Z. Wahrscheinlichkeitstheorie verw. Geb. 5, 187—201 (1966)

On First Hitting Times of some Recurrent Two-dimensional Random Walks

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Received November 2, 1965

1. Introduction

It is well known that under certain conditions a random walk on a two-dimensional lattice is almost sure to hit each point of the lattice. Certain classes of such random walks are considered here in the discrete time parameter (d.p.) and continuous time parameter (c. p.) cases. The main problem is to determine the behaviour of the distribution of the first hitting time of a given point as the distance between that point and the starting point of the random walk becomes large. The method used is also shown to give an analogous result for plane Brownian motion. The random walk result is also applied to a restricted class of three-dimensional random walks: in the latter case, for a given axis of lattice points, the respective behaviours of the distribution of the first hitting place and the joint distribution of the first hitting place and time are examined. The statements of the main results with various preliminaries are now given and proofs follow in the later sections.

For $d = 2, 3$ let the state space of the d-dimensional random walk be the set R_d consisting of all ordered d-tuples with integer components. For any ordered d-tuples $\theta = \{\theta_i\}$ and $\theta' = \{\theta_i'\}$ with real components let

$$\theta.\theta' = \sum_{i=1}^{d} \theta_i \theta_i', \quad |\theta| = (\theta.\theta)^{\frac{1}{2}}, \qquad \theta_* = (\theta_1, \ldots, \theta_{d-1}) .$$

The d-dimensional d.p. random walk is denoted by a family of random variables $\{X(n)\}$, where n runs through the positive integers. In the customary notation the one-step transition probabilities $\{p_{xy}\}$ are assumed to satisfy

$$\sum_{y \in R_d} p_{xy} = 1 , \quad p_{xz} = p_{0,\, z-x} , \qquad x, z \in R_d .$$

The d-dimensional c.p. random walk is similarly denoted by a family of random variables $\{X(t)\}$, where t runs through the non-negative real numbers. The c.p. transition probabilities $\{p_{xy}(t)\}$ are uniquely specified in terms of the transition rates

$$q_{xy} = p_{xy}'(0) , \qquad x, y \in R_d ,$$

assuming that

$$q_{xy} \geqq 0 , \qquad x \neq y ,$$

$$\sum_{y \neq x} q_{xy} = - q_{xx} < \infty , \qquad q_{xy} = q_{0,\, y-x} .$$

For the two-dimensional processes the following conditions, (1d) and (2d) in the d.p. case, and analogously (1c) and (2c) in the c.p. case, are assumed.

(1d) For each x and y in R_2 there is an integer n such that the n-step transition probability p_{xy}^n is positive.

(2d) $$\sum_y p_{xy} y = 0 \quad \text{and} \quad \sum_y p_{xy} |y|^{2+\delta} < \infty$$

for some positive number δ.

(1c) For each x and y in R_2 there exists a finite sequence $x_1, \ldots, x_n$ of points in R_2 such that

$$x_1 = x, \quad x_n = y \quad \text{and} \quad q_{x_1 x_2} \cdots q_{x_{n-1} x_n} > 0 .$$

(2c) $$\sum_{y \neq 0} q_{0y} y = 0 \quad \text{and} \quad \sum_y q_{0y} |y|^{2+\delta} < \infty$$

for some positive number δ.

The transition probabilities $\{p_{xy}(t)\}$ of the two-dimensional c.p. process are obtained later in a form which shows that they are continuous functions of t and that $\lim_{t \to 0} p_{xy}(t) = \delta_{xy}$, where δ_{xy} is Kronecker's symbol. Under these conditions it is known that it is permissable to consider the process $X(t)$ to be Borel measurable and well separable (see CHUNG [1], II. 4). For either type of process it is shown later that the respective conditions (1) and (2) together imply that each lattice point is almost surely visited. Now let the random variable $T(x)$ denote the time at which either type of process first hits the origin starting from the non-zero lattice point x.

Theorem 1. $$P\{T(x) > u\} = \frac{[1 + h(x, u)]\, H(x)}{\log u}, \quad x \neq 0, u > 1,$$

where the functions H and h are defined later and satisfy

(i) $\lim_{u \to \infty} h(x, u) = 0$ *for each fixed non-zero x,*

(ii) $h(x, u) \to 0$ *as* $|x| \to \infty$ *uniformly for all* $u \geqq |x|^2$,

(iii) $[H(x) - 2 \log |x|]$ *is bounded for all non-zero x.*

Corollary. *If under the same hypotheses as Theorem 1, $T(x)$ denotes the first hitting time of any finite set of points in R_2,*

$$P\{T(x) > u\} = \frac{[1 + o(1)]\, 2 \log |x|}{\log u}, \quad u > 1,$$

where the term $o(1)$ tends to zero as $|x| \to \infty$ uniformly for all $u \geqq |x|^2$. In particular for each fixed real number α such that $\alpha \geqq 2$

$$P\{T(x) > |x|^\alpha\} \to 2\alpha^{-1}, \qquad as \quad |x| \to \infty .$$

The behaviour of $P\{T(x) > u\}$ when x is fixed has been investigated for a wider class of d.p. random walks by SPITZER [10], and a more precise result with x fixed is given in [10] for the simple symmetric d.p. random walk.

The method used in the main part of the proof of Theorem 1 provides the following analogous result for plane Brownian motion. Let T_r now denote the first hitting time of a disc in the plane of radius a when the Brownian motion process starts at a distance $r\,(> a)$ from the centre of the disc.

Theorem 2. $P\{T_r > u\} = \frac{[1 + h^*(r, u)]\, 2 \log(r/a)}{\log u}, \quad u > 1,$

where (i) $\lim_{u \to \infty} h^*(r, u) = 0$ *for each fixed* r,

(ii) $h^*(r, u) \to 0$ *as* $r \to \infty$ *uniformly for all* $u \geqq r^2$.

The behaviour of $P\{T_r > u\}$ when r is fixed has been obtained for a more general hitting set by Hunt [5].

