Z. Wahrscheinlichkeitstheorie verw. Gebiete 32, 73–79 (1975)

Strict Disintegration of Measures

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Let R be a compact Hausdorff space, I^k a product of closed unit intervals, M a Baire probability measure on $R \times I^k$, and λ the induced measure on R. Then M has a strict Baire disintegration over λ; that is, for each $r \in R$, there is a Baire probability measure m_r on I^k such that, for each Baire set $E \subseteq R \times I^k$, $m_r\{t \mid (r,t) \in E\}$ defines a Baire measurable function on R, whose integral with respect to λ is $M(E)$. This result generalizes to the case in which R is replaced by an arbitrary measure space.

1. Statement of the Theorems

Let R be an arbitrary set, $\mathscr{B}(R)$ a Borel field of subsets of R, and X the Cartesian product $R \times I^k$ where k is an arbitrary cardinal $\geqq 1$, and I^k is the product of k copies of $I = [0, 1]$. We denote the σ-field of Baire sets in I^k by $\mathscr{B}(I^k)$, and the σ-field $\mathscr{B}(R) \times \mathscr{B}(I^k)$ in X, generated by cylinders in X with bases in $\mathscr{B}(R)$ or in $\mathscr{B}(I^k)$, by $\mathscr{B}(X)$. For arbitrary $E \subseteq X$ and $r \in R$, let $E(r) = \{t \mid t \in I^k, (r,t) \in E\}$ (so that the "r-section" of E is $\{r\} \times E(r)$). It is easy to see that if $E \in \mathscr{B}(X)$ then $E_r \in \mathscr{B}(I^k)$.

Let M be a probability measure on $\mathscr{B}(X)$, and λ the measure induced by M on $\mathscr{B}(R)$ [that is, if $H \in \mathscr{B}(R)$, $\lambda(H) = M(H \times I^k)$]. Let $\mathscr{B}_\lambda(R)$, $\mathscr{B}_M(X)$ be the respective completions of $\mathscr{B}(R)$ and of $\mathscr{B}(X)$, obtained by adjoining subsets of null sets. Clearly $\mathscr{B}_\lambda(R) \times \mathscr{B}(I^k) \subseteq \mathscr{B}_M(X)$, and if $E \in \mathscr{B}_\lambda(R) \times \mathscr{B}(I^k)$, then $E(r) \in \mathscr{B}(I^k)$ for all $r \in R$.

Our object is to prove the following:

Theorem I. *Under the conditions above, for each $r \in R$ there exists a Baire probability measure m_r on $\mathscr{B}(I^k)$ such that, for each $E \in \mathscr{B}_\lambda(R) \times \mathscr{B}(I^k)$,*

(i) *the map $r \mapsto m_r(E(r))$ is $\mathscr{B}_\lambda(R)$-measurable, and*

(ii) $M(E) = \int_R m_r(E(r))\, d\lambda(r)$.

With respect to the whole field $\mathscr{B}_M(X)$, the following assertion follows easily from Theorem I:

Theorem I'. *If $E \in \mathscr{B}_M(X)$, then for almost all $r \in R$ we have $E(r) \in \mathscr{B}_{m_r}(I^k)$, the completion of $\mathscr{B}(I^k)$ with respect to m_r; $m_r(E(r))$ is a λ-measurable function of r; and $M(E) = \int_R m_r(E(r))\, d\lambda(r)$.*

Note that we can replace $X = R \times I^k$ by any of its $\mathscr{B}(X)$-measurable subsets S in Theorem I, if we allow $m_r(S(r))$ to take values $\leqq 1$. For if M is defined on the $\mathscr{B}(X)$-measurable subsets of S, we extend M to X by setting $M(S - X) = 0$, apply

* The author gratefully acknowledges support by the National Science Foundation.

Theorem 1, and re-define m_r on a null set of r's to ensure that $m_r(S(r))=0$ when $S(r)=\emptyset$.

