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Strassen's Marginal Problem in Two or More Dimensions

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Given a subset S of the product of two probability spaces $(X, \mathscr{B}(X), P_1)$ and $(Y, \mathscr{B}(Y), P_2)$, Strassen (1965) asked when there was a law P concentrated on S having P_1 and P_2 as marginals; he obtained results for the case of closed subsets of the product of Polish spaces using Choquet's theory of capacities; a proof based on a game-theoretic argument has been given by Hansel and Troallic (1978). We offer a new argument extending this theorem to a large class of measurable spaces (Theorem 1) and then generalize it beyond the context of products (Theorem 2). Next, the case of more than two factor spaces is taken up in Theorem 3 and finally that of a countable collection of such spaces in Theorem 4.

Hansel and Troallic (1978) also proved a result somewhat "inverse" to Strassen's Theorem (Corollaire 3), and it is this proposition that we generalize to several and finally countably many spaces in Theorems 5 and 6.

The fundamental structures underlying most of what follows are what we shall term *separable spaces*; by this we mean measurable spaces $(X, \mathscr{B})$ with $\mathscr{B}$ countably generated and countaining singletons. We shall often suppress the notation of a σ-algebra, calling the space X alone and indicating its measurable structure with $\mathscr{B} = \mathscr{B}(X)$. If A is a subset of a measurable space $(X, \mathscr{B})$, we shall always consider A as a measurable space with relative σ-algebra $\mathscr{B}(A) = \{A \cap B : B \in \mathscr{B}\}$; under this convention, subsets of separable spaces are again separable. Also, a product of countably many separable spaces is again separable when endowed with the product σ-algebra.

If X is a separable metric space with Borel σ-algebra $\mathscr{B}$, then $(X, \mathscr{B})$ is a separable space. Furthermore, there is a well-known technique due to Marczewski (1938) by which one may introduce metrics on separable spaces compatible with the measurable structure:

Lemma 1. *If $(X, \mathscr{B})$ is a separable space and $\mathscr{C}$ is a countable subset of $\mathscr{B}$, there is a metric d on X such that:*

1. *(X, d) is a totally disconnected metric space with compact completion ((X, d) is totally bounded and hence separable),*

2. *$\mathscr{B}$ is the Borel σ-algebra for (X,d); we say d is a metric for $(X,\mathscr{B})$ whenever this happens,*

3. *the elements of $\mathscr{C}$ are "clopen" (both closed and open) in (X,d), and*

4. *if $\mathscr{C}$ generates $\mathscr{B}$, $\mathscr{C}$ is a base of clopen sets for the topology of (X,d).*

Let J be a countable, non-empty index set, let X_j, $j\in J$, be separable spaces, and put $\mathscr{P}=\Pi\{X_j\colon j\in J\}$. A subset S of $\mathscr{P}$ is *Borel-closed* in $\mathscr{P}$ if it is the complement of a countable union of measurable rectangles, each depending on finitely many coordinates. Lemma 1 and the Lindelöf property for separable metric spaces imply that a subset S of $\mathscr{P}$ is Borel-closed if and only if it is possible to choose metrics d_j for X_j, $j\in J$, making S closed in the corresponding product topology on $\mathscr{P}$.

A separable space $(X,\mathscr{B})$ is *standard* if there is a metric d for X such that (X,d) is a complete, separable metric space (the topology of such a space is called *Polish*); we allow $S=\phi$. The isomorphism types of standard spaces are completely classified by cardinality: every standard space X is Borel isomorphic either with a finite set, the integers, or the real line according as the cardinality of X is finite, countably infinite, or uncountable (see Cohn (1980) p. 275).

We use the terms "probability measure" and "law" interchangeably. If $(X,\mathscr{B},P)$ is a probability space, denote by P_* and P^* the inner and outer measures formed from P; a subset A of X is *P-completion measurable* if $P_*(A)$ $=Pf^{-1}(A)$ for $A\in\mathscr{A}$; $f(P)$ is the *image of P under f*. Given any $A\subset X$ and measurable for all laws P on $(X,\mathscr{B})$. Suppose that $f\colon X\to Y$ is a measurable function between two measurable spaces $(X,\mathscr{B})$ and $(Y,\mathscr{A})$ and that P is a probability on $(X,\mathscr{B})$; then we define $f(P)$ on $(Y,\mathscr{A})$ by the rule $f(P)(A)$ $=Pf^{-1}(A)$ for $A\in\mathscr{A}$; $f(P)$ is the *image of P under f*. Given any $A\subset X$ and probability measure P on $(A,\mathscr{B}(A))$, $\mathscr{B}(A)=\{B\cap A\colon B\in\mathscr{B}(X)\}$, we define the probability $\bar{P}$ *induced by P on X* by the rule $\bar{P}(B)=P(B\cap A)$. A separable metric space (S,d) is *universally measurable* (*u.m.*) if it is universally measurable in its completion $\bar{S}$. If the Borel structure on S is standard, Christensen (1974) Theorem 2.6 implies that S is Borel in $\bar{S}$ and hence is u.m. It is not hard to establish that a separable metric space (S,d) is u.m. if and only if every law P on S is *tight* (for each $\varepsilon>0$, there is a compact $K\subset S$ with $P(K)>1-\varepsilon$). See e.g. Varadarajan (1961) p. 224. This would seem to say that the u.m. property for (S,d) depends on the metric d only through its topology; in fact, more is true.

Lemma 2. *Let X be a set and let d_1 and d_2 be separable metrics on X generating the same Borel σ-algebra. Let Y_1 and Y_2 be completions of X for these respective metrics; then X is u.m. in Y_1 if and only if it is u.m. in Y_2.*

Proof. See Shortt (1983) Theorem 1.

Thus the u.m. property is invariant under choice of metric, depends only on Borel structure, and is therefore properly an attribute of separable measurable (not metric) spaces. We shall call a separable space X *u.m.* if there is a metric d for X for which (X,d) is u.m.

Lemma 3. *A separable space X is u.m. if and only if for every law P on X, there is a set $S \in \mathscr{B}(X)$ with $(S, \mathscr{B}(S))$ standard and $P(S)=1$.*

Proof. See Shortt (1983) Lemma 4.