For the three-dimensional processes the following conditions, (3d), (4d) and (5d) in the d.p. case, and (3c), (4c) and (5c) in the c.p. case, are assumed.

(3d) If $p_{0y} > 0$, one of y_* and y_3 is zero.

(4d) The two-dimensional process $X(n)_*$ satisfies (1d) and (2d).

(5d) $$\sum_{y \in R_3} p_{0y} y_3 = 0, \qquad \sum_{y \in R_3} p_{0y} (y_3)^2 = \sigma^2,$$

where σ^2 is finite and positive.

(3c) If $q_{0y} > 0$, one of y_* and y_3 is zero.

(4c) The random walk on R_2 defined by the transition rates $\{q'_{zw} : z, w \in R_2\}$ satisfies (1c) and (2c) when

$$q'_{zw} = q_{xy}, \qquad z \neq w,\ x = (z_1, z_2, 0),\ y = (w_1, w_2, 0),$$

$$q'_{zz} = -\sum_{w \neq z} q'_{zw}.$$

(5c) $$\sum_{y \in R_3} q_{0y} y_3 = 0, \qquad \sum_{y \in R_3} q_{0y} (y_3)^2 = \sigma^2,$$

where σ^2 is finite and positive.

It is shown later that such a c.p. process may be considered to be Borel measurable and well separable. It is also shown that both types of process almost surely hit the x_3 axis. Now for either type of process let the random variables T and Z denote respectively the time and coordinate of the first hit on the x_3 axis, assuming that the starting point is $(x_1, x_2, 0)$ with $r = (x_1^2 + x_2^2)^{\frac{1}{2}}$.

Lemma 1. *For each fixed real number* α

$$P\{Z \leqq \alpha \sigma \sqrt{T}\} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-\beta^2/2} d\beta, \qquad as \quad r \to \infty.$$

The following results are deduced from this connection between Z and T.

Theorem 3. *For each fixed real number* β *such that* $\beta \geqq 1$,

$$P\{|Z| > r^\beta\} \to \beta^{-1}, \qquad as \quad r \to \infty.$$

Theorem 3 was first proved directly by Doney [2] for the simple symmetric d.p. random walk. The next result and its proof were kindly communicated by Doney.

Theorem 4. *For each pair of fixed real numbers* α *and* β *such that* $\alpha \geqq 2$ *and* $\beta \geqq 1$

$$P\{T > r^\alpha; |Z| > r^\beta\} \to [\max(\alpha/2, \beta)]^{-1}, \qquad as \quad r \to \infty.$$

2. Proof of Theorem 1 (d. p. case)

Two lemmas are obtained before the main part of the proof. Firstly let

$$a_n(x) = P\{T(x) > n\}, \quad \text{and} \quad A(x, \zeta) = \sum_{n=0}^{\infty} a_n(x) \zeta^n, \quad x \in R_2,\ x \neq 0,\ 0 < \zeta < 1,$$

and for each ordered pair of real numbers $\theta = (\theta_1, \theta_2)$ let

$$\varphi(\theta) = \sum_{y \in R_2} p_{0y} e^{i\theta . y}.$$

Lemma 2.
$$A(x, \zeta) = \frac{\int \frac{1 - e^{i\theta . x}}{1 - \zeta\varphi(\theta)} d\theta}{(1-\zeta)\int \frac{d\theta}{1 - \zeta\varphi(\theta)}}, \quad x \neq 0, \ 0 < \zeta < 1,$$

where the integration is with respect to the components θ_1 and θ_2 of θ, and is taken over the square $|\theta_i| \leqq \pi$, $i = 1, 2$, from now on in this section unless otherwise stated.

Proof: Let $\{p_{xy}^n\}$ be the n-step transition probabilities on R_2, and let

$$f_{xy}^n = P\{X(\nu) \neq y, \ 0 < \nu < n, \ X(n) = y \mid X(0) = x\}, \qquad x, y \in R_2, \ n > 0,$$
$$f_{xy}^0 = 0, \qquad x, y \in R_2.$$

It is well known that

$$\sum_{n=0}^{\infty} f_{x0}^n \zeta^n = \sum_{n=0}^{\infty} p_{x0}^n \zeta^n / \sum_{r=0}^{\infty} p_{00}^r \zeta^r, \qquad x \in R_2, \ x \neq 0, \ 0 < \zeta < 1. \tag{6}$$

It is shown independently in Lemma 3 (15) that $\sum_{n=0}^{\infty} p_{00}^n = \infty$, and it is well known that this with condition (1d) implies that $\sum_{n=0}^{\infty} f_{x0}^n = 1$ for each x in R_2. Hence

$$\begin{aligned} A(x, \zeta) &= \sum_{n=0}^{\infty} a_n(x) \zeta^n \\ &= \sum_{n=0}^{\infty} \sum_{r=n+1}^{\infty} f_{x0}^r \zeta^n \\ &= \sum_{r=1}^{\infty} f_{x0}^r (1 - \zeta^r)/(1-\zeta), \end{aligned}$$

so that by (6)

$$A(x, \zeta) = \{\sum_{r=0}^{\infty} p_{00}^r \zeta^r - \sum_{n=0}^{\infty} p_{x0}^n \zeta^n\}/(1-\zeta) \sum_{r=0}^{\infty} p_{00}^r \zeta^r. \tag{7}$$

But $[\varphi(\theta)]^n = \sum_{y \in R_2} p_{0y}^n e^{i\theta . y}$ so that

$$p_{xy}^n = \frac{1}{4\pi^2} \int [\varphi(\theta)]^n e^{-i\theta . (y-x)} \, d\theta, \qquad x, y \in R_2.$$

Hence

$$\begin{aligned} \sum_{n=0}^{\infty} p_{xy}^n \zeta^n &= \sum_{n=0}^{\infty} \left\{\frac{1}{4\pi^2} \int [\zeta\varphi(\theta)]^n e^{-i\theta . (y-x)} d\theta\right\} \\ &= \frac{1}{4\pi^2} \int \left\{\sum_{n=0}^{\infty} [\zeta\varphi(\theta)]^n\right\} e^{-i\theta . (y-x)} d\theta \\ &= \frac{1}{4\pi^2} \int \frac{e^{-i\theta . (y-x)}}{1 - \zeta\varphi(\theta)} d\theta, \quad 0 < \zeta < 1, \end{aligned} \tag{8}$$

where the order of summation and integration may be interchanged since the function considered has modulus at most ζ^n and is therefore absolutely convergent under these operations. Lemma 2 is now completed on substituting the last formula in (7).

