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A Note on R-Recurrence of Markov Chains

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Necessary and sufficient conditions are given for a Markov chain to be R-recurrent and satisfy the Strong Ratio Limit Property, and for a Markov Chain to be R-positive-recurrent.

In this note we deal with aperiodic, irreducible Markov chains (MC). We improve a result of an earlier paper [2], in which criteria for R-recurrence were developed.

Let $P=(P_{ik})_{i,k\in E}$ be a stochastic matrix. (E shall be a countable space.) By P_{ik}^n we denote the n-step transition probabilities. f_{ik}^n is, as normally, the probability that coming from i one reaches k at the n'th step for the first time.

An irreducible, aperiodic MC is characterized by the property that for every pair $(i,k)\in E\times E$ there is a $N(i,k)\in\mathbb{N}$ such that $P_{ik}^n>0$ for all $n\geqq N(i,k)$. (Later we shall denote $N(i,i)$ by $N(i)$.) As is well-known, for MCs with this property

$$\lim_{n\to\infty}\sqrt[n]{P_{ik}^n}=\gamma\leqq 1$$

exists and the limit is independent of i, k. γ is called the convergencenorm, $R=\gamma^{-1}$ is the radius of convergence of the power-series

$$P_{ik}(x)=\sum_{n=0}^{\infty}P_{ik}^n x^n.$$

We now come to the definition of R-recurrence, as given by Vere-Jones [5]. The idea is to characterize those transient, aperiodic, irreducible MCs, which in a sense have similar analytic properties to recurrent MCs. An aperiodic, irreducible MC is called R-recurrent, if

$$P_{ik}(R)=\infty$$

for some pair $(i,k)\in E\times E$ (and then for all $i,k\in E$!). Equivalent is the condition

$$\sum_{k=1}^{\infty}f_{ii}^k R^k=1$$

for all $i \in E$. If $P_{ik}(R) < \infty$ for all $i, k \in E$ or equivalently

$$\sum_{k=1}^{\infty} f_{ii}^k R^k < 1$$

for all $i \in E$, then the MC is called R-transient. Of course 1-recurrence is the same as recurrence. R-positive-recurrent chains are those MCs, which are R-recurrent and for which

$$\sum_{k=1}^{\infty} k f_{ii}^k R^k < \infty$$

is true for one $i \in E$ (and then for all $i \in E$). For R-positive-recurrent MCs

$$\lim_{n \to \infty} R^n P_{ii}^n > 0$$

is true for all $i \in E$, indeed, as in the positive-recurrent case, this property characterizes the R-positive-recurrent MCs.

In [2] we proved that a reversible, aperiodic, irreducible MC, for which

$$\lim_{N \to \infty} \limsup_{n \to \infty} \sum_{k=N}^{n} f_{ii}^k P_{ii}^{n-k} / P_{ii}^n = 0 \tag{1}$$

is true for some $i \in E$, is R-recurrent. (We gave a stronger condition, but in our proof (1) was the essential condition which we needed.) On the other hand Garsia, Orey and Rodemich [1] proved for a recurrent MC that $P_{ii}^{n+1} \sim P_{ii}^n$ if and only if (1) holds. We give a result that generalizes these two statements.

An aperiodic, irreducible MC is said to have the Strong Ratio Limit Property (SRLP) iff there are positive numbers π_i, τ_i $(i \in E)$ such that

$$\lim_{n \to \infty} P_{ik}^{m+n} / P_{jl}^n = \gamma^m \frac{\pi_i \tau_k}{\pi_j \tau_l}.$$

Theorem 1. *For an aperiodic irreducible MC the following conditions are equivalent:*

(i) (1) *holds for all* $i \in E$,

(ii) (1) *holds for some* $i \in E$,

(iii) *the* MC *is* R*-recurrent and has the* SRLP.

Proof. The directions (iii) $\Rightarrow$ (i) and (i) $\Rightarrow$ (ii) are easy to be proved. Thus suppose (1) holds for some $i \in E$.