Perhaps the most significant special case of Theorem I is that in which R is a compact Hausdorff space and $\mathscr{B}(R)$ is the field of Baire sets in R. Then X is also compact Hausdorff, and $\mathscr{B}(X)=\mathscr{B}(R)\times\mathscr{B}(I^k)$ is the field of Baire sets in X. Theorem I reduces to the following:

Theorem II. *Let R be a compact Hausdorff space, k a cardinal $\geqq 1$, M a Baire probability measure on $R\times I^k$, and λ the Baire measure induced on $\mathscr{B}(R)$ by M. Then for each $r\in R$ there exists a Baire probability measure m_r on $\mathscr{B}(I^k)$ such that, for each $U\in\mathscr{B}(R\times I^k)$,*

(1) *the function assigning to r the value $m_r(U(r))$ is Baire measurable on R, and*

(2) $\int_R m_r(U(r))\,d\lambda(r)=M(U)$.

Our strategy is to prove Theorem II, and then to deduce Theorem I from it.

Comparison with some Previous Theorems. Disintegration theorems of this type go back to von Neumann [9], Halmos, and Dieudonné (see [2, Th. 5] and [3]); a sharper form (essentially producing a product decomposition) is in [5, Th. 5]. Two standard formulations (substantially equivalent to each other and to Halmos's theorem as revised in [3]) are given by Bourbaki [1, § 3, Nos. 1 and 3]. In all these theorems, in contrast to Theorem I, the underlying spaces are required to have countable bases. Thus, whereas Bourbaki requires the spaces R and X (there called B and T) to be locally compact and second countable, we impose no cardinality conditions on them. On the other hand, where Bourbaki allows an arbitrary measurable map p from X to R to be given, we require X to be of a special product form and p to be the projection. The measures in Bourbaki are σ-finite, in Theorem I are finite; but that is merely a matter of formulation. The conclusions of Bourbaki's theorems and of Theorem I are essentially the same, except that in the former the measures m_r (there called λ_b) are essentially unique. I do not know whether uniqueness holds in the present Theorem I; this is one of the complications arising from the lack of a countable base.

A proof of Theorem I for the special case $k=1$ (and thus for $k\leqq\aleph_0$) is outlined in [6]; this case furnishes the starting-point for the present proof[1].

After the present paper was written, I learned that Valadier [7, 8] and Graf (unpublished) had independently proved a sharper and more general form of Theorem I, in which I^k is replaced by an arbitrary Hausdorff space U, and "Baire" by "Borel" throughout. (The latter change is the significant one; by itself, the generalization from I^k to U could be accomplished in a standard way, by imbedding U suitably in some I^k.) Nevertheless it is hoped that the present proof may be of interest, since the method is entirely different from that of Valadier and Graf; in particular, the proof of Theorem II makes no use of lifting theory.

2. Further Notation

We regard I^k as $I^A=\prod\{I_a\mid a\in A\}$, where the index set A has cardinal k, and each $I_a=[0,1]=I$. We shall generally use r to denote a point of R; x for a point

[1] See [6, § 4], "A disintegration theorem". A proof can also be deduced without difficulty from Theorem 2b of [5, p. 149], by an argument very similar to that used in § 6 of the present paper.

of I^A; B, C for non-empty subsets of A; z for a point of I^C; U, V, W for subsets of $R\times I^A$, $R\times I^B$, $R\times I^C$ respectively, and F, G, H for subsets of I^A, I^B, I^C respectively. The projection map from $R\times I^A$ to $R\times I^B$ ($B\subseteq A$) is denoted by p_B, and the projection from I^A to I^B by π_B (so that $p_B=i_R\times\pi_B$). We use M^B to denote the probability measure induced on $R\times I^B$ by M; that is, $M^B(V)=M(p_B^{-1}V)=M(V\times I^{A-B})$ if $V\in\mathscr{B}(R\times I^B)$. When B is a singleton, say $\{b\}$, we may write $M^{\{b\}}$ as M^b. We denote $(p_B)^{-1}(\mathscr{B}(R\times I^B))=\{V\times I^{A-B}\,|\,V\in\mathscr{B}(R\times I^B)\}$ by $\mathscr{B}(B,R\times I^A)$, and $(\pi_B)^{-1}(\mathscr{B}(I^B))=\{G\times I^{A-B}\,|\,G\in\mathscr{B}(I^B)\}$ by $\mathscr{B}(B,I^A)$. Note that if $\emptyset\neq C\subseteq B\subseteq A$, then $\mathscr{B}(C,I^A)\subseteq\mathscr{B}(B,I^A)\subseteq\mathscr{B}(A,I^A)=\mathscr{B}(I^A)$, and similarly with I^A replaced by $R\times I^A$.