Lemma 3. *If* $x \neq 0$, $0 < \zeta < 1$ *and* $\eta = 1 - \zeta$

$$A(x, \zeta) = \sum_{n=0}^{\infty} P\{T(x) > n\}\, \zeta^n = \frac{H(x) - \log(1 + \eta |x|^2) + g(x, \eta)}{\eta[-\log \eta + O(1)]},$$

where H and g are defined later and satisfy

(i) $[H(x) - 2\log|x|]$ *is bounded;*

(ii) $g(x, \eta)$ *is bounded for all x and η, and* $\lim_{\eta \to 0+} g(x, \eta) = 0$ *for each fixed x;*

(iii) *the term $O(1)$ is bounded for all x and η.*

Proof: Following a method used by Spitzer [9] it can be shown that

$$\varphi(\theta) = 1 - \tfrac{1}{2} Q(\theta) + O(|\theta|^{2+\delta}), \qquad \text{as} \quad |\theta| \to 0, \tag{9}$$

where $Q(\theta)$ is the positive definite quadratic form $\sum_y p_{0y}(\theta \,.\, y)^2$ and δ is chosen as in (2d), assuming without loss of generality that $\delta < 1$. $\frac{1}{2} Q(\theta)$ may be reduced to the form r^2 by making a suitable transformation in the θ plane, say

$$\begin{aligned} \theta_1 &= r[\gamma_1 \cos\psi_0 \cos\psi + \gamma_2 \sin\psi_0 \sin\psi] \\ \theta_2 &= r[-\gamma_1 \sin\psi_0 \cos\psi + \gamma_2 \cos\psi_0 \sin\psi], \end{aligned} \tag{10}$$

where r and ψ are polar coordinates, and γ_1, γ_2 and ψ_0 are constants with γ_1 and γ_2 both positive. Hence from (9)

$$\varphi(\theta) = 1 - r^2 + O(r^{2+\delta}), \tag{11}$$

$$\frac{1}{1 - \zeta\varphi(\theta)} = \frac{1}{\eta + r^2} + O(r^{\delta - 2}), \tag{12}$$

for all sufficiently small positive r and all η in $(0, 1)$. Also it is easily shown that

$$\theta \,.\, x = cr|x| \cos(\psi - \psi_1), \tag{13}$$

where c and ψ_1 depend on x but not on r and ψ, and c lies between γ_1 and γ_2.

Consider now the denominator and the numerator of the expression for $A(x, \zeta)$ obtained in Lemma 2, assuming from now on that $x \neq 0$ and $0 < \zeta < 1$. Let the positive constant r_0 be chosen such that (11) and (12) hold for all r in $(0, r_0]$ and such that the region for which $r \leqq r_0$ is contained in the square $|\theta_i| \leqq \pi$, $i = 1, 2$. It is known that condition (1d) implies that $\varphi(\theta) = 1$ only if both θ_1 and θ_2 are multiples of 2π (see Spitzer [10], T. 7.1), so that since φ is continuous

$$|\varphi(\theta)| \leqq \text{constant} < 1, \quad \text{if} \quad |\theta_i| \leqq \pi,\ i = 1, 2 \quad \text{and} \quad r > r_0.$$

Hence on dividing the range of integration and using (12)

$$\begin{aligned} \int \frac{d\theta}{1 - \zeta\varphi(\theta)} &= \int_{r=0}^{r_0} \int_{\psi=0}^{2\pi} [(\eta + r^2)^{-1} + O(r^{\delta-2})]\, r\gamma_1\gamma_2\, dr\, d\psi + O(1) \\ &= \pi\gamma_1\gamma_2[-\log\eta + O(1)], \end{aligned} \tag{14}$$

where $r\gamma_1\gamma_2$ is the Jacobian of the transformation (10), and from now on terms

$O(1)$ are assumed to be bounded independently of x and ζ (or η). Hence from (14)

$$(15)\qquad \sum_{n=0}^{\infty} p_{00}^{n} = \lim_{\zeta\to 1-} \sum_{n=0}^{\infty} p_{00}^{n}\zeta^{n} = \lim_{\zeta\to 1-} \int \frac{d\theta}{1-\zeta\varphi(\theta)} = \infty\,,$$

which is the condition required in Lemma 2. By using (12) again

$$(16)\qquad \int \frac{1-e^{i\theta.x}}{1-\zeta\varphi(\theta)}\,d\theta = \int_{r=0}^{r_0}\int_{\psi=0}^{2\pi} (1-e^{i\theta.x})(\eta+r^2)^{-1} r\gamma_1\gamma_2\,dr\,d\psi + O(1)\,.$$