A subset A of a separable space X is *analytic* if it is the measurable image of a standard space (we allow $A=\phi$). It is well-known that analytic spaces are u.m. (cf. Cohn (1980) p. 281 (8.4.3)).

Lemma 4. *Let X be a fixed uncountable standard space; a separable space Y is u.m. if and only if every law P on $X \times Y$ has the following property: if P_1 is the marginal of P on X, and $A \subset X$ is such that $P_1^*(A)=1$, then $P^*(A \times Y)=1$.*

Proof. See Shortt (1983) Theorem 3.

By convergence $P_n \to P$ of (Borel) probability measures on a metric space S we mean convergence "in law" or "weak convergence" defined by bounded continuous functions on S. We shall make use of

Le Cam's Theorem. *Let S be a separable u.m. metric space with laws $P_n \to P$ on S. Then the sequence $P_1, P_2, \ldots$ is uniformly tight.*

Proof. See Le Cam (1957) Theorem 4 or Dudley (1972) p. 10.2.

Lemma 5. *Let X be a metric space on which laws $P_n \to P$. If $A \subset X$ and $P_n^*(A) = P^*(A) = 1$ for all n, then $P_n^* \to P^*$ as laws on A.*

Proof. Closed subsets of A are of the form $F \cap A$, where F is closed in X. For such a set, $\limsup P_n^*(F \cap A) = \limsup P_n(F) \leqq P(F) = P^*(F \cap A)$. Q.E.D.

Lemma 6. *Let X be a separable metrisable space. Then there are, for each positive integer k, partitions $\pi_k(X)$ of X into finitely many Borel sets such that if P, P_k are laws on X with $P_k(A) = P(A)$ for each A in $\pi_k(X)$, then $P_k \to P$.*

Proof. Since X is separable, there is a totally bounded metric d for the topology on X. For each k, choose points $x_1, \ldots, x_n$ (depending on k) such that the open balls $B\left(x_1; \frac{1}{k}\right), \ldots, B\left(x_n; \frac{1}{k}\right)$ cover all of X. Put $A_1 = B\left(x_1; \frac{1}{k}\right)$, and in general

$$A_j = B\left(x_j; \frac{1}{k}\right) \setminus (A_1 \cup A_2 \cup \ldots \cup A_{j-1}) \quad \text{for } j \leqq n.$$

Let $\pi_k(X) = \{A_1, \ldots, A_n\}$. We use criterion ii) of Billingsley (1968) Theorem 2.1 to prove convergence $P_k \to P$. Let g be d-uniformly continuous on X and put $\alpha_j = \inf\{g(x) \colon x \in A_j\}$, $\beta_j = \sup\{g(x) \colon x \in A_j\}$. Then

$$\left|\int g\, dP_k - \int g\, dP\right| \leqq \sum \left|\int_A g\, dP_k - \int_A g\, dP\right| \leqq \sup_j (\beta_j - \alpha_j),$$

where the sum is taken over all A in $\pi_k(X)$. Since g is uniformly continuous and since the diameter of the A in $\pi_k(X)$ is less than $2/k$, $\sup_j(\beta_j - \alpha_j) \to 0$ as $k \to \infty$. Q.E.D.

We are now ready to begin our study of laws with given marginals and supports; the starting point will be the following result for discrete distributions; while it is a special case of Strassen (1965) Theorem 11 or Hansel and

Troallic (1978) Corollaire 2, its proof is new, and it will enable us to generalize their work to the context of separable spaces.

Lemma 7. *Let X be a finite set over which are defined two algebras $\mathscr{A}$ and $\mathscr{B}$; suppose that $\mu_0: \mathscr{A} \cup \mathscr{B} \to \mathbb{R}$ is a finite measure when restricted either to $\mathscr{A}$ or to $\mathscr{B}$. Then a necessary and sufficient condition that μ_0 extend to a measure μ on $\sigma(\mathscr{A}, \mathscr{B})$ is the following:*

(*) *if $B \subset A$ with $A \in \mathscr{A}$, $B \in \mathscr{B}$, then $\mu_0(B) \leqq \mu_0(A)$.*

Proof. Condition (*) is clearly necessary. To prove sufficiency, we may write $X = X_1 \cup \ldots \cup X_n$, where the X_j are the atoms of $\mathscr{A} \cap \mathscr{B}$: solving the problem on each X_j separately, we may and do assume that $\mathscr{A} \cap \mathscr{B}$ is the trivial algebra $\{\phi, X\}$.

Let E be the space of real $\sigma(\mathscr{A}, \mathscr{B})$-measurable functions on X and let P be the cone of everywhere non-negative functions in E. Define $F = \{f + g : f$ is $\mathscr{A}$-measurable, g is $\mathscr{B}$-measurable$\}$, a subspace of E. Let l be the linear functional on F given by $l(f+g) = \int f \, d\mu_0 + \int g \, d\mu_0$; if $f + g = f' + g'$, then $f - f' = g' - g$ is constant, since $\mathscr{A} \cap \mathscr{B} = \{\phi, X\}$. From this it follows that l is well-defined on F.

We now claim that l is non-negative on $P \cap F$, equivalently, if $f \geqq g \geqq 0$, then $l(f) \geqq l(g)$: but

$$l(f) = \int f \, d\mu_0 = \int_0^\infty \mu_0\{t: f(t) > r\} \, dr \geqq \int_0^\infty \mu_0\{t: g(t) > r\} \, dr = \int g \, d\mu_0 = l(g),$$

from condition (*). The Eidelheit Extension Theorem (Kelley and Namioka (1963) 3.3 on p. 20) now applies to give a linear extension of l to all of E which is non-negative on P. If $A \in \sigma(\mathscr{A}, \mathscr{B})$, then $\mu A = l(1_A)$ defines a measure on $\sigma(\mathscr{A}, \mathscr{B})$ with the desired properties. Q.E.D.

Theorem 1. *Let S be a non-empty Borel-closed subset of the product $X \times Y$ of two separable spaces, one of which, say Y, is u.m. Let P_1 and P_2 be laws on X and Y, respectively. Denote by f_1 and f_2 the projection maps from $X \times Y$ onto the respective co-ordinates X and Y. In order that there exist a law P on S with marginals $f_1(P) = P_1$ and $f_2(P) = P_2$, it is necessary and sufficient that for $A \in \mathscr{B}(X)$, $B \in \mathscr{B}(Y)$,*

(*) $(A \times Y) \cap S \subset (X \times B) \cap S$ *implies* $P_1(A) \leqq P_2(B)$.

