Z. Wahrscheinlichkeitstheorie verw. Gebiete
36, 9 – 12 (1976)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Additional Remarks on Convergence of Stochastic Processes

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1. The Additional Remarks

Using the same notations as in [1] let $X \subset \mathbb{R}^T$ $(T \subset \mathbb{R})$ be endowed with the topology $\mathscr{G}$ induced on X by the product topology of $\mathbb{R}^T$. Let μ_n, $n \in \mathbb{N}$, be a sequence of p-measures on the Borel σ-field $\mathscr{B}$ in $(X, \mathscr{G})$ (which are assumed to be $\mathscr{K}$-regular in the sense of [1]). Starting with the assumption that the finite dimensional marginal distributions μ_n^π (pertaining to $\pi = \pi_{\mathbf{t}}: X \to \mathbb{R}^{|\mathbf{t}|}$ for $\mathbf{t} = \{t_1, t_2, \ldots, t_n\}$ with $t_i \in T$) converge (weakly) to a p-measure μ_0^π as $n \to \infty$ (for every π), where the μ_0^π's are finite dimensional marginal distributions of some p-measure μ_0 on $\mathscr{B}$ (being uniquely determined by the μ_0^π's) it was proved in Theorem 2.8 of [1] that the following condition (1.1) imposed on the class $\mathscr{C}_0$ of all open cylindersets $C = \pi_{\mathbf{t}}^{-1}(G)$, G being an open subset of $\mathbb{R}^{|\mathbf{t}|}$ with $\mu_0^{\pi_{\mathbf{t}}}(\partial G) = 0$ (cf. (2.10) in [1]), implies $\lim_{n \to \infty} \mu_n(A) = \mu_0(A)$ for all $A \in \mathscr{B}$:

(1.1) For every monotone increasing sequence $(C_j)_{j \in \mathbb{N}}$ with $C_j = \pi_{\mathbf{t}_j}^{-1}(G_j) \in \mathscr{C}_0$ there exists a monotone decreasing sequence $(C'_j)_{j \in \mathbb{N}}$ with $C'_j = \pi_{\mathbf{t}'_j}^{-1}(G'_j) \in \mathscr{C}_0$ such that $C_j \subset C'_j$ for all $j \in \mathbb{N}$ and $\lim_{j \to \infty} \sup_{n \in \mathbb{N}} \mu_n^{\pi_{\mathbf{s}_j}}(A'_j \setminus A_j) = 0$, where $\mathbf{s}_j = \mathbf{t}_j \cup \mathbf{t}'_j$ and $A'_j = \pi_{\mathbf{s}_j, \mathbf{t}'_j}^{-1}(G'_j)$, $A_j = \pi_{\mathbf{s}_j, \mathbf{t}_j}^{-1}(G_j)$.

But in view of the assertion that (1.1) implies $\lim_{n \to \infty} \mu_n(A) = \mu_0(A)$ for all $A \in \mathscr{B}$ it follows that the limit measure μ_0 must necessarily satisfy the following condition (1.2):

(1.2) For every monotone increasing sequence $(C_j)_{j \in \mathbb{N}}$ with $C_j = \pi_{\mathbf{t}_j}^{-1}(G_j) \in \mathscr{C}_0$ there exists a monotone decreasing sequence $(C'_j)_{j \in \mathbb{N}}$ with $C'_j = \pi_{\mathbf{t}'_j}^{-1}(G'_j) \in \mathscr{C}_0$ such that $C_j \subset C'_j$ for all $j \in \mathbb{N}$ and

$$\lim_{j \to \infty} \mu_0(C'_j \setminus C_j) = \mu_0\Big(\bigcap_{j \in \mathbb{N}} C'_j \setminus \bigcup_{j \in \mathbb{N}} C_j\Big) = 0.$$

We want to show now that (1.2) and therefore also (1.1) cannot be fulfilled in nearly all cases of interest.

(1.1) **Theorem.** If we assume that in the above cited context the space $X \subset \mathbb{R}^T$ with $T \subset \mathbb{R}$ fulfills one of the following conditions

(i) $X = Y^T$ with $Y \subset \mathbb{R}$, $|Y| \geqq 2$ and $|T| \geqq \aleph_0$, or

(ii) $|T| \geqq \aleph_0$, and for any $t_1 < t_2 < \dots < t_m$, $t_i \in T$, and any $(a_1, \dots, a_m) \in \mathbb{R}^m$ there exists $x \in X$ such that $x(t_i) = a_i$, $i = 1, \dots, m$, then (1.2) (and hence (2.10) in [1]) is not fulfilled.

