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Composition Limit Theorems for Multidimensional Probability Generating Functions*

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1. Introduction and Statement of Results

Let $f(s)=(f^1(s), f^2(s), \ldots, f^p(s))$ be a given vector of p-dimensional probability generating functions (p.g.f. vector). Set $\hat{f}_{(n)}(s)$ equal to the composition of $f(s)$ with itself n times. It is a well-known result, [4, p. 42], that under slight regularity conditions on $f(s)$, that $\lim_{n\to\infty} \hat{f}_{(n)}(s)$ exists for all $s\in[0,1]^p$. Suppose we now consider an arbitrary sequence of p.g.f. vectors, $\{f_n(s)\}_{n\geqq 1}$, and again form the composition, $f_{(n)}(s)=f_1(f_2(\ldots f_n(s)))$. It is the purpose of this paper to ascertain conditions for when $\lim_{n\to\infty} f_{(n)}(s)$ exists. Before stating our major results, it is convenient to introduce the following notation.

$$s=(s_1, s_2, \ldots, s_p); \qquad s_i\in[0,1],\ 1\leqq i\leqq p,\ p\geqq 2$$
$$Rs=(s, s, \ldots, s); \qquad s\in[0,1].$$

(Whenever it is not ambiguous we will write 1 for $R1$ and 0 for $R0$.)

Let u, v be any 2 vectors. Then $u \underset{(\geqq)}{\overset{(\leqq)}{<}} v$ means $u_i \underset{(\geqq)}{\overset{(\leqq)}{<}} v_i$; $1\leqq i\leqq p$. $(u,v)=\sum_{i=1}^{p} u_i v_i$; u^t is the transpose of the vector u. If $\{a_n\}$ $n\geqq 1$ is any sequence of vectors in $[0,1]^p$, then $\lim_{n\to\infty} a_n = a$ should be interpreted as a component wise limit. Similar interpretations should be given to $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$. Let $A=(a_{ij})_{1\leqq i,j\leqq p}$ and $B=(b_{ij})_{1\leqq i,j\leqq p}$ be any two matrices. Then,

$$\min[A]=\min_{1\leqq i,j\leqq p} |a_{ij}|, \qquad \max[A]=\max_{1\leqq i,j\leqq p} |a_{ij}|,$$

and

$$A\leqq B \quad \text{means} \quad a_{ij}\leqq b_{ij}, \qquad 1\leqq i,\ j\leqq p.$$

$u\cdot A$ denotes standard matrix multiplication by a vector u. Let $\{f_n(s)\}_{n\geqq 1}$ be any sequence of p.g.f. vectors. Set for each $n\geqq 1$

$$D_j f_n^i(s)=\frac{\partial f_n^i(s)}{\partial s_j}; \qquad 1\leqq i,\ j\leqq p,\ s\in[0,1]^p$$
$$M_n(s)=(D_j f_n^i(s))_{1\leqq i,j\leqq p}; \qquad s\in[0,1]^p$$

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$$\frac{d}{ds} f_n(Rs) = \left(\frac{d}{ds} f_n^1(Rs), \ldots, \frac{d}{ds} f_n^p(Rs)\right); \qquad s \in [0,1]$$

$$\frac{d}{ds} M_n(Rs) = \left(\frac{d}{ds} D_j f_n^i(Rs)\right)_{1 \leqq i, j \leqq p}; \qquad s \in [0,1].$$

Finally we make the following definition.

Definition 1.1. A sequence of p.g.f. vectors, $\{f_n(s)\}_{n \geqq 1}$ will be said to have Property 1 iff for each $1 \leqq i \leqq p$,

(i) $f_n^i(0) < 1;\ n \geqq 1$

and for any sequence of vectors $\{a_{n_k}\} \in [0,1]^p$

(ii) $\lim_{n_k \to \infty} f_{n_k}^i(a_{n_k}) = 1 \Leftrightarrow \lim_{n_k \to \infty} a_{n_k} = R1$.

Condition (i) of the definition is made to guarantee that each component of every p.g.f. vector is a legitimate p.g.f. Condition (ii) is made to compensate for the fact that a p-dimensional p.g.f. is not 1-1. In effect (ii) states that the $f_n^i(s)$ do not approach 1 too quickly.

We now state the results of this paper. Theorem 1, which is our major theorem, gives sufficient conditions for when $\lim_{n \to \infty} f_{(n)}(Rs)$ exists.

Theorem 1. *Let $\{f_n(s)\}_{n \geqq 1}$ be any sequence of p.g.f. vectors satisfying Property 1. Set $f_{(n)}(s) = f_1(f_2(\ldots f_n(s)))$, $n \geqq 1$. Then*

(i) *$\lim_{n \to \infty} f_{(n)}(Rs) = g(s)$ exists for all $s \in [0,1]$ and the convergence is uniform on compact subsets of $[0,1]$.*

(ii) *Either $g^i(s) = g^i(0)$ for all $s \in [0,1)$ or $g^i(s)$ is strictly increasing in $[0,1)$, $1 \leqq i \leqq p$.*

The above theorem is a generalization of a result proven by Church [2] for 1-dimension.