Proof. Necessity is immediate. For sufficiency, we shall first assume that both X and Y are u.m., then treat the general case.

Case I. Both X and Y are u.m. Choose metrics on X and Y making S closed in the corresponding product topology. Now use Lemma 6 to obtain, for each positive integer k, finite partitions $\pi_k(X)$ of X and $\pi_k(Y)$ of Y into Borel sets such that if laws $P_1^{(k)}$ on X and $P_2^{(k)}$ on Y satisfy:

1. $P_1^{(k)}(A) = P_1(A)$ for all $A \in \pi_k(X)$, and
2. $P_2^{(k)}(B) = P_2(B)$ for all $B \in \pi_k(Y)$, then $P_1^{(k)} \to P_1$ and $P_2^{(k)} \to P_2$ as $k \to \infty$.

Using Lemma 7, it becomes possible to construct atomic laws P_k (i.e. laws with finite support) on S such that, for $k \geqq 3$, $P_k((A \times Y) \cap S) = P_1(A)$ and $P_k((X \times B) \cap S) = P_2(B)$ for $A \in \pi_k(X)$, $B \in \pi_k(Y)$ and all k: holding k fixed, let X_0 be the collection of all non-empty sets of the form $(A \times B) \cap S$, $A \in \pi_k(X)$, $B \in \pi_k(Y)$, with

$\mathscr{A}$ generated by sets of the form $\{(A\times B)\cap S\in X_0:\ B\in\pi_k(Y)\}$ for some fixed $A\in\pi_k(X)$ and with $\mathscr{B}$ generated by sets of the form $\{(A\times B)\cap S\in X_0:\ A\in\pi_k(X)\}$ for some $B\in\pi_k(Y)$. Define $\mu_0\{(A\times B)\cap S\in X_0:\ B\in\pi_k(Y)\}=P_1(A)$ for each $A\in\pi_k(X)$ and $\mu_0\{(A\times B)\cap S\in X_0:\ A\in\pi_k(X)\}=P_2(B)$ for each $B\in\pi_k(Y)$; it follows from condition (*) that μ_0 may be further defined on all of $\mathscr{A}\cup\mathscr{B}$ so as to be a probability when restricted to $\mathscr{A}$ or to $\mathscr{B}$ and so that the hypothesis of Lemma 7 obtains. Therefore, μ_0 extends to a probability μ on $\sigma(\mathscr{A},\mathscr{B})=2^{X_0}$. Let P_k now be an atomic law on S such that $P_k((A\times B)\cap S)=\mu\{(A\times B)\cap S\}$ for $(A\times B)\cap S\in X_0$, $A\in\pi_k(X)$, $B\in\pi_k(Y)$, noting that the sets in question are non-empty.

Thus the marginals $P_1^{(k)}$ and $P_2^{(k)}$ of P_k satisfy 1 and 2 above and so converge to P_1 and P_2, respectively. Since X and Y are u.m., Le Cam's Theorem implies that the families $P_1^{(k)}$ and $P_2^{(k)}$ are uniformly tight; the same then holds for the sequence P_k on $X\times Y$.

Consider now the metric completions $\bar{X}$ of X and $\bar{Y}$ of Y; since the P_k are atomic, they are also naturally defined on the complete space $\bar{X}\times\bar{Y}$ and are again uniformly tight on $\bar{X}\times\bar{Y}$; by Prohorov's Theorem (see Billingsley (1968) Theorem 6.1), there is a subsequence $P_{n(k)}\to P_0$ for some law P_0 on $\bar{X}\times\bar{Y}$.

Claim. $P_0^*(X\times Y)=1$, *and* $P=P_0^*$ *is the law on* S *sought*. Given $\varepsilon>0$, choose compact sets $K_1\subset X$ and $K_2\subset Y$ such that $P_k(K_1\times K_2)=P_k((K_1\times K_2)\cap S)>1-\varepsilon$: since S is closed in $X\times Y$, $(K_1\times K_2)\cap S$ is compact and so is closed in $\bar{X}\times\bar{Y}$: then by the Portmanteau Theorem (Billingsley (1968) Theorem 2.1), $P_0((K_1\times K_2)\cap S)\geqq 1-\varepsilon$; letting $\varepsilon\to 0$ proves that $(P_0)_*(S)=(P_0)^*(S)=1$. By Lemma 5, $P_{n(k)}\to P$ on $X\times Y$. Since the marginals of $P_{n(k)}$ converge to P_1 and P_2, there are the marginals of P.

Case II. Only Y is assumed u.m. Again choose metrics for X and Y making S closed for the product topology. If $\bar{X}$ is the metric completion of X, let $\bar{P}_1$ be the law induced by P_1 on $\bar{X}$. We have $S=(X\times Y)\cap\bar{S}$, where $\bar{S}$ is a closed subset of $\bar{X}\times Y$. Now $\bar{S},\bar{X},Y,\bar{P}_1$, and P_2 satisfy the hypotheses of Case I, and so there is a law $\bar{P}$ on $\bar{X}\times Y$ having $\bar{P}_1$ and P_2 as marginals and such that $\bar{P}(\bar{S})=1$.

Claim. $\bar{P}^*(X\times Y)=1$: *this follows from Lemma* 4 *and the fact that* $\bar{P}_1^*(X)=P_1(X)=1$.

Thus we may take $P=\bar{P}^*$ as a law on $X\times Y$; then $P(S)=\bar{P}(\bar{S})=1$, and the marginals of P are P_1 and P_2. Q.E.D.

The following two examples demonstrate that the hypotheses on S and Y cannot simply be eliminated.

Example 1. Let X and Y be copies of the open unit interval (0, 1) under its usual Borel structure and let $S=\{(x,y)\in X\times Y: x>y\}$: there is no law on S with marginals equal to Lebesgue measure on X and Y; condition (*) of Theorem 1 is, however, satisfied. For details, see Kellerer (1964) p. 196. (S is not Borel-closed.)