3. Lemmas

First, a definition. Suppose X is any topological space, m is a finite measure on X, and $\mathscr{C}$ is any σ-field of m-measurable sets in X. We say that m is "$\mathscr{C}$-regular" if, for each $E\in\mathscr{C}$, $m(E)=\inf\{m(0)\,|\,0\in\mathscr{C},\ 0\supseteq E,\ 0 \text{ open in } X\}$.

Now suppose further that a continuous surjection $f\colon X\to X'$ is given, where X' is a topological space and m' is a finite Baire measure on X' (and so necessarily regular). Let $\mathscr{B}(X')$ be the field of Baire sets in X', and $\mathscr{C}=f^{-1}\{\mathscr{B}(X')\}$. Define a measure m on $\mathscr{C}$ by setting $m(f^{-1}E')=m'(E')$ for $E'\in\mathscr{B}(X')$.

Lemma 1. *Under the assumptions above, m is $\mathscr{C}$-regular.*

The verification is routine and is omitted.

Corollary. *Under the same assumptions, every finite measure on $\mathscr{C}$ is $\mathscr{C}$-regular.*

For every such measure m on $\mathscr{C}$ clearly arises from a measure m' on X' as above.

Now apply this Lemma and Corollary to $X=I^A$, $X'=I^B$ (with $\emptyset\neq B\subset A$), and $f=\pi_B$, and we get

Lemma 2. *Every probability measure on $\mathscr{B}(B,I^A)$ is $\mathscr{B}(B,I^A)$-regular.*

4. The Inductive System

Let $\mathscr{X}$ be the family of all ordered pairs (C,m) where C is a non-empty subset of A and m is a real-valued function on $R\times\mathscr{B}(C,I^A)$ such that

(1) for each $r\in R$ the function m_r defined by

$$m_r(F)=m(r,F),\qquad F\in\mathscr{B}(C,I^A)$$

is a probability measure on $\mathscr{B}(C,I^A)$,

(2) for each $U\in\mathscr{B}(C,R\times I^A)$, $m_r(U(r))$ is a Baire measurable function of r on R, and $M(U)=\int_R m_r(U(r))\,d\lambda(r)$.

Note that these conditions on m can be re-phrased in terms of $\mathscr{B}(I^C)$ instead of $\mathscr{B}(C,I^A)$ as follows: Put $\mu_r^C(H)=m_r(\pi_C^{-1}(H))$ $(H\in\mathscr{B}(I^C))$. Then

(a) μ_r^C is, for each $r\in R$, a probability measure on $\mathscr{B}(I^C)$,

(b) if $W\in\mathscr{B}(R\times I^C)$ then $\mu_r^C(W(r))$ is a Baire measurable function of r on R, and $M^C(W)=\int_R\mu_r^C(W(r))\,d\lambda(r)$.

In fact, (a) and (b) are immediate consequences of (1) and (2); conversely, if for each $r\in R$ we have a measure μ_r^C satisfying (a) and (b), we have only to define m by $m(r,F)=\mu_r^C(\pi_C(F))$ $(F\in\mathscr{B}(C,I^A), r\in R)$, to recover (1) and (2).

On $\mathscr{Z}$, define a partial ordering as follows: $(C,m)\leqq(B,m')$ providing

(i) $\emptyset\neq C\subseteq B\subseteq A$,

(ii) for each $r\in R$, the restriction $m'_r|_{\mathscr{B}(C,I^A)}=m_r$. This relation is easily verified to be transitive.