Theorem A. *Let $\{f_n(s)\}_{n \geqq 1}$ be a sequence of 1 dimensional p.g.f's. Set $f_{(n)}(s) = f_1(f_2(\ldots f_n(s)))$. Then,*

(i) *$\lim_{n \to \infty} f_{(n)}(s) = g(s)$ exists for all $s \in [0,1]$.*

(ii) *Either $g(s) = g(0)$ for $s \in [0,1)$ or $g(s)$ is strictly increasing in $[0,1)$ with the former holding if and only if*

$$\lim_{n \to \infty} \prod_{j=1}^{n} \left\{\frac{df_n}{ds}(0)\right\} = 0.$$

The proof of Theorem 1 is very similar in spirit to the proof of Church's result. A simplified proof of Theorem A can be found in [1].

Clearly it would be desirable to have at our disposal simple conditions on the $\{f_n(s)\}_{n \geqq 1}$ which guarantee that Property 1 holds. That is the content of the next result.

Theorem 2. *Let $\{f_n(s))_{n \geqq 1}$ be a sequence of p.g.f. vectors. The following two conditions are sufficient for Property 1 to hold.*

(i) *There exist positive constants* $C, D>0$ *such that for* $n\geqq 1$, $1\leqq i, j, k\leqq p$,

$$C\leqq D_j f_n^i(1)\leqq D \quad and \quad 0<D_{jk}^2 f_n^i(1)\leqq D.$$

$$\left(D_{jk}^2 f_n^i(1)=\frac{\partial^2 f_n^i}{\partial s_j\,\partial s_k}(1)\right).$$

(ii) *There exist positive constants* E, F *such that for* $n\geqq 1$, $1\leqq i, j\leqq p$,

$$E\leqq D_j f_n^i(0)\leqq D_j f_n^i(1)\leqq F.$$

Part (ii) of Church's result gives a necessary and sufficient condition for when the limit function is a constant for $s\in[0, 1)$. Our final result is a generalization of this condition to p-dimensions.

Theorem 3. *Let* $\{f_n(s)\}_{n\geqq 1}$ *be a sequence of p.g.f. vectors satisfying condition* (ii) *of Theorem 2. A necessary and sufficient condition for* $\lim_{n\to\infty} f_{(n)}(R s)=g(s)=\lim_{n\to\infty} f_{(n)}(0)$ *for* $s\in[0, 1)$ *is*

$$\liminf_{n\to\infty}\left(1\cdot\prod_{j=1}^{n} M_j(0), 1\right)=0. \tag{1.1}$$

The motivation behind this research comes from the theory of branching processes with random environments [1]. In [1], Athreya and Karlin used Theorem A to show that the nonzero states of the process were transient. In order to generalize this result to multidimensions, Theorem 1 was necessary. For further details the reader should see [5].

2. Proofs of Theorems 1 and 2

We first prove Theorem 1. To prepare for the proof, we introduce the following definition.

Definition 2.1. Any sequence of p.g.f. vectors, $\{f_n(s)\}_{n\geqq 1}$, will be said to satisfy Condition 1 if for any $s\in[0, 1]^p$, and any subsequence $n_k\to\infty$, either

(i) $\min_{1\leqq i\leqq p}\left[\liminf_{n_k\to\infty}[f_{n_k}^i(s)]\right]=1$

or

(ii) $\max_{1\leqq i\leqq p}\left[\liminf_{n_k\to\infty}[f_{n_k}^i(s)]\right]<1.$

The significance of Condition 1 is the following. Suppose we are given a sequence of p.g.f. vectors, $\{f_n(s)\}_{n\geqq 1}$. By the Helly-Bray Theorem, we can extract a convergent subsequence $\{f_{n_k}(s)\}$. Denote the limit function by $g(s)=(g^1(s),\dots,g^p(s))$ and assume $g(1)=1$. In general we have no way of knowing whether $g^1(s)$ and say $g^2(s)$ depend on the same set of variables. It is quite possible, that in passing to the limit, variables might disappear in one component and not in the other. It is this very thing that Condition 1 prevents. If Condition 1 holds, then all components of $g(s)$ depend on the same set of variables. We note in passing that if $p=1$, then Condition 1 always holds.

Our ultimate goal is to deal with sequences of compositions of functions. If $\{f_n(s)\}_{n\geqq 1}$ is a sequence of p.g.f. vectors and n_k a sequence of integers increasing to ∞, then set

$$h_k(s)=f_{n_k+1}(f_{n_k+2}(\dots f_{n_{k+1}}(s))); \quad k\geqq 1,\ s\in[0, 1]^p.$$

Our first lemma proves that the assumptions of Theorem 1 imply that the collection of p.g.f. vectors $\{h_k(s)\}_{k\geqq 1}$ satisfy Condition 1. Unless otherwise stated, we will always assume that the conditions of Theorem 1 hold.

Lemma 2.1. *The collection of p.g.f. vectors, $\{h_k(s)\}_{k\geqq 1}$ as defined above, satisfy Condition 1.*

Proof. Let $s\in[0,1]^p$. Observe that,

$$h_k(s)=f_{n_k+1}(a_k) \quad \text{where} \quad a_k=f_{n_k+2}\big(f_{n_k+3}(\dots f_{n_{k+1}}(s))\big).$$

The $\{f_{n_k+1}(s)\}$ satisfy Property 1. Therefore for $1\leqq i\leqq p$, and any subsequence $n_{k'}\to\infty$,

$$\liminf_{n_{k'}\to\infty} f^i_{n_{k'}+1}(a_{k'})=1 \Leftrightarrow \lim_{k'\to\infty} a_{k'}=1. \tag{2.1}$$

The validity of Condition 1 follows directly from (2.1). Q.E.D.