Example 2. Let Y be a subset of the interval (0,1) with outer Lebesgue measure $P^*(Y)=1$ and inner measure $P_*(Y)=0$. Put $X=(0,1)\setminus Y$ and $S=\{(x,y)\in X\times Y;\ x\geqq y\}$. Let P_1 and P_2 be P^* on X and Y, respectively: then as in the previous example, noting that $\{(x,x)\in X\times Y\}$ is empty, there is no law on S

with P_1 and P_2 as marginals. (S is closed, and condition (*) is satisfied, but neither X nor Y is u.m.)

Theorem 2. *Let X, Y and S be separable spaces and let there be given laws P_1 on X and P_2 on Y. Let $f\colon S\to X$ and $g\colon S\to Y$ be measurable mappings; then, under the following conditions, there is a law P on S such that $f(P)=P_1$ and $g(P)=P_2$:*

1. *X is u.m. (by symmetry, Y u.m. also suffices),*
2. *$\mathscr{B}(S)=\sigma(f,g)$, the smallest σ-algebra on S making the functions f and g measurable,*
3. *if $f^{-1}(A)\subset g^{-1}(B)$ for $A\in\mathscr{B}(X)$, $B\in\mathscr{B}(Y)$, then $P_1(A)\leqq P_2(B)$, and*
4. *$\{(x,y)\colon f^{-1}(x)\cap g^{-1}(y)\neq\phi\}$ is Borel-closed in $X\times Y$.*

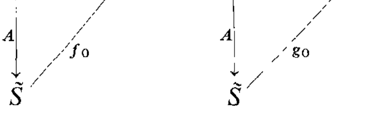
Proof. Define $F\colon S\to X\times Y$ by the rule $F(s)=(f(s),g(s))$; then condition 2 implies that F is injective, in fact a Borel isomorphism onto the image $F(S)$. The set in condition 4 is just $F(S)$; furthermore, if $(A\times Y)\cap F(S)\subset(X\times B)\cap F(S)$ for $A\in\mathscr{B}(X)$, $B\in\mathscr{B}(Y)$, then applying F^{-1} gives $f^{-1}(A)\subset g^{-1}(B)$; condition 3 then implies that $P_1(A)\leqq P_2(B)$. Therefore, the conditions of Theorem 1 are met, so that there is a law Q on $X\times Y$ with $Q(F(S))=1$ and having marginals P_1 and P_2. Then $P=F^{-1}(Q)$ is the law Q.E.I.

If any of the four numbered conditions of this theorem be eliminated, the result will be a false statement. For condition 1, this is shown by Example 2; for condition 4, we have Example 1; as with most of our other results, condition 3 is clearly necessary. With respect to condition 2, we have

Example 3. Let S, X and Y be separable spaces having the unit interval $[0,1]$ as underlying set; let X and Y have the usual Borel structure $\mathscr{B}$ on $[0,1]$. By a well-known result of Banach (1948), the continuum hypothesis implies that there are subsets $H_1, H_2, \ldots$ of $[0,1]$ such that Lebesgue measure P cannot be extended to $\sigma(\mathscr{B}, H_1, H_2, \ldots)$. Give S the Borel structure $\sigma(\mathscr{B}, H_1, H_2, \ldots)$ and let $f\colon S\to X$ and $g\colon S\to Y$ be the identity map on $[0,1]$. All conditions save 2 obtain, but the conclusion of Theorem 3 does not hold. There is, however, at least one legitimate alteration of 2 possible:

Corollary 2.1. *Condition 2 of Theorem 2 may be replaced with the requirement that S be analytic.*

Proof. Define $A\colon S\to\sigma(f,g)$ by the rule $A(s)=\cap\{B\in\sigma(f,g)\colon s\in B\}$, the atom of $\sigma(f,g)$ containing s; since $\sigma(f,g)$ is countably generated, each $A(s)$ is in $\sigma(f,g)$. Put $\tilde{S}=A(S)$; for $\mathscr{B}(\tilde{S})$, take all those $C\subset\tilde{S}$ such that $A^{-1}(C)\in\sigma(f,g)$ (the "quotient structure"). The functions f and g induce measurable mappings $f_0\colon\tilde{S}\to X$ and $g_0\colon\tilde{S}\to Y$ such that the following diagrams commute:



Then X, Y, $\tilde{S}$, P_1, P_2, f_0, and g_0 satisfy the hypotheses of Theorem 2, so there is a law $\tilde{P}$ on $\tilde{S}$ with $f_0(\tilde{P})=P_1$ and $g_0(\tilde{P})=P_2$. Put $P_0=A^{-1}(\tilde{P})$, a probability measure on $(S, \sigma(f,g))$; an application of Varadarajan (1963) p. 194 yields an extension P of $\tilde{P}$ to $\mathscr{B}(S)$; $f(P)=P_1$ and $g(P)=P_2$. Q.E.D.

Corollary 2.2. *Let S be a compact metric space and let f: $S \to X$ and g: $S \to Y$ be continuous surjections of S onto metric spaces X and Y. Suppose that P_1 and P_2 are laws on X and Y, respectively; then there is a law P on S with $f(P)=P_1$ and $g(P)=P_2$ if and only if the following condition obtains:*

(*) *if $f^{-1}(A) \subset g^{-1}(B)$ for $A \in \mathscr{B}(X)$ and $B \in \mathscr{B}(Y)$, then $P_1(A) \leqq P_2(B)$.*

Proof. Since $X=f(S)$ is compact, it is u.m.; since S is compact, it is analytic; condition 3 of Theorem 2 has been assumed in (*); finally, the set of all (x,y) such that $f^{-1}(x) \cap g^{-1}(y) \neq \phi$ is closed in $X \times Y$. Corollary 2.1 now applies to provide the existence of a P. Q.E.D.

Example 1 illustrates how this corollary fails when the hypothesis that S be compact is weakened to S standard, or if the functions f and g are only assumed measurable: in this latter case, we see that since S is standard and uncountable, there is a compact metric for S (S is Borel-isomorphic with $[0,1]$), but under this metric, f and g are no longer continuous.