Our object is, of course, to apply Zorn's Lemma. First, note that $\mathscr{Z}\neq\emptyset$. For choose $a\in A$ and choose $C=\{a\}$; M induces a Baire probability measure M^a on $R\times I_a$, and M^a induces the same measure λ on $\mathscr{B}(R)$ as M does. Now the theorem of § 4 of [6] (the case $k=1$ of the present theorem) applies, giving for each $r\in R$ a Baire measure μ_r^a on I_a satisfying conditions (a) and (b) above. Thus $(\{a\},m)\in\mathscr{Z}$, where $m(r,F)=\mu_r^a(\pi_a F)$ for $r\in R$, $F\in\mathscr{B}(\{a\},I^A)$.

Next we show that $\mathscr{Z}$ is inductive in the partial ordering. Let $\mathscr{L}$ be a non-empty totally ordered subset of $\mathscr{Z}$. Note that if (C,m) and (C,m') are both in $\mathscr{L}$, then $m=m'$ from (ii) above. Put $\mathscr{C}\equiv\{C\,|\,(C,m)\in\mathscr{L}\}$; $C^*=\bigcup\mathscr{C}$, a non-empty subset of A; and $\mathscr{F}=\bigcup\{\mathscr{B}(C,I^A), C\in\mathscr{C}\}$. It is easily seen that $\mathscr{F}$ is a finitely additive sub-field of $\mathscr{B}(C^*,I^A)$. Moreover, the σ-field generated by $\mathscr{F}$ is all of $\mathscr{B}(C^*,I^A)$, since the sets of the form $E\times I^{A-\{a\}}$, where $E\in\mathscr{B}(I_a)$ and $a\in C^*$, generate $\mathscr{B}(C^*,I^A)$ and are in $\mathscr{F}$. It follows from condition (ii) on $\mathscr{Z}$ that for each $r\in R$, the measures m_r on the various fields $\mathscr{B}(C,I^A)$, where $(C,m)\in\mathscr{L}$, are mutually consistent; thus they combine to give a finitely additive measure m'_r on $\mathscr{F}$. We show that (keeping r fixed) m'_r has an extension (necessarily unique) to a countably additive measure m_r^* on $\mathscr{B}(C^*,I^A)$. To do this, it is enough to show that if $F_0, F_1, F_2, \ldots\in\mathscr{F}$, $F_0=\bigcup_{n=1}^\infty F_n$, and $F_1, F_2, \ldots$ are pairwise disjoint, then $m_r(F_0)\leqq\sum_1^\infty m'_r(F_n)$; for the reverse inequality is trivial.

Now $F_n\in\mathscr{B}(C_n,I^A)$ for some $C_n\in\mathscr{C}$ $(n=0,1,2,\ldots)$; let m^n be the corresponding measure function, so that $(C_n,m^n)\in\mathscr{L}$. From Lemma 2, $(m^n)_r$ is $\mathscr{B}(C_n,I^A)$-regular; hence, given $\varepsilon>0$, there exist open sets O_n in $\mathscr{B}(C_n,I^A)$ such that $O_n\supset F_n$ and $(m^n)_r(O_n)<(m^n)_r(F_n)+\varepsilon/2^{n+1}$. Thus $m'_r(O_n)\leqq m'_r(F_n)+\varepsilon/2^{n+1}$. Similarly, there exists a compact $K_0\in\mathscr{B}(C_0,I^A)$ such that $K_0\subset F_0\subset\bigcup_1^\infty O_n$ and $m'_r(K_0)\geqq m'_r(F_0)-\varepsilon/2$. From compactness, $K\subset\bigcup_1^N O_n$ for some N. Therefore $m'_r(K_0)\leqq\sum_1^N m'_r(O_n)$, and the desired inequality follows.

Now we define m^* by $m^*(r,F)=m_r^*(F)$ for $F\in\mathscr{B}(C^*,I^A)$. We shall show that $(C^*,m^*)\in\mathscr{Z}$; since $(C,m)\leqq(C^*,m^*)$ for each $(C,m)\in\mathscr{L}$, (C^*,m^*) will then be the desired upper bound for $\mathscr{L}$ in $\mathscr{Z}$.

We must check that (C^*,m^*) satisfies (1) and (2) above. Condition (1) is clear. To establish (2), let $\mathscr{U}$ be the family of all sets $U\in\mathscr{B}(C^*,R\times I^A)$ for which the conclusions of (2) are true with (C,m) replaced by (C^*,m^*). We must show that $\mathscr{U}$ contains, and so is equal to, $\mathscr{B}(C^*,R\times I^A)$.