We now start the proof of Theorem 1. The proof will be broken into 3 lemmas.

Lemma 2.2. *Let $\{f_n(s)\}_{n\geqq 1}$ be a sequence of p.g.f. vectors satisfying Condition 1. Suppose for some $1\leqq r\leqq p$,*

$$\lim_{n\to\infty} f^r_{(n)}(s)=g(s)=g(s_{i_1}, s_{i_2}, \dots, s_{i_l}); \qquad 1\leqq l\leqq p,\ s\in[0,1)^p \tag{2.2}$$

where the convergence is uniform on compact subsets of $[0,1)^p$. Assume also that $g(s)$ is increasing in each of its components. Then,

$$\lim_{n\to\infty} f_n^{i_j}(Rs)=s \qquad \textit{for } s\in[0,1] \textit{ and } j=1,2,\dots,l.$$

Proof. Without loss of generality assume $i_1=1$, $i_2=2, \dots, i_l=l$. $\{f_n(s)\}_{n\geqq 1}$ is a normal family. Thus, there exists a subsequence $n_k\to\infty$ such that $\lim_{n_k\to\infty} f_{n_k}(s)=w(s)$ and the convergence is uniform on compact subsets of $[0,1]^p$. To prove the lemma, it suffices to show that for $1\leqq i\leqq l$, $w^i(s)$ is a legitimate p.g.f. and linear. Specifically,

$$w^i(s)=\sum_{j=1}^p a^i_j s_j \quad \text{with} \quad \sum_{j=1}^p a^i_j=1; \qquad 1\leqq i\leqq l.$$

We show first that $w(s)$ is a legitimate p.g.f. vector. Condition 1, together with the increasing property of $g(s)$ imply that for any $s\in[0,1)^p$, $w(s)<1$. It follows from the uniform convergence in (2.2) that

$$g(s)=g\big(w(s)\big), \qquad s\in[0,1)^p. \tag{2.3}$$

Relation (2.3) is the key to proving that $w(s)$ is a p.g.f. vector. By letting $s\to 1$ in such a way that $s<1$ always prevails, it is a simple matter to prove that $\lim_{s\to 1} w(Rs)=1$. This implies that $w(s)$ is a p.g.f. vector.

We now demonstrate the linearity of $w(s)$. Condition 1 implies that the matrix

$$H=\left(\frac{\partial w^i}{\partial s_j}(1)\right)_{1\leqq i,\, j\leqq l} \tag{2.4}$$

has all positive entries. It is a well-known result [4], that if (2.4) holds, and at least one of the $w^i(s)$ is not linear, then $\lim_{n\to\infty} w_{(n)}(s)$ exists and necessarily equals a constant so long as $s \neq 1$. $(w_{(n)}(s) = w(w(\dots w(s))))$. Iterating (2.2) we obtain,

$$g(s) = g(w_{(n)}(s)); \quad n \geqq 1. \tag{2.5}$$

However, due to the increasing nature of $g(s)$ and (2.5), it is impossible for $\lim_{n\to\infty} w_{(n)}(s)$ to be a constant. Q.E.D.

Lemma 2.3. *Let $\{f_n(s)\}_{n\geqq 1}$ and $\{g_n(s)\}_{n\geqq 1}$ be two sequences of p.g.f. vectors. Assume $\{g_n(s)\}_{n\geqq 1}$ satisfy Condition 1. Then,*

$$\lim_{n\to\infty} f_n(g_n(Rs)) = Rs \Rightarrow \lim_{n\to\infty} f_n(Rs) = Rs; \quad s \in [0, 1].$$

Proof. It suffices to show

$$\lim_{n\to\infty} f_n^1(g_n(Rs)) = s \Rightarrow \lim_{n\to\infty} f_n^1(Rs) = s.$$

Let

$$Y_n = \begin{cases} \sum_{i=1}^{p} \sum_{j=1}^{N_n^i} (X_{j,n}^i, 1) & \text{if } \sum_{i=1}^{p} N_n^i > 0 \\ 0 & \text{otherwise,} \end{cases}$$

where for each n,

(a) $(X_{j,n}^i, 1)$ and $(N_n^1, \dots, N_n^p)$ are independent variables for all i, j.

(b) The p.g.f. of $(X_{j,n}^i, 1)$ is $g_n^i(Rs)$ for all i, j.

(c) The p.g.f. of $(N_n^1, \dots, N_n^p)$ is $f_n^1(s)$.