The passage to products of more than two spaces is not at all straightforward: as pointed out in Strassen (1965) p. 437 and Kellerer (1964), conditions as simple as (*) in Theorem 1 will not suffice. However, using the techniques we have developed in conjunction with Kellerer (1964) Satz 4.2 it is possible to prove the following multi-dimensional result:

Theorem 3. *Let S be a separable space and let f_j: $S \to X_j$, $j=1, \ldots, n$, be measurable mappings of S to separable u.m. spaces $X_1, \ldots, X_n$ on which are defined laws $P_1, \ldots, P_n$. The following are conditions sufficient to ensure the existence of a law P on S with $f_j(P)=P_j$, $j=1, \ldots, n$:*

1. $\mathscr{B}(S)=\sigma(f_1, \ldots, f_n)$,

2. *$\{(x_1, \ldots, x_n): f_1^{-1}(x_1) \cap \ldots \cap f_n^{-1}(x_n) \neq \phi\}$ is Borel-closed in $X=X_1 \times \ldots \times X_n$, and*

3. *for any measurable partitions $X_j=A_1^{(j)} \cup \ldots \cup A_{m_j}^{(j)}$ of the X_j and any corresponding set of n real vectors $\vec{x}^{(j)}=(x_1^{(j)}, x_2^{(j)}, \ldots, x_{m_j}^{(j)})$, $j=1, \ldots, n$, this inequality obtains:*

$$(*) \qquad \sum_{j=1}^{n} \vec{x}^{(j)} \cdot \vec{P}_j \leqq \sum \left[\sum_{j=1}^{n} x_{i_j}^{(j)} \right]^{+},$$

where $\vec{P}_j=(P_j(A_1^{(j)}), P_j(A_2^{(j)}), \ldots, P_j(A_{m_j}^{(j)}))$; the dot in the first summation is an inner product, and the unmarked summation is taken over all n-tuples $(i_1, \ldots, i_n)$ such that $f_1^{-1}(A_{i_1}^{(1)}) \cap f_2^{-1}(A_{i_2}^{(2)}) \cap \ldots \cap f_n^{-1}(A_{i_n}^{(n)})$ is non-empty.

Corollary 3.1. *Condition* 1 *may be replaced in Theorem* 3 *with the requirement that S be analytic.*

Proof. This again follows *mutatis mutandis* from the "method of quotients" detailed in the proof of Corollary 2.1. Q.E.D.

Corollary 3.2. *In Theorem 3, one may assume that all but one of the X_j are u.m.*

Proof. Entirely analogous to the demonstration of Case II of Theorem 1, using Lemma 4. Q.E.D.

Taking S to be a Borel-closed subset of $X_1 \times \ldots \times X_n$ and defining the f_j as co-ordinate projections leads to an n-dimensional generalization of Theorem 1.

Remark. It has already been pointed out in reference to Theorem 2 that the u.m. requirement for all but one of the X_j in Theorem 3 cannot simply be dropped; the same has been shown for conditions 1 and 2 of Theorem 3. Despite the complexity of condition 3 of this theorem, it is implied by the existence of the sought-after law P.

Theorem 4. *Let S be a separable space and let $f_1: S \to X$, $f_2: S \to X_2, \ldots$ be a sequence of measurable functions on S to separable spaces $X_1, X_2, \ldots$ such that*

1. *all but perhaps one of the X_j are u.m. (say $X_2, X_3, \ldots$ are u.m.),*
2. *$\mathscr{B}(S) = \sigma(f_1, f_2, \ldots)$, and*
3. *$\{(x_1, x_2, \ldots): f_1^{-1}(x_1) \cap f_2^{-2}(x_2) \cap \ldots \neq \phi\}$ is Borel-closed in $X = X_1 \times X_2 \times \ldots$.*

In order that there should exist a law P on S with $f_j(P) = P_j$, $j = 1, 2, \ldots$, it is necessary and sufficient that the same be true for $j = 1, \ldots, n$, for every n.

Proof. Necessity is obvious. For the converse, let Q_n be a sequence of laws on S such that $f_j(Q_n) = P_j$ for $j = 1, \ldots, n$. Define $F: S \to X$ by $F(s) = (f_1(s), f_2(s), \ldots)$; as before, F is a Borel isomorphism of S onto $F(S)$, the set indicated in condition 3. Choose totally bounded metrics for the X_j making $F(S)$ closed in X. Consider the image laws $F(Q_n)$ on X and the laws $\overline{F(Q_n)}$ they induce on the compact completion $\bar{X} = \bar{X}_1 \times \bar{X}_2 \times \ldots$. Then for each n, $\overline{F(Q_n)}^*$ $(F(S)) = 1$, and the marginal of $F(Q_n)$ on $\bar{X}_1$ is $\bar{P}_1$.

Because we have taken $\bar{X}$ compact, Prohorov's Theorem implies that there is a subsequence $\overline{F(Q_{n(k)})} \to \bar{Q}$ for some law $\bar{Q}$ on $\bar{X}$; since the marginals of the $\overline{F(Q_n)}$ on $\bar{X}_1$ are all $\bar{P}_1$, this is the marginal of $\bar{Q}$ on $\bar{X}_1$; since $\bar{P}_1^*(X_1) = 1$, Lemma 4 implies that $\bar{Q}^*(X_1 \times \bar{X}_2 \times \bar{X}_3 \times \ldots) = 1$.

Claim. $\bar{Q}^*(X) = 1$: we know from the u.m. property and Lemma 3 that there are standard subsets $S_j \in \mathscr{B}(\bar{X}_j)$ of X_j, $j \geq 2$, with $\bar{P}_j(S_j) = P_j(S_j) = 1$; as before, we see that the marginal of $\bar{Q}$ on each $\bar{X}_j$ is $\bar{P}_j$, $j = 1, \ldots, n$; therefore, $\bar{Q}(\bar{X}_1 \times S_2 \times S_3 \times \ldots) = 1$ so that $\bar{Q}^*(X) \geq \bar{Q}^*(X_1 \times S_2 \times S_3 \times \ldots) = 1$, as claimed.

Since $\overline{F(Q_{n(k)})} \to \bar{Q}$ on $\bar{X}$ and $\overline{F(Q_{n(k)})}^*(X) = \bar{Q}^*(X) = 1$, Lemma 5 implies that $F(Q_{n(k)}) = \overline{F(Q_{n(k)})}^* \to Q = \bar{Q}^*$ on X. Then $P = F^{-1}(Q)$ serves as the desired law on S. Q.E.D.

