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On the Accuracy of Nonuniform Gaussian Approximation to the Distribution Functions of Sums of Independent and Identically Distributed Random Variables

R. Michel

Mathematisches Institut der Universität Köln,
Weyertal 86, D-5000 Köln 41, Federal Republic of Germany

1. Introduction and Results

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed (i.i.d.) random variables with $EX=0$, $EX^2=1$, and common distribution function F. Write

$$F_n(t)=P\left(n^{-1/2}\sum_{i=1}^{n} X_i<t\right)$$

and denote by Φ the distribution function of the unit normal law. Let

$$\Delta_n(t)=|F_n(t)-\Phi(t)|.$$

Various asymptotically correct bounds for $\sup_{t\in\mathbb{R}} \Delta_n(t)$ have previously been given. It appears that the order of convergence cannot be increased beyond $0(n^{-1/2})$ by assuming appropriate moment conditions for X. To be specific, it is known that $E|X|^{2+c}<\infty$, $0\leqq c\leqq 1$, implies

$$\sup_{t\in\mathbb{R}} \Delta_n(t)=0(n^{-c/2}),$$

and, for $X=(Y-EY)\operatorname{var}(Y)^{-1/2}$, where Y is binomially distributed,

$$\limsup_{n\to\infty}\sup_{t\in\mathbb{R}} n^{1/2}\,\Delta_n(t)>0.$$

Besides this shortcoming of uniform bounds for Δ_n it also appears that these bounds give insufficient results for probabilities of certain deviations (see Michel [6] and [7], where convergence rate problems in this direction are solved under appropriate moment conditions.). Corresponding nonuniform bounds for Δ_n have much wider applicability, such as for obtaining probabilities of moderate deviations, for dealing with L_p metrics, or for approximating certain moments of $n^{-1/2}\sum_{i=1}^{n} X_i$.

A first result in this direction has been obtained by Nagaev ([8], Theorem 2, p. 215) who shows that $E|X|^3<\infty$ implies

$$\sup_{t\in\mathbb{R}}(1+|t|^3)\,\Delta_n(t)\leqq d\,n^{-1/2}, \qquad n\in\mathbb{N}.$$

This has been generalized by Heyde ([3], Theorem 1, p. 903) to

$$\sup_{t\in\mathbb{R}}(1+|t|^{2+c})\,\Delta_n(t)\leqq d\,n^{-c/2}, \qquad n\in\mathbb{N},$$

if $E|X|^{2+c}<\infty$, $0\leqq c\leqq 1$. In Michel ([7], Theorem 3, p. 103) it is shown that $E|X|^{2+c}<\infty$, $c\geqq 0$, gives

$$\sup_{t\in\mathbb{R}}(1+|t|^{2+c})\,\Delta_n(t)\leqq d\,n^{-\frac{1}{2}\min(1,c)}, \qquad n\in\mathbb{N}.$$

(See also Theorem 13 in Petrov [9], p. 125, for the case $c\geqq 1$.)

Concerning sufficient (and necessary) conditions for the existence of non-uniform central limit bounds Bikjalis [1] has proved that

$$\int_{|x|>z} x^2\,dF(x)=0(z^{-c})$$

is sufficient for

$$\sup_{t\in\mathbb{R}}(1+|t|)^{2+c}\,\Delta_n(t)=0(n^{-c/2}), \qquad 0<c<1.$$

(We remark that this can easily be derived from Theorem 2 of Nagaev ([8], p. 215) by truncating the random variable X appropriately and using Lemma 1 below.)

Stimulated by the results by Ibragimov [4], where necessary and sufficient conditions for $\sup_{t\in\mathbb{R}}\Delta_n(t)=0(n^{-c/2})$ are given in the case $0\leqq c\leqq 1$, we present a complete solution to the problem of nonuniform central limit bounds. Our conditions are an extension of Ibragimov's conditions to the case $c\geqq 0$. (For a slightly more general version of Ibragimov's result see Leslie [5], Theorem 1, p. 899.)