The assumptions of the lemma imply that

$$\lim_{n\to\infty} Y_n = 1 \quad \text{in probability}. \tag{2.6}$$

To prove the result it is sufficient to show that

$$\lim_{n\to\infty} P\left\{\sum_{i=1}^{p} N_n^i = 1\right\} = 1.$$

Clearly $\lim_{n\to\infty} P\left\{\sum_{i=1}^{p} N_n^i = 0\right\} = 0$. Otherwise, (2.6) would be violated. To prove that $\lim_{n\to\infty} P\left\{\sum_{i=1}^{p} N_n^i \geqq 2\right\} = 0$, it is enough to show that for any sequence $n_k \to \infty$, there exists a subsequence $n_k' \to \infty$ and an integer $1 \leqq i \leqq p$ such that

$$\limsup_{n_k'\to\infty} P\{(X_{1,n_k'}^i, 1) = 0\} < 1. \tag{2.7}$$

(2.7) together with Condition 1 imply that there exists a constant $\beta > 0$ such that

$$\inf_{\substack{1 \leqq i \leqq p \\ n_k' \geqq 1}} P\{(X_{1,n_k'}^i, 1) \geqq 1\} \geqq \beta.$$

This implies that for all n_k',

$$P\left\{Y_{n_k'} \geqq 2 \,\middle|\, \sum_{i=1}^{p} N_{n_k'}^i \geqq 2\right\} \geqq \beta^2. \tag{2.8}$$

It follows from (2.6) and (2.8) that necessarily,

$$\lim_{n'_k\to\infty} P\left\{\sum_{i=1}^{p} N^i_{n'_k} \geqq 2\right\}=0.$$

So the proof of Lemma 2.3 reduces to proving (2.7). For ease of notation, the subscripted notation will be dropped. We observe that we can write Y_n as

$$Y_n=\sum_{i=1}^{p} W^i_n$$

where

$$W^i_n=\sum_{j=1}^{N^i_n} (X^i_{j,n}, 1).$$

Clearly, not all the W^i_n tend to zero in probability. Suppose W^1_n does not converge to zero in probability. Since $W^1_n \geqq 2 \Rightarrow Y_n \geqq 2$, (2.6) implies that $\lim_{n\to\infty} P\{W^1_n \geqq 2\}=0$. Therefore, there exists a subsequence $n_k \to \infty$ and a random variable W such that

$$\lim_{n_k\to\infty} W^1_{n_k}=W \quad \text{in distribution} \tag{2.9}$$

and

$$P\{W=0\}+P\{W=1\}=1 \quad \text{with} \quad P\{W=1\}>0.$$

Let $h_{n_k}(s)$ be the p.g.f. of $N^1_{n_k}$. (2.9) is equivalent to

$$\lim_{n_k\to\infty} h_{n_k}(g^1_{n_k}(R s))=P\{W=0\}+P\{W=1\}\, s; \qquad s\in[0,1].$$

We can write $W^1_{n_k}$ as

$$W^1_{n_k}=Z^1_{n_k}+Z^2_{n_k}$$

where

$$Z^1_{n_k}=\begin{cases} \sum_{j=1}^{L(k)} (X^1_{j,n_k}, 1) & \text{if } N^1_{n_k} \text{ even} \\ \sum_{j=1}^{L(k)+1} (X^1_{j,n_k}, 1) & \text{if } N^1_{n_k} \text{ odd} \end{cases}$$

and

$$Z^2_{n_k}=W^1_{n_k}-Z^1_{n_k}$$

with

$$L(k)=[N^1_{n_k}/2].$$

Furthermore,

$$Z^1_{n_k}=Z^3_{n_k}+\eta_{n_k}$$

where $Z^3_{n_k}$ has the same distribution as $Z^2_{n_k}$ and

$$\eta_{n_k}=\begin{cases} (X^1_{1,n_k}, 1) & \text{if } N^1_{n_k} \text{ odd} \\ 0 & \text{if } N^1_{n_k} \text{ even}. \end{cases}$$

Observe that $P\{Z^1_{n_k} \geqq 1 | N^1_{n_k}\}$ and $P\{Z^2_{n_k} \geqq 1 | N^1_{n_k}\}$ are both increasing in $N^1_{n_k}$. We now make use of the following well-known lemma.

Lemma. *Let X be a positive random variable and f(x) and g(x) increasing bounded functions. Then*

$$E\{f(X)\,g(X)\} \geqq E\{f(X)\}\,E\{g(X)\}.$$

Applying this lemma to the random variables $P\{Z^1_{n_k} \geqq 1 | N^1_{n_k}\}$ and $P\{Z^2_{n_k} \geqq 1 | N^1_{n_k}\}$ and observing that $Z^1_{n_k}$ and $Z^2_{n_k}$ are conditionally independent given $N^1_{n_k}$, we obtain

$$P\{Z^1_{n_k} \geqq 1, Z^2_{n_k} \geqq 1\} - P\{Z^1_{n_k} \geqq 1\}\,P\{Z^2_{nk} \geqq 1\} \geqq 0.$$

However,

$$\limsup_{n_k\to\infty} P\{Z^1_{n_k} \geqq 1, Z^2_{n_k} \geqq 1\} \leqq \limsup_{n_k\to\infty} P\{W^1_{n_k} \geqq 2\} = 0.$$

Therefore,

$$\lim_{n_k\to\infty} P\{Z^1_{n_k} \geqq 1\}\,P\{Z^2_{n_k} \geqq 1\} = 0. \tag{2.10}$$

It is not too difficult to see that the only way for (2.10) to be true is for

$$\liminf_{n_k\to\infty} P\{Z^1_{n_k} \geqq 1\} > 0 \quad \text{and} \quad \lim_{n_k\to\infty} P\{Z^2_{n_k} \geqq 1\} = 0. \tag{2.11}$$

(2.9) and (2.11) together with the fact that $Z^2_{n_k}$ and $Z^3_{n_k}$ have the same distribution imply that

$$\limsup_{n_k\to\infty} P\{\eta_{n_k} = 0\} < 1. \tag{2.12}$$

Since, $P\{(X^1_{n_k}, 1) = 0\} \leqq P\{\eta_{n_k} = 0\}$, (2.12) implies (2.7). Q.E.D.