Corollary 4.1. *As in Corollaries 2.1 and 3.1, condition 2 may be dropped if S is analytic.*

Corollary 4.2. *By taking $S = S_1 \times S_2 \times \ldots$ and letting the f_j be projections of S onto all finite partial products of the X_j, we obtain Kolmogoroff's Existence Theorem for products of u.m. spaces.*

No one of the itemized conditions in Theorem 4 can simply be dropped; counter-examples follow.

Example 4. The conclusion of Theorem 4 may fail if two of the X_i are not u.m.: let $I=[0,1]$, the unit interval with its usual Borel structure and let λ be Lebesgue measure on I. We require the following

Fact. There are subsets X_1 and X_2 of I with $\lambda^*(X_j)=1$ and $\lambda_*(X_j)=0$, $j=1,2$, and such that the difference set $D=\{h_1-h_2\colon h_1\in X_1, h_2\in X_2\}$ contains no rational point. (Proof is a standard transfinite induction argument.)

Put $S=X_1\times X_2$ under the usual Euclidean metric d, $X_j=\{0,1\}$ for $j\geqq 3$, $f_1\colon S\to X_1$ and $f_2\colon S\to X_2$ the co-ordinate projections and for $n\geqq 3$,

$$f_n(s)=\begin{cases}1 & \text{if } d(s,\Delta)\leqq 1/n\sqrt{2}\\ 0 & \text{if } d(s,\Delta)>1/n\sqrt{2},\end{cases}$$

where Δ is the diagonal line $y=x$ in $I\times I$.

Let $P_1=P_2=\lambda^*$, and $P_j\{1\}=1$, $P_j\{0\}=0$, for $j\geqq 3$. Then:

a) X_1, X_2 are not u.m.; $X_3, X_4, \ldots$ are all u.m.,

b) $\mathscr{B}(S)$ is generated by f_1 and f_2, and so certainly by all the f_j, and

c) $T=\{(x_1,x_2,\ldots)\colon f_1^{-1}(x_1)\cap f_2^{-1}(x_2)\cap\ldots\neq\phi\}$ is closed in $X=X_1\times X_2\times X_3\times\ldots$: since no point of S meets any of the lines $y=x\pm 1/n$, the regions $\{s\in S\colon d(s,\Delta)\leqq 1/n\sqrt{2}\}$ are clopen in S, and the f_j, $j\geqq 3$, are continuous functions on S. Suppose t_n is a sequence in T and $t_n\to t$ in X: then the j^{th}-coordinates $t_n(j)\to t(j)$ for $j\geqq 3$; since $f_j(t_n(1),t_n(2))=t_n(j)$, taking $n\to\infty$ and using the continuity of the f_j establishes that $t\in T$.

There is clearly no law P on S with $f_j(P)=P_j$ for all j, since $\{(x,x)\colon x\in X_1\}\cap S=\phi$; however, only condition 1 of Theorem 4 fails.

Assume the continuum hypothesis:

Example 5. The conclusion of Theorem 4 may fail if $\mathscr{B}(S)$ is not generated by the functions $f_1, f_2, \ldots$: Let C be the Cantor discontinuum, realized as the product of two-point spaces $\{0,1\}=X_1=X_2=\ldots$; let $f_j\colon C\to X_j$ be the j^{th}-coordinate projection. Let S be the separable space with underlying set C and Borel structure $\sigma(\mathscr{B}, H_1, H_2, \ldots)$, where $\mathscr{B}$ is the usual Borel σ-algebra on C and $H_1, H_2, \ldots$ is, as in Example 3, a sequence of C such that no continuous measure may be defined on $\sigma(\mathscr{B}, H_1, H_2, \ldots)$. Define P_j on X_j to be the "fair coin-toss" measure $P_j\{0\}=P_j\{1\}=1/2$, $j=1,2,\ldots$. Then:

a) all of the X_j are u.m. (standard),

b) $\{(x_1,x_2,\ldots)\colon f_1^{-1}(x_1)\cap f_2^{-1}(x_2)\cap\ldots\neq\phi\}=C$, but

c) $\mathscr{B}(S)$ is not generated by the projections f_j.

There is, for each n, an atomic law $P^{(n)}$ on S with $f_j(P^{(n)})=P_j$, $j=1,\ldots,n$, but no P on S such that $f_j(P)=P_j$ for $j\geqq 1$.

Example 6. Conditions 3 in Theorem 4 may not be deleted: Let S be the set of all rational numbers in the half-open interval $[0,1[$, under the usual Euclidean metric. Set $X_n=\{1,\ldots,2^n\}$ under the discrete structure and define $f_n\colon S\to X_n$ by the rule $f_n(s)=[2^n\cdot s]+1$. For laws P_n on X_n, take "uniform distributions" with $P_n\{k\}=2^{-n}$ for $1\leqq k\leqq 2^n$. Then, for each n, there is an atomic law $P^{(n)}$ on S with

$f_j(P^{(n)})=P_j$, $j=1,\ldots,n$, but no law P on S such that $f_j(P)=P_j$ for $j\geqq 1$. In this example:

a) each X_j is an u.m.,

b) $\mathscr{B}(S)$ is generated by the maps f_j, $j\geqq 1$, but

c) $F(S)$ is not Borel-closed in $X=X_1\times X_2\times\ldots$,

where $F\colon S\to X$ is defined as in the proof of Theorem 4: notice that since there is only one metric topology on each X_j, *viz.* the discrete topology, this is equivalent to saying that $F(S)$ is not actually closed in the compact space X. Now F^{-1} is continuous from $F(S)$ onto S; if $F(S)$ were closed, it and its image S would be compact, a contradiction.

The following is a generalization of a result of Hansel and Troallic (1968), Corollaire 3, to products of more than two not necessarily Polish spaces.