Let

$$M = \limsup_{n \to \infty} P_{ii}^{n+1} / P_{ii}^n$$

and choose a subsequent (n') of natural numbers such that

$$h_j = \lim_{n \to \infty} P_{ii}^{n'+j+1} / P_{ii}^{n'+j}$$

exists for all $j \in \mathbb{N}$ and $h_0 = M$. Then one may show from (1)

$$0 < M < \infty,$$

$$h_j = \sum_{k=1}^{\infty} \frac{f_{ii}^k h_{j-k}}{h_{j-k} h_{j-k+1} \dots h_{j-1}}, \qquad \forall j \in \mathbb{N}.$$

$$h_j = M \qquad \forall j \in \mathbb{N}.$$

This is proved for example in [3], p. 90–91. But this implies

$$1 = \sum_{k=1}^{\infty} f_{ii}^k M^{-k}.$$

Now suppose that the MC is R-transient. Then $R < M^{-1}$ and the radius of convergence of the (complex) power-series

$$F_{ii}(z) = \sum_{k=1}^{\infty} f_{ii}^k z^k$$

is at least M^{-1}. Since $f_{ii}^k \geqq 0$ for all $k \in \mathbb{N}$,

$$|F_{ii}(z)| < 1$$

for all z with $|z| < M^{-1}$.

Now by the renewal equation

$$P_{ii}(z) = (1 - F_{ii}(z))^{-1}$$

for $|z| < R$. The right side is a holomorphic function on the disk $\{z \mid |z| < M^{-1}\}$. Thus the radius of convergence of $P_{ii}(z)$ would be $M^{-1} > R$, which is a contradiction.

Thus the MC is R-recurrent. It follows $M^{-1} = R$, thus

$$\limsup_{n \to \infty} P_{ii}^{n+1} / P_{ii}^n = \gamma.$$

By a theorem of Pruitt this implies the SRLP (see [4]). q.e.d.

Corollary. *If for an aperiodic, irreducible* MC

$$\sum_{n=N(i)}^{\infty} f_{ii}^n / P_{ii}^n < \infty$$

then the MC *is R-recurrent and the* SRLP *holds.* (*This generalizes Theorem* 3.2 *of* [2].)

Theorem 2. *For an aperiodic, irreducible* MC *the following conditions are equivalent:*

(i) $\sum_{n=N(i)}^{\infty} n f_{ii}^n / P_{ii}^n < \infty$ *for all* $i \in E$,

(ii) $\sum_{n=N(i)}^{\infty} n f_{ii}^n / P_{ii}^n < \infty$ *for some* $i \in E$,

(iii) *the chain is R-positive-recurrent.*

Proof. (ii) ⇒ (iii): From Corollary 2 we have R-recurrence and the SRLP, thus

$$\lim_{n\to\infty} P_{ii}^{n-k}/P_{ii}^{n} = R^k.$$

From $P_{ii}^{n-k} P_{ii}^{k} \leqq P_{ii}^{n}$ follows for fixed $m \in \mathbb{N}$

$$\sum_{k=N(i)}^{m} k f_{ii}^{k} \frac{P_{ii}^{n-k}}{P_{ii}^{n}} \leqq \sum_{k=N(i)}^{m} k f_{ii}^{k}/P_{ii}^{k}.$$

For $n \to \infty$ follows

$$\sum_{k=N(i)}^{m} k f_{ii}^{k} R^k \leqq \sum_{k=N(i)}^{m} k f_{ii}^{k}/P_{ii}^{k}.$$

Letting $m \to \infty$, we get

$$\sum_{k=N(i)}^{\infty} k f_{ii}^{k} R^k < \infty,$$

thus R-positive-recurrence.

(iii) ⇒ (i): For a R-positive-recurrent MC

$$\lim_{k\to\infty} P_{ii}^{k} R^k > 0$$

and

$$\sum_{k=1}^{\infty} k f_{ii}^{k} R^k < \infty$$

for all $i \in E$. This gives (i). q.e.d.

Corollary. *If an aperiodic, irreducible* MC *satisfies*

$$\limsup_{n\to\infty} \sqrt[n]{f_{ii}^{n}} < \gamma$$

for some $i \in E$, *then the* MC *is* R-*positive-recurrent.*

This generalises Corollary 3.3 of [2] and was used there to construct examples of R-positive-recurrent chains.

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