obtain in the absolutely continuous case

$$v_n(x) - (1 + c_n\,\tilde{\omega}_\delta + \rho_n \tfrac{1}{2}D^2)\,\varphi(x) = \frac{1}{2\pi}\int e^{-itx}\big(f^n(t/a_n) - g_n(t)\big)\,dt \tag{26}$$

where g_n is defined in (20). We choose $\varepsilon > 0$ so small that (12) is true for $1 < |t| < \varepsilon a_n$ and n sufficiently large and divide the integral to the right in (26) into two parts,

$$\int_{|t| < \varepsilon a_n} + \int_{|t| \geqq \varepsilon a_n} .$$

We proceed with the first integral in a similar way as in the proof of Theorem 1 of this paper; with the second as in [4], p. 489.

The modifications necessary when F is a lattice distribution parallels those which permit us to conclude Theorem 3, p. 490 of [4], from Theorem 2, p. 489 of [4].

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Dr. Thomas Höglund
Stockholms Universitet
Hagagatan 25A, Box 6701
S-113 85 Stockholm

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