Proof. (a) For any $y \in X$ and $S \subset T$ let us consider in X the set $Z_y^S = \{x \in X : x(s) = y(s)$ for all $s \in S\}$; then

$(+)$ $Z_y^S \cap Z_{y'}^S = \emptyset \Leftrightarrow \mathrm{rest}_S\, y \neq \mathrm{rest}_S\, y'$ (where $\mathrm{rest}_S\, y$ denotes the restriction of y: $T \to \mathbb{R}$ on S).

(b) According to (i) or (ii) it follows in both cases that at least for $S \subset T$ with $|S| \geqq \aleph_0$ the set $\{Z_y^S : y \in X\}$ contains infinitely many pairwise disjoint elements Z_y^S which implies

$(++)$ For any $S \subset T$ with $|S| \geqq \aleph_0$ and for any $\varepsilon > 0$ there exists $y \in X$ such that $\mu_0(Z_y^S) < \varepsilon$.

(c) Now we claim that the following assertion holds true: For any monotone strictly increasing sequence $(t_n \in T)_{n \in \mathbb{N}}$ there exists a monotone strictly increasing sequence $(n_k \in \mathbb{N})_{k \in \mathbb{N}}$ and a sequence $(y^k \in X)_{k \in \mathbb{N}}$ such that for any $k \in \mathbb{N}$ we have

$(+++)$ $\mu_0(Z_{y^k}^{\mathbf{t}_k^*}) < 2^{-(k+1)}$, where $\mathbf{t}_k^* = \{t_{n_k+1}, \dots, t_{n_{k+1}}\}$, $k \in \mathbb{N}$.

To prove $(+++)$ let

$$T_0 = \{t_n : n \in \mathbb{N}\} \quad \text{(where by assumption } t_1 < t_2 < \dots < t_n < \dots, t_i \in T)$$

and choose $n_1 = 1$. Then by $(++)$ exists $y^1 \in X$ such that $\mu_0(Z_{y^1}^{\{t \in T_0 : t > t_1\}}) < 2^{-(1+1)}$ and therefore there exists $n_2 \in \mathbb{N}$, $n_2 > n_1$, with $\mu_0(Z_{y^1}^{\{t_{n_1+1}, \dots, t_{n_2}\}}) < 2^{-(1+1)}$, i.e. $(+++)$ holds for $k = 1$. Now, assume that for $k \geqq 2$ there are already constructed $n_1, \dots, n_k \in \mathbb{N}$, and $y^1, \dots, y^{k-1} \in X$ for which $(+++)$ holds true; then in the same way as before $(++)$ implies that there exists $y^k \in X$ such that

$$\mu_0(Z_{y^k}^{\{t \in T_0 : t > t_{n_k}\}}) < 2^{-(k+1)}$$

and again we obtain $n_{k+1} \in \mathbb{N}$, $n_{k+1} > n_k$, such that $(+++)$ holds for k. This proves $(+++)$.

(d) Now we fix $T_0 = \{t_n \in T : t_1 < t_2 < \dots < t_n < \dots\}$, $(n_k \in \mathbb{N})_{k \in \mathbb{N}}$ and $(y^k \in X)_{k \in \mathbb{N}}$ with the properties proved under (c). Then, considering again $\mathbf{t}_k^* = \{t_{n_k+1}, \dots, t_{n_{k+1}}\}$, we obtain $\mu_0^{\pi_{\mathbf{t}_k^*}}(\{y^k(t_{n_k+1}), \dots, y^k(t_{n_{k+1}})\}) < 2^{-(k+1)}$. Since $\mu_0^{\pi_{\mathbf{t}_k^*}}$ is a regular measure on the Borel sets in $\mathbb{R}^{d_k}$ (with $d_k = n_{k+1} - n_k$) and since the class of all open sets G in $\mathbb{R}^{d_k}$ with $\mu_0^{\pi_{\mathbf{t}_k^*}}(\partial G) = 0$ form a base, it follows that there exists an open set G_k^* in $\mathbb{R}^{d_k}$ with

$$(y^k(t_{n_k+1}), \dots, y^k(t_{n_{k+1}})) \in G_k^*, \quad \mu_0^{\pi_{\mathbf{t}_k^*}}(\partial G_k^*) = 0 \quad \text{and} \quad \mu_0^{\pi_{\mathbf{t}_k^*}}(G_k^*) < 2^{-(k+1)}, \quad k \in \mathbb{N}.$$