But from (13)

$$\begin{aligned}\int_{\psi=0}^{2\pi}(1-e^{i\theta.x})\,d\psi &= 2\pi - \int_{\psi=0}^{2\pi}\exp[icr\,|x|\cos(\psi-\psi_1)]\,d\psi\\ &= 2\pi - 4\int_{\psi=0}^{\pi/2}\cos(cr\,|x|\cos\psi)\,d\psi\\ &= 2\pi[1-J_0(cr\,|x|)]\,,\end{aligned}$$

where J_0 is a Bessel function of the first kind (see WATSON [*11*]). Now since

$$J_0(u) = 1 + O(u^2),\quad \text{as}\quad u\to 0+,\quad\text{and}\quad J_0(u) = O(u^{-\frac{1}{2}}),\quad\text{as}\quad u\to+\infty\,,$$

and since c lies between γ_1 and γ_2,

$$\begin{aligned}(17)\qquad &\int_{r=0}^{r_0}\int_{\psi=0}^{2\pi}(1-e^{i\theta.x})(\eta+r^2)^{-1} r\,dr\,d\psi\\ &= 2\pi\int_{r=0}^{r_0}[1-J_0(cr\,|x|)](\eta+r^2)^{-1} r\,dr\\ &= 2\pi\int_{u=0}^{cr_0|x|}[1-J_0(u)](u^2+c^2\eta\,|x|^2)^{-1}u\,du\\ &= \pi[2\log|x| - \log(1+\eta\,|x|^2) + O(1)]\,.\end{aligned}$$

Let $H(x)$ be defined by

$$(18)\qquad \pi\gamma_1\gamma_2 H(x) = \lim_{\zeta\to 1-}\int\frac{1-e^{i\theta.x}}{1-\zeta\varphi(\theta)}\,d\theta\,,$$

where the limit exists by virtue of (11). Then if $g(x,\eta)$ is defined by

$$(19)\qquad \int\frac{1-e^{i\theta.x}}{1-\zeta\varphi(\theta)}\,d\theta = \pi\gamma_1\gamma_2[H(x) - \log(1+\eta\,|x|^2) + g(x,\eta)]\,,$$

the required properties of H and g follow from (16), (17) and (18). The lemma is completed on combining (14) with (19).

Main Proof: In the Tauberian theorems 98 and 100 in HARDY [*4*] a method due to KARAMATA is used to obtain the behaviour under certain conditions of a function $B(t)$, say, as $t\to+\infty$ from the behaviour of its Laplace-Stieltjes transform $\int_0^\infty e^{-\lambda t} d_t B(t)$ as $\lambda\to 0+$. Now let

$$B(x,t) = \sum_{n\leqq t} a_n(x)\,,\qquad t\geqq 0,\ x\in R_2,\ x\neq 0\,.$$

By using a modification of the method of KARAMATA the behaviour of $B(x, t)$ as $|x|$ and $t \to +\infty$ is now obtained from the behaviour of its Laplace-Stieltjes transform

$$A(x, e^{-\lambda}) = \int_0^\infty e^{-\lambda t} d_t B(x, t), \qquad \lambda > 0,$$

as $|x| \to \infty$ and $\lambda \to 0+$. The behaviour of $a_n(x)$ then follows from the monotonicity of $B(x, t)$ as t varies.

It is known that if g is a given real function which is Riemann integrable on $(0, 1)$ and ε is a given positive number, there exist polynomials

$$p(u) = \sum_{s=0}^{j} p_s u^s \quad \text{and} \quad q(u) = \sum_{s=0}^{k} q_s u^s$$

such that

$$p < g < q \quad \text{and} \quad \int_0^\infty e^{-t}[q(e^{-t}) - p(e^{-t})]\, dt < \varepsilon. \tag{20}$$

Hence, since $B(x, t)$ increases as t increases,

$$\begin{aligned} \int_0^\infty e^{-\lambda t} g(e^{-\lambda t})\, d_t B(x,t) &\geqq \int_0^\infty e^{-\lambda t} p(e^{-\lambda t})\, d_t B(x, t) \\ &= \sum_{s=0}^{j} p_s \int_0^\infty e^{-(s+1)\lambda t} d_t B(x, t) \\ &= \sum_{s=0}^{j} p_s A\{x, e^{-(s+1)\lambda}\}. \end{aligned} \tag{21}$$

But by Lemma 3 for $0 \leqq s \leqq j$

$$A\{x, e^{-(s+1)\lambda}\} = \frac{H(x)[1 + h_s(x, \lambda)]}{-(s+1)\lambda \log \lambda}, \qquad 0 < \lambda < 1, \tag{22}$$

where $\lim_{\lambda \to 0+} h_s(x, \lambda) = 0$ for each fixed x, and $h_s(x, \lambda) \to 0$ as $|x| \to \infty$ uniformly for λ satisfying $\lambda |x|^2 \leqq 1$. Now let $g(u) = u^{-1}$ when $e^{-1} \leqq u \leqq 1$ and 0 otherwise, and let polynomials $p(u)$ and $q(u)$ be chosen as in (20). Then

$$\begin{aligned} \sum_{s=0}^{j} p_s (s+1)^{-1} &= \int_0^\infty e^{-t} \sum_{s=0}^{j} p_s e^{-st}\, dt \\ &= \int_0^\infty e^{-t} p(e^{-t})\, dt \\ &\geqq 1 - \varepsilon. \end{aligned}$$

Since $B(x, 0) = 0$, it follows from (21) and (22) that if $0 < \lambda < 1$ and $\lambda |x|^2 \leqq 1$,

$$\begin{aligned} B(x, \lambda^{-1}) &= \int_0^\infty e^{-\lambda t} g(e^{-\lambda t})\, d_t B(x, t) \\ &\geqq \sum_{s=0}^{j} \frac{p_s H(x)[1 + h_s(x, \lambda)]}{-(s+1)\lambda \log \lambda} \\ &\geqq \frac{H(x)}{-\lambda \log \lambda} \left[1 - \varepsilon + \sum_{s=0}^{j} \frac{p_s h_s(x, \lambda)}{(s+1)}\right]. \end{aligned}$$

Hence on replacing λ^{-1} by t it follows that

$$B(x, t) \geqq \frac{(1 - 2\varepsilon)\, t H(x)}{\log t}$$

for (i) all large enough t when x is fixed, and (ii) all large enough $|x|$ provided that $t \geqq |x|^2$. By making a similar calculation based on the inequality $g < q$ in (20) it follows that

$$B(x, t) \leqq \frac{(1 + 2\varepsilon)\, t H(x)}{\log t}$$

under either of the last conditions (i) and (ii). Since $a_n(x) = P\{T(x) > n\}$ decreases as n increases

$$\frac{B(x, n)}{n} \geqq a_n(x) \geqq \frac{B(x, n^{1+\delta}) - B(x, n)}{n^{1+\delta}}$$

for each positive number δ. Theorem 1 now follows in the d. p. case on applying the above estimates for $B(x, t)$ to the last inequality and noting that ε and δ may be chosen arbitrarily small.