Theorem 5. *Let $X_1,\ldots,X_n$ be separable spaces with $X_2,\ldots,X_n$ u.m. and let $f_1\colon X_1\to Z,\ldots,f_n\colon X_n\to Z$ be measurable surjections taking laws $P_1,\ldots,P_n$ on $X_1,\ldots,X_n$ to a single law Q on a separable space Z; then there is a law P on $X_1\times\ldots\times X_n$ with marginals P_j on X_j, $j=1,\ldots,n$, such that $P(S)=1$, where S is the set $\{(x_1,\ldots,x_n)\colon f_1(x_1)=\ldots=f_n(x_n)\}$.*

Proof. Case I: X_1 is also u.m. Firstly, note that the complement of S in $X=X_1\times\ldots\times X_n$ is a countable union of rectangles of the form $f_1^{-1}(A_1)\times\ldots\times f_n^{-1}(A_n)$ for $A_1,\ldots,A_n\in\mathscr{B}(Z)$: S is the inverse image of the diagonal in Z^n under the mapping $(x_1,\ldots,x_n)\to(f_1(x_1),\ldots,f_n(x_n))$ on X, and the diagonal of Z^n is Borel-closed.

Let $P_1(x_1,\cdot),\ldots,P_n(x_n,\cdot)$ be almost everywhere proper regular conditional probabilities for $P_1,\ldots,P_n$ given $f_1,\ldots,f_n$ (see Blackwell and Ryll-Nardzewski (1963) for definitions and a discussion): then there is a measurable subset B of Z with $P_j(x_j,f_j^{-1}(A))=0$ whenever $A\in\mathscr{B}(Z)$ and $x_j\in f_j^{-1}(B)\setminus f_j^{-1}(A)$ and such that $Q(B)=P_j(f_j^{-1}(B))=1$. Since for each A_j, $P_j(\cdot,A_j)$ is $f_j^{-1}(\mathscr{B}(Z))$-measurable, we may write $P_j(x_j,A_j)=g_j(f_j(x_j),A_j)$ for some measurable real function $g_j(\cdot,A_j)$ on Z. Since each f_j is surjective, $g_j(z,\cdot)$ is a law on X_j for each $z\in Z$.

A law P on X may now be defined so that $P(A_1\times\ldots\times A_n)=\int_Z g_1(z,A_1)\ldots g_n(z,A_n)\,dQ(z)$ for $A_j\in\mathscr{B}(X_i)$, $j=1,\ldots,n$; then P has the correct marginals; furthermore, $P(S)=1$, since if $f_1^{-1}(A_1)\times\ldots\times f_n^{-1}(A_n)\subset X\setminus S$, $A_j\in\mathscr{B}(Z)$, then $P(f_1^{-1}(A_1)\times\ldots\times f_n^{-1}(A_n))=\int_B g_1(z,f_1^{-1}(A_1))\ldots g_n(z,f_n^{-1}(A_n))\,dQ(z)=0$, since for each $z\in B$ and $(x_1,\ldots,x_n)\in f_1^{-1}(z)\times\ldots\times f_n^{-1}(z)$, there is a j such that $x_j\notin f_j^{-1}(A_j)$.

Case II. X_1 is not assumed u.m.; we claim that there are u.m. spaces $Y_1,\ldots,Y_n$ and Z_0 and measurable surjections $\bar{f}_j\colon Y_j\to Z_0$, $j=1,\ldots,n$, such that $X_j\subset Y_j$, $Z\subset Z_0$ and $\bar{f}_j(x)=f_j(x)$ whenever $x_j\in X_j$, $j=1,\ldots,n$: select metrics for Z and the X_j and let $\bar{Z}$ and $\bar{X}_1,\ldots,\bar{X}_n$ be their completions; by an extension theorem of Kuratowski (1966) p. 434, there are measurable $\bar{f}_j\colon\bar{X}_j\to\bar{Z}$ extending the f_j, $j=1,\ldots,n$. Put $Z_0=\bigcap_j\bar{f}_j(\bar{X}_j)$ and $Y_j=\bar{f}_j^{-1}(Z_0)$ for $j=1,\ldots,n$: since Z_0 is analytic, so are the Y_j (Parthasarathy (1967) p. 17 (3.4)), and hence they are u.m.

Define $\bar{P}_j$ and $\bar{Q}$ to be the laws induced by the P_j and Q on Y_j and Z_0, respectively; we now see that $\bar{f}_j(\bar{P}_j)=\bar{Q}$, $j=1,\ldots,n$. Case I applies to guarantee the existence of a law $\bar{P}$ on $Y_1 \times \dots \times Y_n$ with marginals $\bar{P}_1, \dots, \bar{P}_n$ such that $\bar{P}(\bar{S}) = 1$, where $\bar{S} = \{(y_1, \dots, y_n): \bar{f}_1(y_1) = \dots = \bar{f}_n(y_n)\}$. Note that $S = \bar{S} \cap (X_1 \times \dots \times X_n)$.

Claim 1. $\bar{P}^*(X_1 \times Y_2 \times \dots \times Y_n) = 1$: this follows from Lemma 4, the u.m. property for $Y_2 \times \dots \times Y_n$, and the fact that $\bar{P}_1^*(X_1) = 1$.

Claim 2. $\bar{P}^*(X_1 \times \dots \times X_n) = 1$: since $X_2, \dots, X_n$ are u.m., and since $\bar{P}_j^*(X_j) = 1$ for $j = 2, \dots, n$, there are sets $C_j \subset X_j$, $C_j \in \mathscr{B}(Y_j)$ with $\bar{P}_j(C_j) = 1$ for $j = 2, \dots, n$; thus, $\bar{P}(Y_1 \times C_2 \times \dots \times C_n) = 1$; combining this with Claim 1 yields

$$\bar{P}^*(X_1 \times \dots \times X_n) \geqq \bar{P}^*((X_1 \times Y_2 \times \dots \times Y_n) \cap (Y_1 \times C_2 \times \dots \times C_n)) = 1,$$

as claimed.

Taking $P = \bar{P}^*$ on $X = X_1 \times \dots \times X_n$, we note that $P(S) = \bar{P}(\bar{S}) = 1$ and that P has marginal P_j on X_j, $j = 1, \dots, n$. Q.E.D.

Example 7. As with our other results, if two of the the X_j are not u.m., then even the well-behavedness of Z will not necessarily ensure the validity of this theorem's conclusion: as in Shortt (1983) Theorem 5, one may exhibit separable spaces A_1, A_2, A_3 with A_2 standard, but neither A_1 nor A_3 u.m., such that there are laws P_1 on $X_1 = A_1 \times A_2$ and P_2 on $X_2 = A_2 \times A_3$ having a common projection Q on $Z = A_2$, but such that there is no law on $A_1 \times A_2 \times A_3$ with P_1 and P_2 as marginals.