We furthermore give a characterization of Ibragimov's essential condition (i.e., Condition (2) below) which yields a proper interpretation of our theorem.

Theorem. *Assume that $EX=0$, $EX^2=1$. Let $c\geqq 0$ be given. Then there exists a positive constant d (depending only on F and c) such that for all $n\in\mathbb{N}$,*

$$\sup_{t\in\mathbb{R}}(1+|t|^{2+c})\,\Delta_n(t)\leqq d\,n^{-\frac{1}{2}\min(1,c)} \tag{1}$$

if and only if

$$\int_{|x|>z} x^2\,dF(x)=0(z^{-c}), \qquad z\to\infty, \tag{2}$$

and

$$\int_{-z}^{z} x^3\,dF(x)=0(1), \qquad z\to\infty, \tag{3}$$

holds in addition, if $c=1$.

2. Discussions

(i) Using Ibragimov's results we may conclude from our theorem the interesting fact that on the assumptions $EX=0$, $EX^2=1$ for $0 \leqq c \leqq 1$,

$$\sup_{t \in \mathbb{R}} \Delta_n(t) = 0(n^{-c/2}) \quad \text{iff } \sup_{t \in \mathbb{R}} (1+|t|^{2+c}) \Delta_n(t) = 0(n^{-c/2}).$$

(ii) The conditions of the theorem are no moment conditions, indeed. Condition (2) gives evidence about the tails of the distribution H defined by $H(x) = \int_{-\infty}^{x} u^2 \, dF(u)$ and allows the following intuitively clear interpretation of the theorem: The more is known of the speed of convergence of $\int_{-z}^{z} x^2 \, dF(x)$ towards $EX^2=1$ the more can be said about the rate of convergence in the Central Limit Theorem. Observe that this assertion needs some clarification. The nonuniform bounds being considered in this paper immediately yield corresponding uniform bounds. As preceding remarks show we only have a limited possibility to increase the order of convergence with respect to n. In the case $c>1$ the power of n in (1) remains fixed (being equal to $-\frac{1}{2}$) and the influence of knowing how fast $\int_{-z}^{z} x^2 \, dF(x)$ converges towards 1 is only apparent in the increase of the power of $|t|$ (and vice versa).

A further possibility of how to interprete our result follows from the characterization of Condition (2) given in Lemma 2 below: Here we prove that

(2) *is equivalent to* $P(|X|>z) = 0(z^{-2-c}), \quad z \to \infty$.

From this statement together with the theorem we conclude that *the existence of nonuniform central limit bounds for* $n^{-1/2} \sum_{i=1}^{n} X_i$ *heavily depends on the behavior of the tails of the distribution of* X.

(iii) In Lemma 1 we draw some conclusions about absolute moments of truncated versions of X, provided (2) is fulfilled.

Observe that Condition (2) is actually weaker than the assumption $E|X|^{2+c} < \infty$ in Theorem 3 of Michel [7], p. 103. (It appears that the crucial point in the proof of our theorem is sufficiency.): If $X = \operatorname{Var}(Y)^{-1/2} Y$, where the distribution of Y admits a Lebesgue-density proportional to $(1+|y|^{3+c})^{-1}$, $c>0$, then $E|X|^{2+c} = \infty$, whereas (2) is fulfilled. Recall that (3) holds true for distributions which are symmetric about zero.

(iv) The considerable significance the case $c=1$ has in Ibragimov's as well as in our result is manifested in the fact that in the expansion of the characteristic functions we have to bound $\left| \int_{|x| \leqq h} x^3 \, dF(x) \right|$ for a suitable truncation point h. Lemma 1 shows that from $\int_{|x|>z} x^2 \, dF(x) = 0(z^{-1})$ we can only conclude that $\int_{|x| \leqq h} |x|^3 \, dF(x) = 0(\log h)$ in contrast to the desirable $0(1)$. This is the reason, too, why in the proof of our theorem the cases $c=1$ and $c \neq 1$ need to be discussed separately.