3. Proof of Theorem 1 (c. p. case)

It is shown the theorem may be proved in this case by methods similar to those used in the d. p. case.

Since the transition rates q_{xy} are bounded, it follows from the theory of semigroups of linear operators that if $P(t) = \{p_{xy}(t)\}$ and $Q = \{q_{xy}\}$, $P(t)$ is determined uniquely and $P(t) = e^{tQ}$. Hence if

$$\Phi(\theta) = \sum_x q_{0x}\, e^{i\theta . x},$$

$$p_{xy}(t) = \frac{1}{4\pi^2} \int \exp[t\Phi(\theta) - i\,\theta.(y - x)]\, d\theta, \qquad x, y \in R_2,$$

where the integration is with respect to the components θ_1 and θ_2 of θ and is taken over the square $|\theta_i| \leqq \pi$, $i = 1, 2$, from now on in this section unless otherwise stated. Then since the real part of $\Phi(\theta)$ is not positive

$$\int_0^\infty e^{-\lambda t} p_{xy}(t)\, dt = \frac{1}{4\pi^2} \int \frac{e^{-i\theta.(y-x)}}{\lambda - \Phi(\theta)}\, d\theta, \qquad \lambda > 0,\ x, y \in R_2. \tag{23}$$

Now consider a d. p. random walk defined on R_2 by the one-step transition probabilities

$$p'_{xy} = q_{xy}(1 - \delta_{xy})(-q_{xx})^{-1}, \qquad x, y \in R_2.$$

(1c) implies that (1d) holds for this new process and it follows that

$$\sum_y p'_{0y}\, e^{i\theta . y} = 1$$

only if both θ_1 and θ_2 are multiples of 2π (see Spitzer [*10*], T. 7.1). Hence $\Phi(\theta) = 0$ only if both θ_1 and θ_2 are multiples of 2π. By using methods similar to those which obtained (15) in Lemma 3 it follows from (2c) and (23) that

$$\int_0^\infty p_{00}(t)\,dt = \lim_{\lambda\to 0+} \int_0^\infty e^{-\lambda t} p_{00}(t)\,dt = \infty .$$

Also by (1c), for each x and y in R_2 there exists t such that $p_{xy}(t) > 0$. These last two facts imply that the process is recurrent and that $P\{T(x) < \infty\} = 1$ for each x in R_2 (see CHUNG [1], II. 10). Now let

$$\hat{p}_{x0}(\lambda) = \int_0^\infty e^{-\lambda t} p_{x0}(t)\,dt, \qquad \lambda > 0, \quad x \in R_2,$$

$$F_{x0}(t) = P\{T(x) \leqq t\}, \quad \hat{F}_{x0}(\lambda) = \int_0^\infty e^{-\lambda t} d_t F_{x0}(t), \qquad \lambda > 0, \quad x \in R_2, \quad x \neq 0 .$$

It is known (see CHUNG [1], II (12.4)) that

$$\hat{F}_{x0}(\lambda) = \hat{p}_{x0}(\lambda)/\hat{p}_{00}(\lambda)$$

so that

$$\int_0^\infty e^{-\lambda t} P\{T(x) > t\}\,dt = \frac{\hat{p}_{00}(\lambda) - \hat{p}_{x0}(\lambda)}{\lambda\,\hat{p}_{00}(\lambda)}, \qquad x \neq 0 .$$

By using (23) and (2c) the proof proceeds from this point by methods similar to those used for Lemma 3 and the main proof in the d.p. case.

4. Proof of Corollary of Theorem 1

In this section let $X(t)$ denote the position at time t of either a discrete or a continuous parameter process, and for convenience put

$$P_x\{A\} = P\{A \mid X(0) = x\}$$

for any x in R_2 and any event A. Let L be any fixed finite set of points in R_2 and select any point y in L. Then by Theorem 1, if $x \notin L$

$$\begin{aligned} (24) \qquad P_x\{X(t) \notin L \text{ for all } t \text{ in } (0,u]\} &\leqq P_x\{X(t) \neq y \text{ for all } t \text{ in } (0,u]\} \\ &= \frac{[1+o(1)]\,2\log|x|}{\log u}, \qquad u > 1, \end{aligned}$$

where the term $o(1)$ tends to zero as $|x| \to \infty$ uniformly for all u such that $u \geqq |x|^2$. Conversely, by applying the strong Markov property (see CHUNG [1], I. 13 and II. 9), if y_0 is some fixed point not in L and $u > 1$

$$\begin{aligned} (25) \quad & P_x\{X(s) = y_0 \text{ for some } s \text{ in } (0, 2u]\} \\ & \geqq P_x\{\text{For some } t \text{ in } (0,u],\ X(\nu) \notin L \text{ for all } \nu \text{ in } (0,t) \text{ and } X(t) \in L; \\ & \qquad\qquad X(t+s) = y_0 \text{ for some } s \text{ in } (0,u]\} \\ & = \sum_{y\in L} P_x\{\text{For some } t \text{ in } (0,u],\ X(\nu) \notin L \text{ for all } \nu \text{ in } (0,t) \text{ and } X(t) = y; \\ & \qquad\qquad X(t+s) = y_0 \text{ for some } s \text{ in } (0,u]\} \\ & = \sum_{y\in L} P_x\{\text{For some } t \text{ in } (0,u],\ X(\nu) \notin L \text{ for all } \nu \text{ in } (0,t) \text{ and } X(t) = y\} \times \\ & \qquad\qquad \times P_y\{X(s) = y_0 \text{ for some } s \text{ in } (0,u]\} . \end{aligned}$$