Put $\mathscr{H}=\bigcup\{\mathscr{B}(C,R\times I^A)\,|\,C\in\mathscr{C}\}$. It is easy to see that $\mathscr{H}\subseteq\mathscr{U}$, that $\mathscr{H}$ is a finitely additive field of sets, and that the σ-field generated by $\mathscr{H}$ is $\mathscr{B}(C^*,R\times I^A)$. The implication $U\in\mathscr{U}\Rightarrow R\times I^A-U\in\mathscr{U}$ is trivial, and a routine calculation shows that if $U_1\subseteq U_2\subseteq\cdots$ and $U_n\in\mathscr{U}$ $(n=1,2,\ldots)$ then $\bigcup_n U_n\in\mathscr{U}$. Thus $\mathscr{U}$ is a monotone class containing $\mathscr{H}$; and it follows immediately from the statements above that $\mathscr{U}\supseteq\mathscr{B}(C^*,R\times I^A)$, as required. Hence $(C^*,m^*)\in\mathscr{Z}$.

5. Proof of Theorem II Concluded

By Zorn's Lemma, $\mathscr{X}$ has a maximal element, say (C, m). We shall show that $C=A$, from which the theorem follows. Suppose, then, that $a\in A-C$; we derive a contradiction. Put $B=C\cup\{a\}$; thus $R\times I^B=(R\times I^C)\times I_a=R'\times I$. Apply the known case $k=1$ of the present theorem to the measure M^B on $R'\times I$; note that the induced measure on R' (corresponding to "λ") is here M^C. We obtain, for each $(r,z)\in R\times I^C$, a Baire probability measure $\nu_{r,z}$ on $I_a=I$, such that for each $V\in\mathscr{B}(R\times I^B)$, $\nu_{r,z}(V(r,z))$ is a Baire measurable function of (r,z), and

$$M^B(V)=\int_{R\times I^C}\nu_{r,z}(V(r,z))\,dM^C(r,z).$$

Now for arbitrary $F\in\mathscr{B}(B,I^A)$ we note that $F=G\times I^{A-B}$ where $G=\pi_B F\in\mathscr{B}(I^B)$, and define, for each $r\in R$, $m'_r(F)=\int_{I^C}\nu_{r,z}G(z)\,d\mu_r^C(z)$, where $G=\pi_B F\subseteq I^B=I^C\times I_a$, and $G(z)$ means $\{t\mid t\in I_a, (z,t)\in G\}$. We must check that the integrand is Baire measurable on I^C (for fixed $r\in R$). But from the definition of $\nu_{r,z}$ above, $\nu_{r,z}(V(r,z))$ is Baire measurable in (r,z), and so in z for fixed r, for all $V\in\mathscr{B}(R\times I^B)$; now apply this with $V=R\times G$.

We put $m'(r,F)=m'_r(F)$ $(F\in\mathscr{B}(B,I^A))$ and prove that $(B,m')\in\mathscr{X}$. This is most conveniently done in terms of the measures μ_r^B where (as in § 4)

$$\mu_r^B(G)=m'_r(G\times I^{A-B})=\int_{I^C}\nu_{r,z}G(z)\,d\mu_r^C(z),\qquad \text{for } G\in\mathscr{B}(I^B) \text{ and } r\in R.$$

We must show that conditions (a) and (b) of § 4 hold. Condition (a) follows easily from the definition and the above remarks on Baire measurability. To see that (b) holds, suppose $V\in\mathscr{B}(R\times I^B)$; we must show that $M^B(V)=\int_R\mu_r^B(V(r))\,d\lambda(r)$ (first showing that the integrand is Baire measurable). Now, $\mu_r^B(V(r))=\int_{I^C}\nu_{r,z}V(r,z)\,d\mu_r^C(z)$. First we show that for an arbitrary bounded Baire measurable function f on $R\times I^C$,

$$\int_{R\times I^C} f(r,z)\,dM^C(r,z)=\int_R\Big\{\int_{I^C} f(r,z)\,d\mu_r^C(z)\Big\}\,d\lambda(r),$$

and that the inner integral on the right is a Baire measurable function of r. This follows by a routine argument from the special case in which f is the characteristic function of a set $W\in\mathscr{B}(R\times I^C)$. To prove this special case, define $U=W\times I^{A-C}$; then $U\in\mathscr{B}(C,R\times I^A)$, and $M^C(W)=M(U)$. Because $(C,m)\in\mathscr{X}$, condition (2) now gives

$$M^C(W)=\int_R m_r(W(r)\times I^{A-C})\,d\lambda(r)=\int_R\mu_R^C(W(r))\,d\lambda(r),$$

as desired.