3. Proof of the Theorem

(i) Since in the case $c=0$ the assertions immediately result from Chebyshev's inequality, we assume $c>0$ in the following. Furthermore, throughout the paper, $d>0$ denotes a generic constant only depending on the distribution function F and the (fixed) c.

(ii) Necessity. Since $Y\geqq 0$ implies $EY=\int_0^\infty P(Y>t)\,dt$, we have

$$\int_{|x|>z} x^2\,dF(x)=\int_0^\infty P(X^2 1_{\{|X|>z\}}>t)\,dt=\int_0^\infty P(|X|>\max(t^{1/2},z))\,dt.$$

Using (1) for $n=1$ we therefore obtain for $z>0$,

$$\begin{aligned}\int_{|x|>z} x^2\,dF(x)&\leqq \int_{|x|>z} x^2\,d\Phi(x)+d\int_0^\infty \{1+[\max(t^{1/2},z)]^{2+c}\}^{-1}\,dt\\ &\leqq \int_{|x|>z} x^2\,d\Phi(x)+dz^{-2-c}\int_0^{z^2} dt+d\int_{z^2}^\infty t^{-1-c/2}\,dt.\end{aligned}$$

Hence, (2) follows.

Furthermore, (3) follows from Ibragimov [2], since (1) for $c=1$ implies that $\sup_{t\in\mathbb{R}} \Delta_n(t)=0(n^{-1/2})$.

(iii)–(ix). Sufficiency.

(iii) By Ibragimov's result it suffices to consider the case $|t|\geqq 1$. Furthermore, if the case $t\geqq 1$ has been discussed, then for $t\leqq -1$, $\Delta_n(t)=|\tilde F_n(-t+0)-\Phi(-t)|$, where $\tilde F_n$ denotes the distribution function of $n^{-1/2}\sum_{i=1}^n(-X_i)$.

Hence, by $EX=0$, we may assume in the following w.l.o.g. that $t\geqq 1$.

(iv) $1\leqq t\leqq ((c+2)\log n)^{1/2}$. W.l.o.g. we may assume $t\geqq r^{-1}$, where

$$r=\tfrac{1}{4}(c+2)^{-1}\min(1,c), \tag{4}$$

and $n\geqq N_0$ (n sufficiently large: see (vii).)

Let

$$\bar X=X1_{\{|X|\leqq h\}} \quad \text{with} \quad h=rn^{1/2}t. \tag{5}$$

(This truncation implies in particular that $p(|t|)\exp[2hn^{-1/2}t]\leqq d\exp[t^2/4]$ for a polynomial p.)

Let $\bar F$ denote the distribution function of $\bar X$, $\bar F_n$ the distribution function of $n^{-1/2}\sum_{i=1}^n \bar X_i$. We remark that $\int f(x)\,d\bar F(x)=\int_{|x|\leqq h} f(x)\,dF(x)+f(0)\,P(|X|>h)$ for any measurable function f.

Evidently,

$$\Delta_n(t)=|1-F_n(t)-\Phi(-t)|\leqq|1-\bar F_n(t)-\Phi(-t)|+nP(|X|>h) \tag{6}$$

and

$$P(|X|>h) \leqq h^{-2} \int_{|x|>h} x^2 \, dF(x) \leqq d h^{-(2+c)}. \tag{7}$$

With

$$s = n^{-1/2} t \quad \text{let } \beta = \int \exp[s x] \, d\bar{F}(x). \tag{8}$$