But by Theorem 1

$$P_y\{X(s) = y_0 \text{ for some } s \text{ in } (0, u]\} = 1 + O[(\log u)^{-1}], \qquad \text{as } u \to \infty,$$

so that from (25)

$$\begin{aligned} &P_x\{X(s) = y_0 \text{ for some } s \text{ in } (0, 2u]\} \\ &\qquad \geqq P_x\{X(t) \in L \text{ for some } t \text{ in } (0, u]\} + O[(\log u)^{-1}], \qquad u > 1, \end{aligned}$$

where the terms $O[.]$ are independent of x. Hence by applying Theorem 1 again

$$\begin{aligned} (26) \qquad &P_x\{X(t) \notin L \text{ for all } t \text{ in } (0, u]\} \\ &\qquad \geqq P_x\{X(s) \neq y_0 \text{ for all } s \text{ in } (0, 2u]\} + O[(\log u)^{-1}] \\ &\qquad = \frac{[1 + o(1)]\, 2 \log |x|}{\log u}, \qquad u > 1, \end{aligned}$$

where the term $o(1)$ tends to zero as $|x| \to \infty$ uniformly for all u such that $u \geqq |x|^2$. The corollary follows on combining (24) with (26).

5. Proof of Theorem 2

For the Brownian motion process in the plane let T_r denote the first entrance time of a disc of radius a when the starting point of the process is at a distance $r (> a)$ from the centre of the disc. In [8] Spitzer showed that

$$\int_0^\infty e^{-\lambda t} P\{T_r \leqq t\}\, dt = \frac{K_0(r\sqrt{2\lambda})}{\lambda K_0(a\sqrt{2\lambda})}, \qquad \lambda > 1,$$

where K_0 is a modified Bessel function of the second kind. Since

$$K_0(u) = -\log u + \text{constant} + O(u), \qquad \text{as } u \to 0+,$$

it is then easily shown that

$$\begin{aligned} \int_0^\infty e^{-\lambda t} P\{T_r > t\}\, dt &= \frac{K_0(a\sqrt{2\lambda}) - K_0(r\sqrt{2\lambda})}{\lambda K_0(a\sqrt{2\lambda})} \\ &= \frac{\log(r/a) + O(r\sqrt{\lambda})}{\lambda[-\log \lambda + O(1)]}, \qquad \text{if } r\sqrt{\lambda} \leqq \text{constant}. \end{aligned}$$

Theorem 2 now follows by applying the same method as used in the main proof for the discrete parameter random walk.

6. Proof of Lemma 1 (d. p. case)

Let the process be initially at $(x_1, x_2, 0)$ with $r = (x_1^2 + x_2^2)^{1/2}$. Of the random number T of steps taken before the x_3 axis is first hit, let H be the total number of zero steps and steps parallel with the $x_1 x_2$ plane, and let V be the total number of non-zero steps parallel with the x_3 axis. Thus from (3d) $T = H + V$. Also let the random variable V_n be the total number of the first n steps which are non-zero and parallel with the x_3 axis.

By (3d) the one-step transition probabilities $\{p_{xy} : x, y \in R_3\}$ may be denoted by (i) p^*_{ab} for zero steps and steps parallel with the $x_1 x_2$ plane, where the suffices of p^*_{ab} are in R_2, and (ii) $\bar{p}_{wz}$ for non-zero steps parallel with the x_3 axis, where the suffices of $\bar{p}_{wz}$ are integers. Then by (3d) and (5d)

$$\sum_z \bar{p}_{0z} z = 0\,, \quad \sigma^2 = \sum_z \bar{p}_{0z} z^2 < \infty \quad \text{and} \quad 0 < \varrho < 1\,, \tag{27}$$

where $\varrho = \sum_z \bar{p}_{0z}$. Also if the process is initially at $(x_1, x_2, 0)$ and $a = (x_1, x_2)$

$$P\{H = h;\, Z = z;\, V = v\} \tag{28}$$

$$= \sum_{a_i \neq 0} \sum_{z_j} \binom{h + v - 1}{v} p^*_{a a_1} p^*_{a_1 a_2} \cdots p^*_{a_{h-1} 0}\, \bar{p}_{0 z_1} \bar{p}_{z_1 z_2} \cdots \bar{p}_{z_{v-1} z}\,.$$

Let ϱ_1 and ϱ_2 be chosen such that $0 < \varrho_1 < \varrho < \varrho_2 < 1$. Then for each fixed real number α

$$\begin{aligned} P\{Z \leqq \alpha\sigma\sqrt{T}\} &\leqq P\{Z \leqq \alpha\sigma\sqrt{T};\, V > \varrho_1 T\} + P\{V \leqq \varrho_1 T\} \\ &\leqq P\{Z \leqq \alpha\sigma\sqrt{V/\varrho_1};\, V > \varrho_1 T\} + P\{V \leqq \varrho_1 T\} \\ &\leqq P\{Z \leqq \alpha\sigma\sqrt{V/\varrho_1}\} + P\{V \leqq \varrho_1 T\}, \end{aligned} \tag{29}$$

$$\begin{aligned} P\{Z \leqq \alpha\sigma\sqrt{T}\} &\geqq P\{Z \leqq \alpha\sigma\sqrt{T};\, V \leqq \varrho_2 T\} \\ &\geqq P\{Z \leqq \alpha\sigma\sqrt{V/\varrho_2};\, V \leqq \varrho_2 T\} \\ &\geqq P\{Z \leqq \alpha\sigma\sqrt{V/\varrho_2}\} - P\{V > \varrho_2 T\}. \end{aligned} \tag{30}$$

Consider firstly the inequality

$$P\{V \leqq \varrho_1 T\} \leqq P\{V \leqq \varrho_1 T;\, T > r^2\} + P\{T \leqq r^2\}\,. \tag{31}$$