Lemma 2.4. *Let $\{f_n(s)\}_{n \geqq 1}$ be a sequence of p.g.f. vectors satisfying the conditions of Theorem 1. Let $f_{(n)}(s) = f_1(f_2(\dots f_n(s)))$. Then for $1 \leqq i \leqq p$, $\lim_{n\to\infty} f^i_n(Rs)$ exists and converges to either a constant or an increasing function.*

Proof. The following arguments hold for $1 \leqq i \leqq p$. Since $f^i_{(n)}(0)$ is increasing in n, $\lim_{n\to\infty} f^i_{(n)}(0)$ exists. Without loss of generality, we can assume that there exists an $s_0 \in [0, 1)$ such that

$$\alpha = \lim_{n\to\infty} f^i_{(n)}(0) < \limsup_{n\to\infty} f^i_{(n)}(R s_0) = \beta.$$

Choose a subsequence $n_k \to \infty$ such that $\lim_{n_k\to\infty} f^i_{(n_k)}(R s_0) = \beta$. $\{f^i_{(n_k)}(s)\}$ is a normal family. Thus there exists a further subsequence $\{n'_k\}$ such that

$$\lim_{n'_k\to\infty} f^i_{(n'_k)}(s) = g(s_{i_1}, \dots, s_{i_l}); \qquad 1 \leqq l \leqq p$$

and the convergence is uniform on compact subsets of $[0, 1)^p$. Without loss of generality, assume $i_1 = 1$, $i_2 = 2, \dots, i_l = l$. Since $g(R s_0) = \beta > \alpha = g(R0)$ we know that $g(s)$ is increasing in each of its variables. Observe that,

$$f^i_{(n'_k)}(s) = w^i_0(w_1(\dots w_{k-1}(s)))$$

where

$$w_j(s) = f_{n'_j+1}(f_{n'_j+2}(\dots f_{n'_{j+1}}(s))), \qquad j = 1, 2, \dots, k-1$$
$$w_0(s) = f_1(f_2(\dots f_{n'_1}(s))).$$

By Lemma 2.1, the $\{w_k(s)\}$ satisfy Condition 1. Lemma 2.2 implies that

$$\lim_{n\to\infty} w^j_n(Rs) = s; \qquad j = 1, \dots, l,\ s \in [0, 1]. \tag{2.13}$$

Let q_m be any sequence of integers converging to ∞. To prove the lemma it suffices to show that

$$\lim_{m\to\infty} f^i_{(q_m)}(Rs)=g(Rs); \quad s\in[0,1). \tag{2.14}$$

For each integer $m\geqq 1$, we associate the integer $k(m)$ so that

$$n'_{k(m)}\leqq q_m\leqq n'_{k(m)+1}.$$

Also,

$$w_{k(m)}(s)=d_m(t_m(s))$$

where

$$d_m(s)=f_{n'_{k(m)}+1}(f_{n'_{k(m)}+2}(\dots f_{q_m}(s)))$$

and

$$t_m(s)=f_{q_m+1}(f_{q_m+2}(\dots f_{n'_{k(m)+1}}(s))).$$

Again by Lemma 2.1, the $\{t_m(s)\}$ satisfy Condition 1. Using (2.13) and Lemma 2.3, we conclude that

$$\lim_{m\to\infty} d^j_m(Rs)=s; \quad j=1,\dots,l, \quad s\in[0,1].$$

But,

$$f^i_{(q_m)}(Rs)=f^i_{(n'_{k(m)})}(d_m(Rs)).$$

Thus by uniform convergence,

$$\lim_{m\to\infty} f^i_{(q_m)}(Rs)=g(Rs).$$

This proves (2.14). Q.E.D.

Theorem 1 is a direct consequence of Lemma 2.4.

We now turn our attention to the proof of Theorem 2. For any function f of p variables, let $\operatorname{grad} f|_s$ denote the gradient vector of f evaluated at the point s. Using Taylor's Theorem, the monotonicity of $f^i_n(s)$, and the assumptions of (i) and (ii) of Theorem 2, we obtain the existence of a constant $A>0$ such that

$$0\leqq 1-f^i_n(s)\leqq A(1,1-s); \quad n\geqq 1, \quad 1\leqq i\leqq p, \quad s\in[0,1]^p. \tag{2.15}$$

One direction of the theorem is a consequence of (2.15). Indeed, it follows from (2.15) that for any sequence of vectors $\{a_{n_k}\}\in[0,1]^p$

$$\lim_{n_k\to\infty} a_{n_k}=1 \Rightarrow \lim_{n_k\to\infty} f_{n_k}(a_{n_k})=1.$$

We now consider the other direction. For convenience the subscripted notation will be dropped. Suppose for some i, $\liminf_{n\to\infty} f^i_n(a_n)=1$ and $\liminf_{n\to\infty} a_n \neq 1$. Then some component of the $\{a_n\}$, say the first, has a convergent subsequence $\{a^1_{n_k}\}$ such that

$$\lim_{n_k\to\infty} a^1_{n_k}=b_0<1.$$

The collection of functions $\{f^i_{n_k}(s,1,\dots,1)\}$ is a normal family and thus has a subsequence $\{f^i_{n'_k}(s,1,\dots,1)\}$ converging to a limit function $h(s)$, uniformly on compact subsets of $[0,1]$. However,

$$0=\lim_{n'_k\to\infty}\{1-f^i_{n'_k}(a_{n'_k})\}\geqq \lim_{n'_k\to\infty}\{1-f^i_{n'_k}(a^1_{n'_k},1,\dots,1)\}=1-h(b_0).$$