Furthermore, set

$$G_n(x) = \beta^{-n} \int_{-\infty}^{x} \exp[t u] \, d\bar{F}_n(u). \tag{9}$$

Using

$$1 - \bar{F}_n(t) = \beta^n \int_t^\infty \exp[-t x] \, dG_n(x)$$

and

$$\Phi(-t) = \exp[t^2/2] \int_t^\infty \exp[-t x] \, d\Phi(x-t)$$

we therefore obtain from $|ab - cd| \leqq a\,|b-d| + cd\,|ac^{-1} - 1|$ that

$$\begin{aligned} &|1 - \bar{F}_n(t) - \Phi(-t)| \\ &\leqq \beta^n \left| \int_t^\infty \exp[-t x] \, d(G_n(x) - \Phi(x-t)) \right| + |\beta^n \exp[-t^2/2] - 1| \, \Phi(-t) \\ &\leqq \beta^n \exp[-t^2] \sup_{x \in \mathbb{R}} |G_n(x) - \Phi(x-t)| + |\beta^n \exp[-t^2/2] - 1| \, \Phi(-t). \end{aligned} \tag{10}$$

(v) With

$$c^* = \tfrac{1}{2} \min(1, c) \tag{11}$$

we have by Lemma 3,

$$\begin{aligned} |\beta - 1 - \tfrac{1}{2} n^{-1} t^2| &\leqq \left| \int_{|x| \leqq h} \exp[s x] \, dF(x) - 1 - \tfrac{1}{2} n^{-1} t^2 \right| + P(|X|>h) \\ &\leqq d n^{-1-c^*} \exp[2 r t^2]. \end{aligned} \tag{12}$$

Together with $n \geqq N_0$ this implies by standard arguments,

$$|\beta^n \exp[-t^2/2] - 1| \leqq d n^{-c^*} \exp[2 r t^2] \leqq d n^{-c^*} \exp[t^2/4]. \tag{13}$$

Using $t^2 \leqq (c+2) \log n$, (4), and (11) we furthermore obtain from the first inequality in (13) that

$$\beta^n \leqq d \exp[t^2/2] \tag{14}$$

(vi) We shall show now that

$$\sup_{x \in \mathbb{R}} |G_n(x) - \Phi(x-t)| \leqq d n^{-c^*} \exp[t^2/4]. \tag{15}$$

It seems to be most natural to prove (15) by using the Berry-Esséen Theorem, but it appears that in the case $c=1$ neither the Berry-Esséen Theorem (see part (iv) of the discussions) nor Ibragimov's Theorem (here one cannot see how the bound for $\sup_{t \in \mathbb{R}} \Delta_n(t)$ depends on the bounds which are given in the "moment conditions") give bounds which are appropriate for proving (15). Hence, for $c=1$ we have to show (15) directly by applying the usual Berry-Esséen techniques via characteristic functions. Since in the other cases only minor modifications are needed we prefer (for unity of presentation) to prove (15) directly for $c \neq 1$, too.

With

$$G(x)=\beta^{-1} \int_{-\infty}^{x} \exp[s u]\, d\bar{F}(u) \tag{16}$$

let

$$f(u)=\int e^{iux}\, dG(x).$$

In (vii) we show that for $u \in \mathbb{R}$,

$$|f(u)-1-ius+\tfrac{1}{2}u^2| \leqq R_0(un^{1/2}) \tag{17}$$

with

$$R_0(u)=d_0\,|u|\, n^{-1-c^*}(1+|u|)^2 \exp[t^2/4]. \tag{18}$$

Choose $\gamma \in (0,1)$ such that for $u \in \mathbb{R}$,

$$\gamma(3d_0+2)(1+|u|)^2 \leqq (1+u^2)/4 \tag{19}$$

and such that $|u| \leqq T$, where

$$T=\gamma n^{c^*} \exp[-t^2/4], \tag{20}$$

implies

$$|v|\, s+\tfrac{1}{2}v^2+R_0(u) \leqq 1/2, \tag{21}$$

where

$$v=n^{-1/2}u. \tag{22}$$

Then we obtain from (17) and (21) for $|u| \leqq T$,

$$|f(v)-1| \leqq 1/2. \tag{23}$$

Hence, for the same range of u,

$$\log f(v)=\log[1-(1-f(v))]=-\sum_{k=1}^{\infty} \frac{1}{k}(1-f(v))^k$$

which implies by (17) and (23),

$$\begin{aligned} &|\log f(v)-ivs+\tfrac{1}{2}v^2| \\ &\leqq |f(v)-1-ivs+\tfrac{1}{2}v^2|+\tfrac{1}{2}\sum_{k=2}^{\infty}|1-f(v)|^k \\ &\leqq R_0(u)+|1-f(v)|^2. \end{aligned} \tag{24}$$