By observing only the process $X(n)_*$ it follows from (4d) and Theorem 1 that

$$P\{T < \infty\} = 1\,, \quad \lim_{r \to \infty} P\{T \leqq r^2\} = 0\,. \tag{32}$$

Also

$$\{V \leqq \varrho_1 T;\, T > r^2\} \subseteqq \bigcup_{n > r^2} \{V_n \leqq \varrho_1 n\}\,,$$

and the probability of the latter event tends to zero as $r \to \infty$ by the strong law of large numbers, since in fact $V = V_T$ and V_n is the sum of n independent identically distributed random variables each having expectation ϱ. Thus

$$P\{V \leqq \varrho_1 T;\, T > r^2\} \to 0\,, \qquad \text{as } r \to \infty\,.$$

Hence from (31) and (32)

$$P\{V \leqq \varrho_1 T\} \to 0\,, \qquad \text{as } r \to \infty\,. \tag{33}$$

Similarly it can be shown that

$$P\{V > \varrho_2 T\} \to 0\,, \qquad \text{as } r \to \infty\,. \tag{34}$$

From (28) it is easily shown that

$$\begin{aligned} P\{Z = z \mid V = v\} &= \sum_h P\{H = h;\, Z = z;\, V = v\} \Big/ \sum_{k,w} P\{H = k;\, Z = w;\, V = v\} \\ &= \sum_{z_i} (\bar{p}_{0 z_1}/\varrho) \cdots (\bar{p}_{z_{v-1} z}/\varrho)\,, \end{aligned}$$

where the terms $(\tilde{p}_{wz}/\varrho)$ are the conditional one-step transition probabilities of non-zero steps parallel with the x_3 axis. It follows from (27) that the conditional distribution of Z given that $V = v$ is the same as the distribution of the sum of v independent identically distributed random variables, each having zero expectation and finite variance σ^2/ϱ. Hence by the central limit theorem (see GNEDENKO and KOLMOGOROV [3], § 35, Th. 4)

$$P\{Z \leqq \alpha' \sigma\sqrt{V/\varrho} \mid V = v\} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha'} e^{-\beta^2/2} d\beta, \qquad \text{as } v \to \infty,$$

for each fixed real number α'. From (32) and (33) it follows that $P\{V < \infty\} = 1$, and $V \to \infty$ as $r \to \infty$ in probability. Hence for each fixed real number α'

$$P\{Z \leqq \alpha' \sigma\sqrt{V/\varrho}\} = \sum_v P\{Z \leqq \alpha' \sigma\sqrt{V/\varrho} \mid V = v\} P\{V = v\}$$

$$\to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha'} e^{-\beta^2/2} d\beta, \qquad \text{as } r \to \infty.$$

The lemma is now completed in the d.p. case by applying the last limit, (33) and (34) to (29) and (30), and noting that ϱ_1 and ϱ_2 may be chosen arbitrarily close to ϱ.

7. Proof of Lemma 1 (c. p. case)

For any ordered triple of real numbers $\theta = (\theta_1, \theta_2, \theta_3)$ let

$$\Psi(\theta) = \sum_{y \in R_3} q_{0y} e^{i\theta . y}.$$

Then by the same argument as used in the two-dimensional case the transition probabilities $\{p_{xy}(t)\}$ are given by

$$(35) \qquad p_{xy}(t) = \frac{1}{8\pi^3} \int \exp[t\Psi(\theta) - i\theta . (y - x)]\, d\theta \qquad x, y \in R_3,\ t \geqq 0,$$

where the integration is with respect to the components of θ and is taken over the cube $|\theta_i| \leqq \pi$, $i = 1, 2, 3$. By (3c)

$$(36) \qquad \Psi(\theta) = h(\theta_*) + v(\theta_3),$$

where $h(\theta_*)$ and $v(\theta_3)$ are respectively functions of θ_* only and θ_3 only, and are defined by

$$(37) \qquad \begin{aligned} h(\theta_*) &= \sum_{y_* \neq 0} q_{0y}(e^{i\theta . y} - 1), \\ v(\theta_3) &= \sum_{y_3 \neq 0} q_{0y}(e^{i\theta . y} - 1). \end{aligned}$$

Hence from (35) and (36) $p_{xy}(t)$ can be expressed as the product of two transition probabilities, i.e.

$$(38) \qquad p_{xy}(t) = {}^h p_{xy}(t) \times {}^v p_{xy}(t)$$

where

$$(39) \qquad \begin{aligned} {}^h p_{xy}(t) &= \frac{1}{4\pi^2} \int \exp[t h(\theta_*) - i\theta_* . (y_* - x_*)]\, d\theta_*, \\ {}^v p_{xy}(t) &= \frac{1}{2\pi} \int \exp[t v(\theta_3) - i\theta_3 . (y_3 - x_3)]\, d\theta_3, \end{aligned}$$

the integration being firstly with respect to θ_1 and θ_2, taken over $|\theta_i| \leqq \pi$, $i = 1, 2$, and secondly taken over $|\theta_3| \leqq \pi$. Thus ${}^h p_{xy}(t)$ depends on x_* and y_* only and ${}^v p_{xy}(t)$ depends on x_3 and y_3 only. Let these transition probabilities specify processes $(X(t)_1, X(t)_2)$ on R_2 and $X(t)_3$ on the integers, where as in the earlier two-dimensional c. p. case it may be assumed that these processes are each Borel measurable and well separable. From (38) it follows that the two processes are independent, and the proof of the lemma proceeds from this fact.