To derive (b), apply the result just established to $f(r,z)=\nu_{r,z}(V(r,z))$; we obtain

$$\int_{R\times I^C}\nu_{r,z}(V(r,z))\,dM^C(r,z)=\int_R\Big\{\int_{I^C}\nu_{r,z}(V(r,z))\,d\mu_r^C(z)\Big\}\,d\lambda(r),$$

that is (since $(V(r))(z)=V(r,z)$),

$$M^B(V)=\int_R\mu_r^B(V(r))\,d\lambda(r),$$

completing the proof of (b).

Thus $(B, m')\in\mathscr{Z}$. We claim that $(C, m)\leqq(B, m')$. Since $C\subseteq B$, we need only check that, for each $r\in R$ and $F\in\mathscr{B}(C, I^A)$, $m'_r(F)=m_r(F)$. Now $F=H\times I^{A-C}$, where $H=\pi_C F\in\mathscr{B}(I^C)$, and by definition $m'_r(F)=\int_{I^C} v_{r,z}(H\times I_a)(z)\,d\mu_r^C(z)$. The integrand here is just the characteristic function of H, so $m'_r(F)=\mu_r^C(H)=m_r(H\times I^{A-C})=m_r(F)$, as required.

This contradicts the maximality of (C, m). Thus $C=A$, and Theorem II follows.

6. Proof of Theorem I

We now deduce Theorem I from Theorem II, and return to the notation of § 1. R is now an arbitrary set (with no topology), λ is the measure induced on the Borel field $\mathscr{B}(R)$ in R, and $\mathscr{B}_\lambda(R)$ is the completion of $\mathscr{B}(R)$ with respect to the measure λ. Let $\mathscr{M}$ be the algebra of measurable modulo null sets of $\mathscr{B}_\lambda(R)$, and S the representation space of $\mathscr{M}$ equipped with the usual (compact, extremally disconnected) topology, and the usual measure, which we call $\tilde{\lambda}$. That is, for $a\in\mathscr{M}$, we denote the open-closed set corresponding to a by $S(a)$, set $\tilde{\lambda}(S(a))=\lambda(a)$, and extend $\tilde{\lambda}$ to the field $\mathscr{B}(S)$ of Baire sets in S. Let $\rho\colon \mathscr{M}\to\mathscr{B}_\lambda(R)$ be a lifting of $\mathscr{M}$ [4]. It is easy to verify that, for each $r\in R$, the family $\{a \mid a\in\mathscr{M}, r\in\rho(a)\}$ is an ultrafilter in $\mathscr{M}$, and so a point of S. Thus if we define, for $r\in R$,

$$\theta(r)=\bigcap\{S(a) \mid a\in\mathscr{M}, r\in\rho(a)\},$$

θ is a well defined map of R into S. It is easy to see that, for $a\in\mathscr{M}$, $\theta^{-1}S(a)=\rho(a)$, and so $\tilde{\lambda}(S(a))=\lambda(\theta^{-1}(S(a)))$; and it follows at once that, for all $H\in\mathscr{B}(S)$, $\theta^{-1}(H)\in\mathscr{B}_\lambda(R)$ and $\lambda(\theta^{-1}H)=\tilde{\lambda}(H)$. Hence if $\tilde{f}$ is $\mathscr{B}(S)$-measurable in S, and if f is the function on R such that $f(r)=\tilde{f}(\theta r)$, then f is $\mathscr{B}_\lambda(R)$-measurable and