Furthermore, by (17) and (22),

$$|1-f(v)| \leqq R_0(u)+|v|\, s+\tfrac{1}{2}v^2.$$

Since $|u| \leqq T$ implies $|v| = |u|\, n^{-1/2} \leqq \gamma \exp[-t^2/4] < 1$ and

$$\max(R_0(u), |v|\, s + \tfrac{1}{2} v^2) \leqq 1/2$$

(see (21)), we therefore obtain for $|u| \leqq T$,

$$\begin{aligned} |1 - f(v)|^2 &\leqq \tfrac{1}{2} R_0(u) + R_0(u) + |v|(s + \tfrac{1}{2}|v|)^2 \\ &= \tfrac{3}{2} R_0(u) + |u|\, n^{-1/2}(t + \tfrac{1}{2}|u|)^2\, n^{-1} \\ &\leqq 2R_0(u) + |u|\, n^{-1-c^*}(1 + |u|)^2\, t^2 \\ &\leqq 2R_0(u) + 2\,|u|\, n^{-1-c^*}(1 + |u|)^2 \exp[t^2/4]. \end{aligned} \tag{25}$$

Let

$$R_1(u) = (3\, d_0 + 2)\, |u|\, n^{-1-c^*}(1 + |u|)^2 \exp[t^2/4]. \tag{26}$$

Then (24), (25), (18), (26), and (22) give for $|u| \leqq T$,

$$|\log f(v) - i v s + \tfrac{1}{2} v^2| \leqq R_1(u). \tag{27}$$

With G_n defined in (9) let

$$f_n(u) = \int e^{iux}\, dG_n(x)$$

and set

$$h(u) = \int e^{iux}\, d\Phi(x - t) = \exp[iut - \tfrac{1}{2} u^2].$$

From $s = n^{-1/2} t$ and the definitions of G_n and G (see (8), (9), and (16)) we obtain $f_n(u) = f(u n^{-1/2})^n = f(v)^n$.

Hence, $|u| \leqq T$ implies by (27),

$$\begin{aligned} &|u|^{-1}\, |f_n(u) - h(u)| \\ &= |u|^{-1} \exp[-u^2/2]\, |\exp[n \log f(v) - iut + \tfrac{1}{2} u^2] - 1| \\ &\leqq |u|^{-1} \exp[-u^2/2]\, n R_1(u) \exp[n R_1(u)]. \end{aligned}$$

Hence by Esséen's Lemma ([2], p. 32), (26), (19), and (20),

$$\begin{aligned} &\sup_{x \in \mathbb{R}} |G_n(x) - \Phi(x - t)| \\ &\leqq d \int_{-T}^{T} |u|^{-1}\, |f_n(u) - h(u)|\, du + d T^{-1} \\ &\leqq d\, n^{-c^*} \exp[t^2/4]. \end{aligned}$$

This is the assertion of (15).