Therefore, $h(b_0)=1$. Since $h(s)$ is either a constant or an increasing function, we conclude that $h(s)=1$. This implies that

$$\lim_{n'_k\to\infty} \frac{\partial}{\partial s} f^i_{n'_k}(s, 1, \ldots, 1)=0; \qquad s\in[0, 1). \tag{2.16}$$

It is not difficult to see that (2.16) contradicts both Conditions (i) and (ii) of Theorem 2. Q.E.D.

3. Proof of Theorem 3

We now turn our attention to proving a necessary and sufficient condition for when $\lim_{n\to\infty} f_{(n)}(Rs)=g(s)$ is a constant vector for $s\in[0,1)$. The conditions of Theorem 3 will be assumed throughout this section.

We will first show the necessity of (1.1). Suppose some component of $g(s)$ is a constant. It then follows from the definition of $g(s)$ and the conditions of Theorem 3 that all the components of $g(s)$ are constants. Therefore,

$$0=\frac{d}{ds}(g(s), 1)=\lim_{n\to\infty}\left(1\cdot\prod_{j=1}^{n} M_j(a_j(n)), 1\right) \tag{3.1}$$

where,

$$a_j(n)=f_{j+1}(f_{j+2}(\ldots f_n(Rs))); \qquad 1\leqq j\leqq n-1$$

$$a_n(n)=Rs; \qquad s\in[0,1).$$

Also,

$$\left(1\cdot\prod_{j=1}^{n} M_j(a_j(n)), 1\right)\geqq\left(1\cdot\prod_{j=1}^{n} M_j(0), 1\right). \tag{3.2}$$

(1.1) follows from (3.1) and (3.2).

In order to prove the sufficiency of (1.1), we need to introduce the following functions and prove certain properties about them. Define

$$f_{(j,k)}(s)=f_j(f_{j+1}(\ldots f_{j+k}(s))); \qquad j\geqq 1, \quad k\geqq 0, \quad s\in[0,1]^p$$

and

$$e_j(s)=\lim_{k\to\infty} f_{(j,k)}(Rs); \qquad j\geqq 1, \quad s\in[0,1). \tag{3.3}$$

The existence of the limit in (3.3) follows from Theorem 1. If $g^i(s)$ is assumed increasing for some i, it is a simple matter to show that each component of $e_j(s)$ is increasing. Therefore, $e_j(s)<1$ for $s\in[0,1)$. It follows from the definition of the $\{e_j(s)\}_{j\geqq 1}$ that,

$$e_j(s)=f_j(f_{j+1}(\ldots f_k(e_{k+1}(s)))); \qquad 1\leqq j<k<\infty \tag{3.4}$$

and

$$\frac{d}{ds} e^i_j(s)=\sum_{l=1}^{p} D_l f^i_j(e_{j+1}(s))\frac{d}{ds} e^l_{j+1}(s); \qquad 1\leqq i\leqq p, \quad j\geqq 1, \quad s\in[0,1). \tag{3.5}$$

Lemma 3.1. *Assume* $g^1(s)$ *is increasing. Let* $s_0\in[0,1)$. *Then*

$$\max_{1\leqq i\leqq p}\left[\limsup_{j\to\infty} e^i_j(s_0)\right]<1. \tag{3.6}$$

Proof. It follows from the definition of $g^1(s)$ that

$$g^1(s)=\lim_{n\to\infty} f^1_{(n)}(R\,s)=f^1_{(n)}(e_{n+1}(s)); \qquad n\geqq 1. \tag{3.7}$$

Suppose that (3.6) is false. Then for some i, there exists a subsequence $n_k\to\infty$ such that,

$$\lim_{n_k\to\infty} e^i_{n_k}(s_0)=1. \tag{3.8}$$

From (3.4) we obtain,

$$e^i_{n_k}(s_0)=f^i_{n_k}(e_{n_k+1}(s_0)). \tag{3.9}$$

(3.8) and (3.9) together with Theorem 2 imply that

$$\lim_{n_k\to\infty} e^j_{n_k+1}(s_0)=1; \quad 1\leqq j\leqq p. \tag{3.10}$$

Using (3.7), it is easy to see that (3.10) is incompatable with the assumption that $g^1(s)$ is strictly increasing. Q.E.D.

