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A Characterization of the Normal Distribution

(A note on a paper of Kozin)

Rudolf Borges

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1. Introduction

P. Lévy [4, p. 509] proved the following theorem in an elementary way aside of technical improvements by G. Baxter [1], F. Kozin [2], and K. Krickeberg [3, p. 172]:

Theorem A. *Let* $(z_t)_{0 \leqq t \leqq 1}$ *be a Gaussian process with stationary independent increments and with* $z_0 = 0$ *and* $V z_1 = 1$. *Let parameter points* $0 = t_{n0} < t_{n1} < \cdots < t_{nk_n} = 1$ *be given, such that there exists a sequence* (γ_n) *of numbers with the properties:*

$$\gamma_n > 0\,, \quad \lim_{n\to\infty} (\gamma_n \delta_n) = 0\,, \quad \sum_{n=1}^{\infty} \gamma_n^{-1} < \infty \tag{1}$$

for $\delta_n := \max\{(t_{ni} - t_{ni-1}) : i = 1, \ldots, k_n\}$.

Then with probability one it holds that

$$\lim_{n\to\infty} \sum_{i=1}^{k_n} (z_{t_{ni}} - z_{t_{ni-1}})^2 = 1\,. \tag{2}$$

Recently Kozin's paper [2] has again be called [6, p. 103] "an extension (of the results of Lévy and Baxter) to processes with stationary independent increments". Hence it may be worthwhile to show that this is not the case. Therefore we shall prove in section 3 the following characterization of the normal distribution. Condition (c) is included, since it is explicitly used in the proof of Theorem A.

Theorem 1. *Let* z *be a random variable with infinitely divisible distribution and characteristic function* $E \exp iuz = \exp \psi(u)$. *Then the following conditions are equivalent:*

(a) z is normally distributed,
(b) $d^{2n}\psi(u)/du^{2n}|_{u=0} = 0$ for some $n > 1$,
(c) $E(z - Ez)^4 = 3(Vz)^2$.

Kozin's assumption of Theorem A reads as [2, p. 961] "Let $(z_t)_{0 \leqq t \leqq 1}$ be a process with stationary independent increments having characteristic function $\exp t\psi(u)$. Let Ez_t^4 exist and $d^4\psi(u)/du^4|_{u=0} = 0$". Hence the above theorem implies that Kozin's assumption is only satisfied in the Gaussian case.

2. A Lemma

The assumption of Theorem 1 implies the Lévy-Chintchin formula (see for example [5])

$$\psi(u) = i u \alpha + \int_{-\infty}^{+\infty} \left(e^{iux} - 1 - \frac{iux}{1+x^2}\right) \frac{1+x^2}{x^2}\, dG(x)\,, \tag{3}$$

where G is monotone non-decreasing continuous on the right and bounded, α is a constant and the value of the integrand at $x = 0$, defined by continuity, is $-u^2/2$. Condition (a) is equivalent to the following condition: $G(x)$ is constant except possibly at the point $x = 0$ and $G(+0) - G(-0) = Vz$. We denote by $\psi^{(n)}(u)$ the n-th derivative of ψ and by $\Delta_h^n \psi(u)$ its n-th symmetric difference.

Lemma. *Let the Lévy-Chintchin formula* (3) *be given and consider the conditions*

(α) $\int |x|^n dG(x) < \infty$,

(β) $|\psi^{(n)}(0)| < \infty$.

Then 1. For each $n \geqq 1$ *condition* (α) *implies that for* $m \leqq n$ *the m-th differentiation can be interchanged with the integration in* (3).

2. *For each* $n \geqq 1$ *condition* (α) *implies* (β).

3. *For even* $n > 1$ *condition* (β) *implies* (α).

Proof. 1. For $m \geqq 2$ we obtain for the m-th symmetric difference

$$\Delta_h^m \psi(u) = \int_{-\infty}^{+\infty} e^{iux} \left(\frac{e^{ihx} - e^{-ihx}}{2hx}\right)^m (1 + x^2)\, x^{m-2}\, dG(x) \tag{4}$$

$$= i^m \int_{-\infty}^{+\infty} e^{iux} \left(\frac{\sin hx}{hx}\right)^m (x^m + x^{m-2})\, dG(x).$$

Hence condition (α) for $n \geqq m$ implies by the dominated convergence theorem, that the limit as $h \to 0$ can be interchanged with the integral, that is

$$\psi^{(m)}(u) = i^m \int_{-\infty}^{+\infty} e^{iux} (x^m + x^{m-2})\, dG(x) \quad \text{for} \quad m \geqq 2. \tag{5}$$

The proof for the case $m = 1$ runs similarly and thus assertion 1 holds.

2. Put $u = 0$ and $n = m \geqq 2$ in (5). Assertion 2 follows for $n \geqq 2$. The proof for the case $n = 1$ is similar.

3. Put $u = 0$ and $n = m$ even in (4). Condition (β) implies

$$\left| \int_{-k}^{+k} \left(\frac{\sin hx}{hx}\right)^n (x^n + x^{n-2})\, dG(x) \right| \leqq \left| \int_{-\infty}^{+\infty} \left(\frac{\sin hx}{hx}\right)^n (x^n + x^{n-2})\, dG(x) \right|$$
$$\leqq |\psi^{(n)}(0)| + 1$$

for $0 < h \leqq h_0$ for some $h_0 > 0$. Hence the monotone convergence theorem implies for each $k > 0$

$$\int_{-k}^{+k} (x^n + x^{n-2})\, dG(x) \leqq |\psi^{(n)}(0)| + 1$$

and thus letting $k \to \infty$ condition (α).

Remark. The equivalence of (α) and (β) for $n = 2$ is well known. See for example Loève [*5*, p. 299].

3. Proof of Theorem 1

At first we prove that (b) implies (a): Condition (b) implies by the Lemma formula (5) for $u = 0$ and $m = 2n > 2$, that is

$$\int_{-\infty}^{+\infty} (x^{2n} + x^{2n-2})\, dG(x) = 0.$$

Hence $G(x)$ is constant except possibly at the point $x = 0$, that is, z is normally distributed. This proves that (b) implies (a). Secondly it is well known that condition (a) implies (c).

Finally we prove that (c) implies (b): Without loss of generality we can assume that $Ez = 0$, since $\psi''(u)$ is independent of Ez. Hence it is easily seen that $Ez^2 = -\psi''(0)$ and $Ez^4 = 3(\psi''(0))^2 + \psi^{(4)}(0) = 3(Ez^2)^2 + \psi^{(4)}(0)$. Hence condition (c) for the case $Ez = 0$ implies $\psi^{(4)}(0) = 0$, that is condition (b). This proves the theorem.

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Mathematisches Institut
63 Gießen, Arndtstr. 2