(vii) It remains to show (17).

a) $0 < c \neq 1$. We have

$$\begin{aligned} &|f(u) - 1 - ius + \tfrac{1}{2} u^2| \\ &\leqq \int |e^{iux} - 1 - iux + \tfrac{1}{2} u^2 x^2|\, dG(x) + |u|\, |s - \int x\, dG(x)| \\ &\quad + \tfrac{1}{2} u^2\, |1 - \int x^2\, dG(x)| \\ &\leqq \tfrac{1}{6} |u|^3 \int |x|^3\, dG(x) + |u|\, |s - \int x\, dG(x)| + \tfrac{1}{2} u^2\, |1 - \int x^2\, dG(x)|. \end{aligned} \tag{28}$$

From Lemmas 1 and 3 and the remarks preceding (5) and (6), respectively,

$$\begin{aligned}\int |x|^3\, dG(x) &= \beta^{-1} \int |x|^3 \exp[sx]\, d\bar{F}(x)\\ &\leqq 2\int |x|^3 \exp[sx]\, d\bar{F}(x)\\ &= 2 \int_{|x|\leqq h} |x|^3 \exp[sx]\, dF(x)\\ &\leqq 2\exp[hs] \int_{|x|\leqq h} |x|^3\, dF(x)\\ &\leqq d\, h^{1-2c^*} \exp[hs]. \end{aligned} \tag{29}$$

Furthermore, by Lemma 3,

$$\begin{aligned}|s-\int x\, dG(x)| &= |s-\beta^{-1} \int_{|x|\leqq h} x \exp[sx]\, dF(x)|\\ &\leqq d\, h^{-1-2c^*} \exp[2hs]\end{aligned} \tag{30}$$

and

$$|1-\int x^2\, dG(x)| \leqq d\, h^{-2c^*} \exp[2hs]. \tag{31}$$

Hence (28)–(31) imply

$$\begin{aligned}&|f(u)-1-ius+\tfrac{1}{2}u^2|\\ &\leqq d\, h^{-1-2c^*} |u|(1+2|u|\, h+u^2 h^2) \exp[2hs].\end{aligned}$$

This together with (5), (4), and (8) implies (17). (We have used $t \geqq 1$, $r<1$, and $t^{1-2c^*} \exp[2rt^2] \leqq d \exp[t^2/4]$.)

b) $c=1$. Obviously,

$$\begin{aligned}&|\int e^{iux}\, dG(x)-1-iu\int x\, dG(x)+\tfrac{1}{2}u^2 \int x^2\, dG(x)|\\ &\leqq \int_{|xu|\leqq 1} |e^{iux}-1-iux+\tfrac{1}{2}u^2x^2+\tfrac{1}{6}iu^3x^3|\, dG(x)\\ &\quad +\tfrac{1}{6}|u|^3 |\int_{|xu|\leqq 1} x^3\, dG(x)| + \int_{|xu|>1} |e^{iux}-1-iux|\, dG(x)\\ &\quad +\tfrac{1}{2}u^2 \int_{|xu|>1} x^2\, dG(x)\\ &\leqq u^2 \int_{|xu|>1} x^2\, dG(x)+\tfrac{1}{6}|u|^3 |\int_{|xu|\leqq 1} x^3\, dG(x)|\\ &\quad +\tfrac{1}{24}u^4 \int_{|xu|\leqq 1} x^4\, dG(x).\end{aligned}$$

From this we obtain (17) in the same way as in the preceding case using Lemmas 1 and 3 together with

$$\int_{|xu|>1} x^2\, dG(x) \leqq d\,|u| \exp[hs],$$

$$|\int_{|xu|\leqq 1} x^3\, dG(x)| \leqq d \exp[2hs],$$

and

$$|u| \int_{|xu|\leqq 1} x^4\, dG(x) \leqq d \exp[hs]$$

(viii) From (6), (7), (10), (12), (13), (14), (15), and $\Phi(-t) \leqq d \exp[-t^2/2]$ we obtain for $1 \leqq t \leqq ((c+2)\log n)^{1/2}$, $n \in \mathbb{N}$,

$$\Delta_n(t) \leqq d n^{-c^*} \exp[-t^2/4] \leqq d t^{-2-c} n^{-c^*}. \tag{32}$$

(ix) $((c+2)\log n)^{1/2} \leqq t$. Let

$$h = r n^{1/2} t, \quad \text{where} \quad r = (c+2)^{-2} \min(1, c). \tag{33}$$

Since $h < 1$ implies $t < r^{-1} n^{-1/2} \leqq d$, we may assume w.l.o.g. that $h \geqq 1$.