Lemma 3.2. *Assume $g^1(s)$ is strictly increasing. Given any sequence $n_k\to\infty$, we can always extract a subsequence $n'_k\to\infty$ such that for some $1\leqq j\leqq p$,*

$$\lim_{n'_k\to\infty} e^j_{n'_k}(s)=s; \qquad s\in[0,1). \tag{3.11}$$

Proof. We can assume that $n_k=k$ for all k. The collection of p.g.f. vectors, $\{f_{(k-1)}(s)\}_{k\geqq 2}$ is a normal family. Therefore, there exists a convergent subsequence $\{f_{(n_k-1)}(s)\}$. Assume,

$$\lim_{n_k\to\infty} f^1_{(n_k-1)}(s)=h^1(s_{i_1}, s_{i_2}, \dots, s_{i_l}); \qquad 1\leqq l\leqq p. \tag{3.12}$$

Since, $h^1(R\,s)=g^1(s)$, we know that h^1 is not a constant. Without loss of generality assume that $i_1=1, i_2=2, \dots, i_l=l$. Also note that for $n_j>n_k$,

$$f^1_{(n_j-1)}(s)=f^1_{(n_k-1)}(h_{(k,j)}(s)) \tag{3.13}$$

where

$$h_{(k,j)}(s)=f_{n_k}(f_{n_k+1}(\dots f_{n_j-1}(s))); \qquad s\in[0,1]^p.$$

For each k, the collection, $\{h_{(k,j)}(s)\}_{n_j>n_k}$ is a normal family. Thus, there exists a function $h_k(s)$ and a sequence $j(k)\to\infty$ such that

$$h_k(s)=\lim_{j(k)\to\infty} h_{(k,j(k))}(s); \qquad s\in[0,1]^p.$$

It follows from (3.12) and (3.13) that

$$h(s_1,\dots,s_l)=f^1_{(n_k-1)}(h_k(s)); \qquad k\geqq 1, \quad s\in[0,1]^p. \tag{3.14}$$

(3.14) implies that for each k, $h_k(s)$ depends only on the variables $s_1, s_2, \dots, s_l$. It should also be noted that

$$h_k(R\,s)=e_{n_k}(s); \qquad k\geqq 1.$$

Thus, if we can show that,

$$\lim_{k\to\infty} h^1_k(R\,s)=s; \qquad s\in[0,1) \tag{3.15}$$

we are done. The collection of functions $\{h_k(s)\}_{k\geq 1}$ is a normal family. Thus, we can always find a function $w(s)$ and a subsequence $k'\to\infty$ such that

$$w(s)=\lim_{k'\to\infty} h_{k'}(s).$$

To prove (3.15), it is sufficient to show that $w(s)$ is always linear, i.e.,

$$w^i(s)=\sum_{j=1}^{l} a_j^i s_j \quad \text{and} \quad \sum_{j=1}^{l} a_j^i=1; \quad 1\leqq i\leqq p.$$

Let $s<1$. By Lemma 3.1, we know that $w(s)<1$. Thus by uniform convergence in (3.12), we obtain,

$$h^1(s_1, s_2, \ldots, s_l)=h^1(w(s)); \quad s\in[0,1)^p.$$

We can now proceed exactly as in Lemma 2.2, and conclude that (3.15) is valid, providing we can show that the collection of functions $\{h_k(s)\}_{k\geq 1}$ satisfies Condition 1. However, that follows from the definition of $h_k(s)$ and Theorem 2. Q.E.D.

We now prove some corollaries of Lemma 3.2. For each of the next four corollaries $g^1(s)$ is assumed to be increasing.

Corollary 3.1. *For $s\in[0,1)$ there exist constants α_1, β_1 depending on s such that,*

$$0<\alpha_1<\inf_{j\geq 1}\left(1, \frac{d}{ds} e_j(s)\right)\leqq \sup_{j\geq 1}\left(1, \frac{d}{ds} e_j(s)\right)<\beta_1<\infty.$$

Proof. Since $g^1(s)$ is increasing, we know that $0<\left(1, \frac{d}{ds} e_j(s)\right)<\infty$ for all j. Suppose the result was false and that $\sup_{j\geq 1}\left(1, \frac{d}{ds} e_j(s)\right)=\infty$. Then there exist a subsequence, $j_k\to\infty$ such that

$$\lim_{j_k\to\infty}\left(1, \frac{d}{ds} e_{j_k}(s)\right)=\infty. \tag{3.16}$$

By Lemma 3.2 there exists a subsequence of the $\{j_k-1\}$, say $\{j'_k-1\}$, and an $1\leqq i\leqq p$ such that

$$\lim_{j'_k-1\to\infty} e^i_{j'_k-1}(s)=s; \quad s\in[0,1).$$

It follows that

$$\lim_{j'_k-1\to\infty} \frac{d}{ds} e^i_{j'_k-1}(s)=1; \quad s\in[0,1) \tag{3.17}$$

(3.5), (3.17) together with the assumptions of Theorem 3 imply that for j'_k sufficiently large,

$$\frac{1}{2F}<\left(1, \frac{d}{ds} e_{j'_k}(s)\right)\leqq\frac{3}{2E}.$$

This contradicts (3.16). The left hand inequality is established in the same way. Q.E.D.

Corollary 3.2. *For $s\in[0,1)$ there exist constants α_2 and β_2 depending on s such that*

$$0<\alpha_2\leqq \inf_{\substack{1\leqq i\leqq p\\ j\geq 1}} \frac{d}{ds} e^i_j(s)\leqq \sup_{\substack{1\leqq i\leqq p\\ j\geq 1}} \frac{d}{ds} e^i_j(s)\leqq\beta_2<\infty.$$

Proof. Omitted

Corollary 3.3. *For $s \in [0, 1)$ there exist constants α_3 and β_3 depending on s such that*

$$0 < \alpha_3 \leqq \inf_{\substack{1 \leqq m < n \\ n \geqq 1}} \left(1, \prod_{j=m}^{n} M_j(e_{j+1}(s)) \cdot 1^t\right)$$

$$\leqq \sup_{\substack{1 \leqq m < n \\ n \geqq 1}} \left(1, \prod_{j=m}^{n} M_j(e_{j+1}(s)) \cdot 1^t\right) \leqq \beta_3 < \infty.$$