Unlike Theorem 4 (cf. Example 4), the "countable" case allows more freedom with the u.m. hypothesis:

Theorem 6. *Let $X_1, X_2, \dots$ and Z be separable spaces on which are defined laws $P_1, P_2, \dots$ and Q, respectively. Suppose that $f_1: X_1 \to Z, f_2: X_2 \to Z, \dots$ are measurable surjections such that $f_1(P_1) = f_2(P_2) = \dots = Q$; suppose further that all but finitely many of the X_j are* u.m.; *then a necessary and sufficient condition that there exist a law P on $X = X_1 \times X_2 \times \dots$ with marginals $P_1, P_2, \dots$ and with* $P(S) = 1$, *where* $S = \{(x_1, x_2, \dots): f_1(x_1) = f_2(x_2) = \dots\}$, *is given by the following:*

(∗) *for each positive integer n, there is a law Q_n on $X_1 \times \dots \times X_n$ with marginals $P_1, \dots, P_n$ and such that $Q_n(S_n) = 1$, where $S_n = \{(x_1, \dots, x_n): f_1(x_1) = \dots = f_n(x_n)\}$.*

Proof. Necessity is clear. Suppose now that (∗) holds. Choose a matric for Z and choose totally bounded metrics for the X_j which make the f_j continuous: e.g. let the inverse images under the f_j of elements of a countable base for the topology of Z be included as open sets for the metrics on X_j; then S and the S_n are closed in the associated product topologies.

Case I. Suppose that each X_j is u.m., $j = 1, 2, \dots$: then let $a_1 \in X_1$, $a_2 \in X_2, \dots$ be arbitrary and $\bar{X}_1, \bar{X}_2, \dots$ the (compact) completions of $X_1, X_2, \dots$. Define laws R_n on $X = X_1 \times X_2 \times \dots$ by the rule $R_n = Q_n \otimes \delta_n$, where δ_n is the (Dirac) point mass at $(a_{n+1}, a_{n+2}, \dots)$ and let $\bar{R}_n$ be the law induced by R_n on $\bar{X} = \bar{X}_1 \times \bar{X}_2 \times \dots$; then the marginals of $\bar{R}_n$ on $\bar{X}_1, \dots, \bar{X}_n$ are $\bar{P}_1, \dots, \bar{P}_n$, the laws induced by $P_1, \dots, P_n$.

Since $\bar{X}$ is compact, there is a subsequence $\bar{R}_{n(k)} \to \bar{P}$ for some law $\bar{P}$ on $\bar{X}$. Because the marginals of the $\bar{R}_{n(k)}$ on $\bar{X}_j$ eventually equal $\bar{P}_j$, these are the marginals of $\bar{P}$. Given $\varepsilon>0$, choose compact $K_j \subset X_j$ with $\bar{P}_j(K_j)>1-\varepsilon\cdot 2^{-j}$, making sure that $a_j \in K_j$ for $j=1,2,\ldots$: then $\bar{P}^*(X) \geqq \bar{P}(K_1 \times K_2 \times \ldots) \geqq \limsup \bar{R}_{n(k)}(K_1 \times K_2 \times \ldots) \geqq 1-\varepsilon$; letting $\varepsilon \to 0$ shows that $\bar{P}^*(X)=1$. Take $P=\bar{P}^*$ as a law on X: the marginals of P are the P_j.

Since $\bar{R}_{n(k)} \to \bar{P}$, and $\bar{R}^*_{n(k)}(X)=\bar{P}^*(X)=1$, Lemma 5 implies that $R_{n(k)} \to P$ on X. Now $S_n \times X_{n+1} \times X_{n+2} \times \ldots$ is a sequence of closed subsets of X decreasing to S; for each n,

$$P(S_n \times X_{n+1} \times X_{n+2} \times \ldots) \geqq \limsup R_{n(k)}(S_n \times X_{n+1} \times X_{n+2} \times \ldots)=1,$$

so that $P(S)=1$, as desired.

Case II. Suppose that there are no restrictions on the spaces $X_1, \ldots, X_n$, but that $X_{n+1}, X_{n+2}, \ldots$ are all u.m. Define

$$S_0=\{(x_{n+1}, x_{n+2}, \ldots): f_{n+1}(x_{n+1})=f_{n+2}(x_{n+2})=\ldots\} \subset X_{n+1} \times X_{n+2} \times \ldots,$$

and let $F\colon S_n \to Z$ and $G\colon S_0 \to Z$ be the surjections given by $F(x_1, \ldots, x_n)=f_1(x_1)$ and $G(x_{n+1}, \ldots)=f_{n+1}(x_{n+1})$. From Case I, we know that there exists a law Q_0 on S_0 with marginals P_j on X_j, $j=n+1, n+2, \ldots$. Now $F(Q_n)=G(Q_0)=Q$, and since S_0 is u.m., there is, by Theorem 5, a law P on $S_n \times S_0 \subset X$ with $P(S)=1$ and having marginals Q_n on S_n and Q_0 on S_0, hence with marginals P_j on X_j, $j=1,2,\ldots$. Q.E.D.

Example 8. As in Example 3, let $I=[0,1]$, $\mathscr{B}$ the Borel σ-algebra on I and $H_1, H_2, \ldots$ a sequence of subsets of I to which Lebesgue measure Q cannot be extended. Put $\mathscr{B}_n=\sigma(\mathscr{B}, H_1, \ldots, H_n)$ and let P_n be extensions of Lebesgue measure to $\mathscr{B}_n$ such that $P_n=P_m$ on $\mathscr{B}_m$ if $m<n$. Put $X_j=(I, \mathscr{B}_j)$ $j=1,2,\ldots$, $Z=(I, \mathscr{B})$ and let f_j be the identity function on I. Then all the hypotheses of Theorem 6 are satisfied, except that the X_j are not u.m., and the conclusion fails to hold. (S is the diagonal in $I \times I \times \ldots$, Borel isomorphic with $(I, \sigma(\mathscr{B}, H_1, H_2, \ldots))$.)

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