Now, if we put $\mathbf{t}_j = \{t_1, t_2, \dots, t_{n_{j+1}}\}$, then for $k \leqq j$ $G_{jk} = \pi_{\mathbf{t}_j, \mathbf{t}_k^*}^{-1}(G_k^*)$ is an open subset of $\mathbb{R}^{n_{j+1}}$ with $\mu_0^{\pi_{\mathbf{t}_j}}(\partial G_{jk}) = 0$ and $\mu_0^{\pi_{\mathbf{t}_j}}(G_{jk}) < 2^{-(k+1)}$. Hence $G_j = \bigcup_{k \leqq j} G_{jk}$ is an open subset of $\mathbb{R}^{n_j+1}$ with $\mu_0^{\pi_{\mathbf{t}_j}}(\partial G_j)=0$, and therefore $C_j=\pi_{\mathbf{t}_j}^{-1}(G_j)$, $j\in\mathbb{N}$, is a monotone increasing sequence of sets belonging to $\mathscr{C}_0$ as it occurs in (1.2).

(e) Assume that there exists a cylinderset $C\neq X$ with $\bigcup_{j\in\mathbb{N}} C_j\subset C$. Then $C=\{x\in X:(x(s_1),\ldots,x(s_r))\in B\}$ for some $s_1<s_2<\cdots<s_r$, $s_i\in T$, $r\in\mathbb{N}$, and some Borel subset B of $\mathbb{R}^r$, where $\mathbb{R}^r\setminus B\neq\emptyset$ (resp. $Y^r\setminus B\neq\emptyset$ in case (i)) since $C\neq X$. According to the choices of T_0, $(n_k\in\mathbb{N})_{k\in\mathbb{N}}$, and $(y^k\in X)_{k\in\mathbb{N}}$ (see (d)) there exists $p\in\mathbb{N}$ such that $\{s_1,\ldots,s_r\}\cap\{t_n\in T:n>p\}=\emptyset$ and we may define for some $(a_1,\ldots,a_r)$ $\mathbb{R}^r\setminus B$ (resp. for some $(a_1,\ldots,a_r)\in Y^r\setminus B$ in case (i)) and for $j\in\mathbb{N}$ so large that $n_j>p$

$$\begin{aligned}Z=\{x\in X:&(x(s_1),\ldots,x(s_r),x(t_{n_j+1}),\ldots,x(t_{n_{j+1}}))\\&=(a_1,\ldots,a_r,y^j(t_{n_j+1}),\ldots,y^j(t_{n_{j+1}}))\}\end{aligned}$$

to obtain a cylinderset Z in X with $Z\neq\emptyset$ according to (i) or (ii) and $Z\subset C_j\subset C$, $j\in\mathbb{N}$, whereas $Z\cap C=\emptyset$. Hence $C=X$ is the only cylinderset containing $\bigcup_{j\in\mathbb{N}} C_j$.

(f) The assertion of the theorem follows now immediately from (d) and (e), because

$$\begin{aligned}\mu_0\Big(\bigcup_{j\in\mathbb{N}} C_j\Big)&=\lim_{i\to\infty}\mu_0\Big(\bigcup_{j\leq i} C_j\Big)=\lim_{i\to\infty}\mu_0\Big(\pi_{\mathbf{t}_i}^{-1}\Big(\bigcup_{j\leq i} G_{ij}\Big)\Big)\\&\leq\lim_{i\to\infty}\sum_{j\leq i}\mu_0^{\pi_{\mathbf{t}_i}}(G_{ij})<\sum_{j=1}^{\infty}2^{-(j+1)}=\tfrac{1}{2}.\end{aligned}$$

As to the case where one allows T to be at most countable one obtains the following theorem due to Le Cam:

(1.2) **Theorem.** *Let T be at most countable, consider $X=(X,\mathscr{G})\subset\mathbb{R}^T$ and suppose that the following condition*