Let $\bar{X} = X 1_{\{|X| \leqq h\}}$ and denote by $\bar{F}_n$ the distribution function of $n^{-1/2} \sum_{i=1} \bar{X}_i$. By Markov's inequality,

$$1 - \bar{F}_n(t) \leqq \beta^n \exp[-s n^{1/2} t], \tag{34}$$

where

$$s = n^{-1/2} t^{-1} (c \log n + \tfrac{1}{2}(c+2)^2 \log t) \tag{35}$$

and

$$\beta = \int \exp[s x] \, d\bar{F}_1(x).$$

By Lemma 3, and the remark preceding formula (6),

$$\begin{aligned} \beta &\leqq 1 + \tfrac{1}{2} s^2 + d h^{-2(1+c^*)} \exp[2sh] + P(|X| > h) \\ &\leqq 1 + \tfrac{1}{2} s^2 + d h^{-2(1+c^*)} \exp[2sh], \end{aligned} \tag{36}$$

where c^* is given by (11). (Recall that $d > 0$ is a generic constant.) Using $t^2 \geqq (c+2) \log n$ we immediately obtain from (33) and (35) that

$$s^2 \leqq n^{-1} c (\log n + (c+2) \log t + d) \tag{37}$$

and

$$d h^{-2(1+c^*)} \exp[2sh] \leqq n^{-1} d. \tag{38}$$

Hence, by (36)–(38), using $\beta^n \leqq \exp[n(\beta - 1)]$,

$$\beta^n \leqq d n^{c/2} t^{c(c+2)/2}. \tag{39}$$

Together with (34) and (35), (39) implies

$$1 - \bar{F}_n(t) \leqq d n^{-c^*} t^{-(2+c)}. \tag{40}$$

Since $t \geqq ((c+2)\log n)^{1/2}$ implies $\Phi(-t) \leqq d \exp[-\frac{1}{2} t^2] \leqq d n^{-c^*} t^{-(2+c)}$, we obtain from (6), (7), (33), and (40)

$$\Delta_n(t) \leqq d n^{-c^*} t^{-(2+c)} \tag{41}$$

for $t \geqq ((c+2)\log n)^{1/2}$, $n \in \mathbb{N}$.

Sufficiency now follows from (iii), (32), and (41).

4. Lemmas

The following Lemmas 1 and 2 give an idea of how to interpret the essential condition $\int_{|x|>z} x^2 \, dF(x) = 0(z^{-c})$, $z \to \infty$, of our theorem.

Lemma 1. *Assume that* $EX^2=1$ *and* $R(z)=\int_{|x|>z} x^2\,dF(x)\leqq b z^{-c}$, $z>0$, *for positive constants* b *and* c. *Then*

(i) $E|X|^\alpha \leqq 1+b(\alpha-2)(c+2-\alpha)^{-1}$, *if* $\alpha\in(2, 2+c)$

(ii) $\int_{-z}^{z} |x|^{2+c}\,dF(x)=0(\log z)$, $z\to\infty$.

(iii) $\int_{-z}^{z} |x|^\alpha\,dF(x)=0(z^{\alpha-2-c})$, $z\to\infty$, *if* $\alpha>2+c$.