A continuous parameter central limit theorem is now applied to $X(t)_3$. From (39), and by the uniqueness of the characteristic function,

$$E\{\exp[i\,u\,X(t)_3] \,|\, X(0)_3 = 0\} = e^{tv(u)}$$

for each real number u. Furthermore from (5c) and (37), and by using say Theorem 2.3.3 in Lukacs [7], it can easily be shown that

$$v(u) = -\tfrac{1}{2}\sigma^2 u^2 + o(u^2), \qquad \text{as } u \to 0.$$

Hence for each fixed real number u

$$E\{\exp[i\,u\,X(t)_3/\sigma\sqrt{t}] \,|\, X(0)_3 = 0\} = \exp[t\,v(u/\sigma\sqrt{t})] \to e^{-u^2/2}, \qquad \text{as } t \to \infty,$$

where the limit is the characteristic function of the normal distribution with zero expectation and unit variance. It follows from the continuity theorem for characteristic functions (e. g. see Lukacs [7], Th. 3.6.1, where the method of proof holds for the c. p. case) that for each fixed real number α

$$(40) \qquad P\{X(t)_3 \leqq \alpha\,\sigma\sqrt{t} \,|\, X(0)_3 = 0\} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-\beta^2/2}\, d\beta, \qquad \text{as } t \to \infty.$$

Let the process $X(t)$ be initially at $(x_1, x_2, 0)$ and let $r = (x_1^2 + x_2^2)^{1/2}$. Suppose that Ω_h and Ω_v are the respective spaces of events generated by the independent random variables $(X(t)_1, X(t)_2)$ and $X(t)_3$. Let T be the time at which $(X(t)_1, X(t)_2)$ is first zero, and let $Z = X(T)_3$. Suppose that α is any fixed real number and consider the event

$$(41) \qquad \{Z \leqq \alpha\,\sigma\sqrt{T}\}$$

in the product space $\Omega_h \times \Omega_v$. The left hand side of (40) is the probability measure of the section of the event (41) at any 'elementary event' in the space Ω_h for which $T = t$. Also from (4c), and by applying Theorem 1 to the process $(X(t)_1, X(t)_2)$, it follows that $P\{T < \infty\} = 1$, and $T \to \infty$ as $r \to \infty$ in probability. Hence from (40) and the product measure theorem (e. g. see Loève [6])

$$P\{Z \leqq \alpha\,\sigma\sqrt{T}\} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-\beta^2/2}\, d\beta, \qquad \text{as } r \to \infty.$$

8. Proofs of Theorems 3 and 4 (d. p. and c. p. cases)

These theorems can now be proved as consequences of Theorem 1 and Lemma 1. Let Z and T be the random variables defined in Lemma 1, supposing that the process is initially at $(x_1, x_2, 0)$ with $r = (x_1^2 + x_2^2)^{1/2}$. Then for each positive

ε and λ

$$\begin{aligned}P\{|Z| > u\} &\leqq P\{|Z| > u;\ T \leqq u^2\lambda^{-2}\} + P\{T > u^2\lambda^{-2}\}\\ &\leqq P\{|Z| > \lambda\sqrt{T};\ T \leqq u^2\lambda^{-2}\} + P\{T > u^2\lambda^{-2}\}\\ &\leqq P\{|Z| > \lambda\sqrt{T}\} + P\{T > u^2\lambda^{-2}\},\\ P\{|Z| > u\} &\geqq P\{|Z| > u;\ T > u^2\varepsilon^{-2}\}\\ &\geqq P\{|Z| > \varepsilon\sqrt{T};\ T > u^2\varepsilon^{-2}\}\\ &\geqq P\{T > u^2\varepsilon^{-2}\} - P\{|Z| \leqq \varepsilon\sqrt{T}\}.\end{aligned}$$

It follows from conditions (4) and Theorem 1, and by observing only the x_1 and x_2 components of the process, that the terms $P\{T > u^2\lambda^{-2}\}$ and $P\{T > u^2\varepsilon^{-2}\}$ are of the form

$$\frac{[1 + o(1)]\log r}{\log u}, \qquad u > 1,$$

where the term $o(1)$ depends respectively on λ and ε, and tends to zero as $r \to \infty$ uniformly for all u satisfying $u \geqq r$. According to Lemma 1 the terms

$$\lim_{r\to\infty} P\{|Z| > \lambda\sqrt{T}\}, \quad \lim_{r\to\infty} P\{|Z| \leqq \varepsilon\sqrt{T}\}$$

can be made arbitrarily small by making suitable choices of λ and ε. Hence

$$P\{|Z| > u\} = \frac{[1 + o(1)]\log r}{\log u} + o(1), \qquad u > 1,$$

where the terms $o(1)$ tend to zero as $r \to \infty$ uniformly for all u satisfying $u \geqq r$. Theorem 3 follows immediately.

To prove Theorem 4 suppose firstly that $\alpha > 2\beta \geqq 2$. Then by Theorem 1

$$\begin{aligned}P\{T > r^\alpha;\ |Z| > r^\beta\} &= P\{T > r^\alpha\} - P\{T > r^\alpha;\ |Z| \leqq r^\beta\}\\ &\to 2\alpha^{-1}, \qquad \text{as } r \to \infty,\end{aligned}$$

since by Lemma 1

$$P\{T > r^\alpha;\ |Z| \leqq r^\beta\} \leqq P\{|Z| \leqq r^{\beta-\alpha/2}\sqrt{T}\} \to 0, \qquad \text{as } r \to \infty.$$

Similarly if $2\beta > \alpha \geqq 2$, by Theorem 3

$$\begin{aligned}P\{T > r^\alpha;\ |Z| > r^\beta\} &= P\{|Z| > r^\beta\} - P\{T \leqq r^\alpha;\ |Z| > r^\beta\}\\ &\to \beta^{-1}, \qquad \text{as } r \to \infty,\end{aligned}$$

since by Lemma 1

$$P\{T \leqq r^\alpha;\ |Z| > r^\beta\} \leqq P\{|Z| > r^{\beta-\alpha/2}\sqrt{T}\} \to 0, \qquad \text{as } r \to \infty.$$

This completes the proof of Theorem 4.

Acknowledgements. The author is grateful to Professor G. E. H. Reuter for suggesting this topic and giving much helpful advice. The author is also grateful to Dr. R. A. Doney for discussion of some three-dimensional aspects of this work and for allowing the Theorem 4 to be given here. Thanks are also due to the Department of Scientific and Industrial Research for their financial support.

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