(iii) *$\{x\}\notin\mathscr{G}$ for all $x\in X$*

holds true. Then (1.2) *(and hence* (2.10) *in* [1]) *is not fulfilled.*

Proof. We remark first that either one of the two conditions (i) or (ii) of Theorem 1.1 implies (iii). Now, in the present case $\mathbb{R}^T$ and therefore $X=(X,\mathscr{G})$ become separable metrizable spaces. Furthermore, since $\mathscr{C}_0$ forms a base of $\mathscr{G}$, it follows that for any $G\in\mathscr{G}$ there exist $C_n\in\mathscr{C}_0$, $n\in\mathbb{N}$, which may be assumed without loss of generality to be monotone increasing, such that $G=\bigcup_{n\in\mathbb{N}} C_n$. Therefore according to Lemma 2.2 (1.2) would imply that $\mu_0(\partial G)=0$ for all $G\in\mathscr{G}$, which is in contradiction with Lemma 2.1; this proves the theorem.

2. Auxiliary Lemmata

(2.1) **Lemma.** *Let $X=(X,\mathscr{G}(X))$ be an arbitrary separable metrizable space and suppose that the one point sets of X are not open. Then for any probability measure μ defined on the Borel σ-field $\mathscr{B}(X)$ in X there exists a $G\in\mathscr{G}(X)$ with $\mu(\partial G)>0$.*

Proof. Assume that $\mu(\partial G)=0$ for all $G\in\mathscr{G}(X)$; since the one point sets of X are not open, we have $\partial(X\setminus\{x\})=\{x\}$ and therefore $\mu(\{x\})=0$ for all $x\in X$.

Let $\{x_j : j \in \mathbb{N}\}$ be a countable dense subset of X. Since μ is regular there exist $G_j \in \mathscr{G}(X)$ such that $x_j \in G_j$ and $\mu(G_j) < 2^{-(j+1)}$ for every $j \in \mathbb{N}$. Then it follows with $G = \bigcup_{j \in \mathbb{N}} G_j$ that $G \in \mathscr{G}(X)$ with $\partial G = X \setminus G$ and $\mu(\partial G) \geqq 1 - \sum_{j \in \mathbb{N}} 2^{-(j+1)} > 0$.

(2.2) **Lemma.** *Condition* (1.2) *implies that for any monotone increasing sequence* $(C_j \in \mathscr{C}_0)_{j \in \mathbb{N}}$ *occuring there one has* $\mu_0(\partial(\bigcup_{j \in \mathbb{N}} C_j)) = 0$.

Proof. Let $(C_j \in \mathscr{C}_0)_{j \in \mathbb{N}}$ be monotone increasing, $C_j = \pi_{\mathbf{t}_j}^{-1}(G_j)$ with G_j open in $\mathbb{R}^{|\mathbf{t}_j|}$ and $\mu_0^{\pi_{\mathbf{t}_j}}(\partial G_j) = 0$; according to (1.2) there exists a monotone decreasing sequence $(C'_j \in \mathscr{C}_0)_{j \in \mathbb{N}}$, $C'_j = \pi_{\mathbf{t}'_j}^{-1}(G'_j)$ with G'_j open in $\mathbb{R}^{|\mathbf{t}'_j|}$ and $\mu_0^{\pi_{\mathbf{t}'_j}}(\partial G'_j) = 0$ such that for all $j \in \mathbb{N}$ $C_j \subset C'_j$ and $\lim_{j \to \infty} \mu_0(C'_j \setminus C_j) = 0$. But this implies that

$$\mu_0(\partial(\bigcup_{j \in \mathbb{N}} C_j)) \leqq \lim_{j \to \infty} \mu_0(C_j'^c \setminus C_j) = \lim_{j \to \infty} \mu_0(C'_j \setminus C_j) = 0,$$

where $C_j'^c$ denotes the closure of C'_j and where the equality of the last two terms is due to the fact that $\mu_0(\partial C'_j) \leqq \mu_0^{\pi_{\mathbf{t}'_j}}(\partial G'_j) = 0$.

Acknowledgement. The authors are very grateful to Professor Lucien Le Cam for his constructive remarks on an earlier version of the manuscript.

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Received January 15, 1974; in revised form December 20, 1974