$$\int_R f(r)\,d\lambda(r)=\int_S \tilde{f}(S)\,d\tilde{\lambda}(s).$$

Now let $T=S\times I^k$ (a compact Hausdorff space) and let $\mathscr{B}(T)$ be the σ-field of Baire sets in T, so that $\mathscr{B}(T)$ is generated by sets of the form $S(a)\times L$, where $a\in\mathscr{M}$, $L\in\mathscr{B}(I^k)$. Define $\psi\colon R\times I^k\to S\times I^k$ by setting $\psi(r,t)=(\theta r, t)$. Then for all $H\subseteq S\times I^k$ and $r\in R$, $H(\theta r)=\{t \mid t\in I^k, (\theta r, t)\in H\}=\{t \mid t\in I^k, (r,t)\in\psi^{-1}H\}=(\psi^{-1}H)(r)$. Also from the properties mentioned above for θ, it follows (by a simple Borel induction) that if $H\in\mathscr{B}(T)$, then $\psi^{-1}(H)\in\mathscr{B}_\lambda(R)\times\mathscr{B}(I^K)\subseteq\mathscr{B}_M(X)$.

Set $\tilde{M}(H)=M(\psi^{-1}H)$ for $H\in\mathscr{B}(T)$; then $\tilde{M}$ is a Baire measure on T. Moreover, for each $L\in\mathscr{B}(S)$, $\tilde{M}(L\times I^k)=M(\psi^{-1}(L\times I^k))=M[\theta^{-1}L\times I^k]=\lambda(\theta^{-1}L)=\lambda(L)$.

Now Theorem II applies to $(T, \mathscr{B}(T), \tilde{M})$, so that for each $s\in S$ there is a probability measure $\tilde{m}_s$ on $\mathscr{B}(I^k)$ such that, for $H\in\mathscr{B}(T)$,

(i) $s\mapsto\tilde{m}_s(H(s))$ is Baire measurable in S, and

(ii) $\int_S\tilde{m}_s(H(s))\,d\tilde{\lambda}(s)=\tilde{M}(H)$.

For $r\in R$ and $L\in\mathscr{B}(I)$, set $m_r L=\tilde{m}_{\theta r}(L)$. We show that, with this choice of m_r, Theorem I holds. First suppose $E\in\mathscr{B}_\lambda(R)\times\mathscr{B}(I^k)$ and is of the form $\theta^{-1}H$, where $H\in\mathscr{B}(T)$. From the considerations above, for each r, $E(r)=H(\theta r)\in\mathscr{B}(I^k)$; and by definition of m_r, $m_r E(r)=\tilde{m}_{\theta r}H(\theta r)$. Thus if we set $\tilde{f}(s)=\tilde{m}_s H(s)$ and $f(r)=m_r E(r)$ we have $f(r)=\tilde{f}(\theta r)$. Hence $m_r E(r)$ is $\mathscr{B}_\lambda(R)$-measurable in r, and

$$\int_R m_r E(r)\,d\lambda(r)=\int_S\tilde{m}_s(H(s))\,d\tilde{\lambda}(s)=\tilde{M}(H)=M(\psi^{-1}H)=M(E).$$

Finally, for each $E \in \mathscr{B}_\lambda(R) \times \mathscr{B}(I^k)$, we can find a set $H \in \mathscr{B}(T)$ and a null set $N \subset R$ such that the symmetric difference $E \Delta \psi^{-1}(H) \subseteq N \times I$. For if E is of the form $A \times L$, where $A \in \mathscr{B}_\lambda(R)$, $L \in \mathscr{B}(I^k)$, we take $H = S(a) \times L$ where a is the measure class of A, so that $\psi^{-1}(H) = \rho(a) \times L$; and $N = A \Delta \rho(a)$. The assertion carries over to all $E \in \mathscr{B}_\lambda(R) \times \mathscr{B}(I^k)$ by Borel induction. But if E, H and N are related as above, then on $R - N$, $E(r) = (\psi^{-1}H)(r)$ and so $m_r(E_r) = m_r(\psi^{-1}H)(r)$. Therefore, since Theorem I holds for $\psi^{-1}H$, it holds for E, and the proof is now complete.

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(Received April 29, 1974; in revised form January 31, 1975)