Proof. The assertions follow from

$$\begin{aligned}\int_{-z}^{z} |x|^\alpha\,dF(x) &= -\int_0^z x^{\alpha-2}\,dR(x)\leqq(\alpha-2)\int_0^z x^{\alpha-3} R(x)\,dx\\ &\leqq(\alpha-2)\left(R(0)\int_0^1 x^{\alpha-3}\,dx+\int_1^z x^{\alpha-3} R(x)\,dx\right)\\ &\leqq 1+b(\alpha-2)\int_1^z x^{\alpha-3-c}\,dx.\end{aligned}$$

Lemma 2. *Assume that* $EX^2=1$. *Let* $c>0$ *be given. Then*

$$\int_{|x|>z} x^2\,dF(x)=0(z^{-c}),\quad z\to\infty,\qquad \textit{iff}\quad P(|X|>z)=0(z^{-2-c}),\quad z\to\infty.$$

Proof. Set $S(z)=P(|X|>z)$ and assume that there exists a positive constant b with $S(z)\leqq b z^{-2-c}$, $z>0$. Then,

$$\begin{aligned}\int_{|x|>z} x^2\,dF(x) &= -\int_z^\infty x^2\,dS(x)=z^2 S(z)+2\int_z^\infty x S(x)\,dx\\ &\leqq b z^{-c}+2b\int_z^\infty x^{-1-c}\,dx=b c^{-1}(c+2) z^{-c}.\end{aligned}$$

Since the other conclusion is obvious, the assertion follows.

Lemma 3. *Assume that* $EX=0$, $EX^2=1$, $\int_{|x|>z} x^2\,dF(x)=0(z^{-c})$, $z\to\infty$, *for some* $c>0$, *and that* $\int_{-z}^{z} x^3\,dF(x)=0(1)$, $z\to\infty$, *in addition, if* $c=1$. *Then there exists a constant* $b>0$ *such that for all* $s>0$, $h\geqq 1$, *and* $m\in\{0, 1, 2, 3\}$,

$$\begin{aligned}&\left|\int_{|x|\leqq h} x^m \exp[s x]\,dF(x)-(\delta_{0m}+\delta_{2m}+s\delta_{1m}+\tfrac{1}{2}s^2\delta_{0m})\right|\\ &\leqq b h^{m-2-\min(1,c)}\exp[2hs],\end{aligned}$$

where δ_{ij} *denotes Kronecker's symbol.*

Proof. We shall give a proof for $m=0$ only, as the other cases can be handled by using the same arguments. Let

$$T(s,h)=\left|\int_{|x|\leqq h} \exp[s x]\,dF(x)-1-\tfrac{1}{2}s^2\right|$$

(i) $0<c\neq 1$. By a Taylor expansion of $u\to\exp[u]$ around $u=0$ we obtain from $EX=0$ and $EX^2=1$,

$$T(s,h)\leqq P(|X|>h)+s\int_{|x|>h}|x|\,dF(x)+\tfrac{1}{2}s^2\int_{|x|>h}x^2\,dF(x)$$
$$+\tfrac{1}{6}s^3\exp[hs]\int_{|x|\leqq h}|x|^3\,dF(x).$$

The first three terms of the r.h.s. of this inequality are bounded by

$$(h^{-2}+sh^{-1}+\tfrac{1}{2}s^2)\int_{|x|>h}x^2\,dF(x).$$

Furthermore, by Lemma 1,

$$\int_{|x|\leqq h}|x|^3\,dF(x)\leqq d\,h^{1-\min(1,c)}.$$

Hence, using $h\geqq 1$ and $hs>0$,

$$T(s,h)\leqq d\,h^{-2-\min(1,c)}(1+hs+\tfrac{1}{2}(hs)^2+\tfrac{1}{6}(sh)^3)\exp[hs],$$

which implies the assertion.

(ii) $c=1$. By adding one additional term to the Taylor expansion of $\exp[u]$ we obtain in the same way

$$T(s,h)\leqq(h^{-2}+sh^{-1}+\tfrac{1}{2}s^2)\int_{|x|>h}x^2\,dF(x)$$
$$+\tfrac{1}{6}s^3\Big|\int_{|x|\leqq h}x^3\,dF(x)\Big|+\tfrac{1}{24}s^4\exp[hs]\int_{|x|\leqq h}x^4\,dF(x),$$

which again yields the assertion of the lemma, as $\Big|\int_{|x|\leqq h}x^3\,dF(x)\Big|\leqq d$.

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