Proof. Omitted

Corollary 3.4. *For $s \in [0, 1)$ there exist constants α_4 and β_4 depending on s such that*

$$0 < \alpha_4 \leqq \sup_{\substack{1 \leqq m < n \\ n \geqq 1}} \left[\min\left[\prod_{j=m}^{n} M_j(e_{j+1}(s))\right]\right]$$

$$\leqq \sup_{\substack{1 \leqq m < n \\ n \geqq 1}} \left[\max\left[\prod_{j=m}^{n} M_j(e_{j+1}(s))\right]\right] \leqq \beta_4 < \infty.$$

Proof. This result follows from Corollary 3.3 and the following inequality.

$$\min\left[\prod_{j=m}^{n} M_j(e_{j+1}(s))\right] \leqq \left(1, \prod_{j=m}^{n} M_j(e_{j+1}(s)) \cdot 1^t\right)$$

$$\leqq p^2 \max\left[\prod_{j=m}^{n} M_j(e_{j+1}(s))\right] \tag{3.18}$$

$$\leqq p^2 \left(\frac{F}{E}\right)^2 \min\left[\prod_{j=m}^{n} M_j(e_{j+1}(s))\right].$$

(3.18) is a simple consequence of the conditions of Theorem 3. Q.E.D.

We are finally ready to prove the sufficiency of (1.1). Suppose that $g^1(s)$ is increasing and (1.1) holds. Consider the ratio.

$$B(n) = \frac{\left(1, \prod_{j=1}^{n} M_j(e_{j+1}(s_0)) \cdot 1^t\right)}{\left(1, \prod_{j=1}^{n} M_j(0) \cdot 1^t\right)}; \quad n \geqq 1, \quad s_0 \in [0, 1).$$

Suppose that

$$\limsup_{n \to \infty} B(n) < \infty. \tag{3.19}$$

(3.19) is sufficient to prove the result. To see this observe that

$$\left(1, \prod_{j=1}^{n} M_j(0) \cdot 1^t\right) = \frac{1}{B(n)} \left(1, \prod_{j=1}^{n} M_j(e_{j+1}(s_0)) \cdot 1^t\right). \tag{3.20}$$

From 3.19, 3.20 and Corollary 3.3, we conclude that

$$\liminf_{n \to \infty} \left(1, \prod_{j=1}^{n} M_j(0) \cdot 1^t\right) > 0.$$

This contradicts (1.1). So all that remains is to prove (3.19). Using Taylor's theorem and the monotonicity of the M_j's we deduce that for $j \geqq 1$,

$$M_j(e_{j+1}(s_0)) \leqq M_j(0) + P_j$$

where

$$P_j = ((\operatorname{grad} D_k f_j^i|_{e_{j+1}(s_0)}, 1))_{1 \leqq i, k \leqq p}.$$

It follows from Corollary 3.2 and the assumptions of Theorem 3 that

$$P_j \leqq \alpha_2 \frac{d}{ds}[M_j(e_{j+1}(s_0))] \leqq \frac{\max\left[\frac{d}{ds} M_j(e_{j+1}(s_0))\right] M_j(0)}{E \alpha_2^{-1}}.$$

We, therefore, obtain the inequality:

$$B(n) \leqq \prod_{j=1}^{n} \left\{1 + \alpha_2 E \max\left[\frac{d}{ds} M_j(e_{j+1}(s_0))\right]\right\}. \tag{3.21}$$

So to prove (3.19), it is enough to show that

$$\sum_{j=1}^{\infty} \max\left[\frac{d}{ds} M_j(e_{j+1}(s_0))\right] < \infty. \tag{3.22}$$

By Lemma 3.2 there exists a subsequence $n_k \to \infty$ and an integer i such that for $s \in [0, 1)$

$$\lim_{n_k \to \infty} e_{n_k}^i(s) = s.$$

By uniform convergence, it follows that for $s \in [0, 1)$

$$\lim_{n_k \to \infty} \frac{d}{ds} e_{n_k}^i(s) = 1 \quad \text{and} \quad \lim_{n_k \to \infty} \frac{d^2}{ds^2} e_{n_k}^i(s) = 0. \tag{3.23}$$

Using 3.4, 3.23, and Corollary 3.4, it is not difficult to show that

$$\lim_{n_k \to \infty} \sum_{j=n_k}^{\infty} \left(1 \cdot \frac{d}{ds} M_j(e_{j+1}(s)), 1\right) = 0. \tag{3.24}$$

(3.22) is a direct consequence of (3.24).

This completes the proof of Theorem 3.

Remarks. 1. In the situation when $\lim_{n \to \infty} f_{(n)}(Rs) = g(s)$ is a constant vector, it can be shown that $\lim_{n \to \infty} f_{(n)}(s)$ exists for any $s \in [0, 1]^p$ and is equal to the constant vector.

2. It would be desirable to show that it is possible that $\lim_{n \to \infty} f_{(n)}(Rs)$ converges but $\lim_{n \to \infty} f_{(n)}(s)$ does not for arbitrary $s \in [0, 1]^p$. Although no example of this type of behavior is known, I do believe that it can occur.

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