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Generalized Maximum Likelihood Estimators for Ranked Means*

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Suppose that observations from populations $\pi_1, \ldots, \pi_k$ $(k \geqq 2)$ are normally distributed with unknown means $\mu_1, \ldots, \mu_k$ (respectively) and a common known variance σ^2. Let $\mu_{[1]} \leqq \cdots \leqq \mu_{[k]}$ denote the ranked means. Several ranking and selection procedures take n independent observations from each population, denote the sample mean of the n observations from π_i by $\bar{X}_i$ $(i=1, \ldots, k)$, and utilize the ranked sample means $\bar{X}_{[1]} \leqq \cdots \leqq \bar{X}_{[k]}$. (See [2] for details.) We assume throughout that both the numerical values of $\mu_1, \ldots, \mu_k$ and the pairings of the $\mu_{[1]}, \ldots, \mu_{[k]}$ with the populations $\pi_1, \ldots, \pi_k$ are completely unknown and consider problems of estimation of $\mu_{[i]}$ $(1 \leqq i \leqq k)$ based on the statistics provided by the single-stage rule stated above, and utilizing recent work of Weiss and Wolfowitz.

Generalized maximum likelihood estimators (GMLE's), introduced by Weiss and Wolfowitz [6], provide (where available) asymptotically efficient estimators, whereas this is not always true for MLE's even if the latter can be found. As noted in [1], for the case of estimating $\mu_{[1]}, \ldots, \mu_{[k]}$, what is meant by "the MLE" is not clear. One possibility, the IMLE, is difficult to compute and may or may not possess desirable properties. Most classical MLE theory assumes i.i.d. observations and is therefore not applicable in our case, since the IMLE is in this case the MLE based on non-i.i.d. observations: the ranked data. The theory of Weiss and Wolfowitz [6] allows for more general situations, although most of their applications are to i.i.d. "non-regular" cases. (Corrections to Weiss and Wolfowitz [6] are contained in Weiss and Wolfowitz [7], in Weiss and Wolfowitz [9], and below. An additional example is given in Weiss and Wolfowitz [8].)

We first summarize the results of Weiss and Wolfowitz [6] for the case $k=2$.

(1) *Definition.* Let Θ be a closed region in $\mathscr{R}^2$, $\Theta \subseteq \bar{\Theta}$ with $\bar{\Theta}$ a closed region such that every finite boundary point of Θ is an inner point of $\bar{\Theta}$.

(2) *Definition.* For each n let $X(n)$ denote the (finite) vector of r.v.'s of which the estimator is to be a function.

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(3) *Definition.* Let $K_n(x|\theta)$ be the density, with respect to a σ-finite measure μ_n, of $X(n)$ at the point x (of the appropriate space) when θ is the "true" value of the unknown parameter.

(4) *Definition.* Let $r=(r_1, r_2)$ be fixed and positive. $\{Z_{n1}(X(n), r), Z_{n2}(X(n), r)\}$ is a sequence of GMLE's if, for each $\theta=(\theta_1, \theta_2)\in\Theta$, (A') and (B') below are satisfied.

(5) *Condition* (A'). There is a sequence of positive constants $\{k_1(n), k_2(n)\}$ such that $k_1(n)\to\infty$, $k_2(n)\to\infty$, and a function $L(y_1, y_2|\theta)$ such that $L(\cdot|\theta)$ is a continuous d.f., and, for any $y=(y_1, y_2)$ and *any* integers h_1 and h_2

$$\lim_{n\to\infty} P_{\theta_1+\frac{h_1 r_1}{k_1(n)}, \theta_2+\frac{h_2 r_2}{k_2(n)}}\left[k_1(n)\left(Z_{n1}-\theta_1-\frac{h_1 r_1}{k_1(n)}\right)\leqq y_1, k_2(n)\left(Z_{n2}-\theta_2-\frac{h_2 r_2}{k_2(n)}\right)\leqq y_2\right]$$
$$=L(y_1, y_2|\theta_1, \theta_2).$$

(6) *Condition* (B'). For *any* integers h_1, h_2 there exists a set $S_n(\theta, h_1, h_2)$ in the space of $X(n)$ such that

(7) $\lim P_{\alpha_{ij}}[X(n)\in S_n(\theta, h_1, h_2)]=1 \quad (i, j=0, 1)$,

where

(8) $$\alpha_{ij}=\left(\theta_1+\frac{(h_1+i)r_1}{k_1(n)}, \theta_2+\frac{(h_2+j)r_2}{k_2(n)}\right),$$

and there exist sequences

(9) $\{a_{nij}(X(n), \theta, h_1, h_2)\} \quad (i, j=0, 1)$

of (two-dimensional) r.v.'s such that, as $n\to\infty$, $a_{nij}=(a_{nij1}, a_{nij2})$ converges stochastically to zero when α_{ij} is the parameter of the density of $X(n)$, and such that, whenever $X(n)\in S_n(\theta, h_1, h_2)$, we have the following: Let

(10) $M=\max\{K_n(X(n)|\alpha_{ij}), (i, j=0, 1)\}$,

(11) $$m=(m_1, m_2)=\left(\theta_1+\frac{(h_1+\frac{1}{2})r_1}{k_1(n)}, \theta_2+\frac{(h_2+\frac{1}{2})r_2}{k_2(n)}\right).$$

Then, where "$(a<b, c<d)$" means "$(a\leqq b, c<d)$ or $(a<b, c\leqq d)$,"

(12a) $$M=K_n(X(n)|\alpha_{00}) \Rightarrow \left(Z_{n1}<m_1+\frac{a_{n001}}{k_1(n)}, Z_{n2}<m_2+\frac{a_{n002}}{k_2(n)}\right),$$

(12b) $$M=K_n(X(n)|\alpha_{01}) \Rightarrow \left(Z_{n1}<m_1+\frac{a_{n011}}{k_1(n)}, Z_{n2}>m_2+\frac{a_{n012}}{k_2(n)}\right),$$

(12c) $$M=K_n(X(n)|\alpha_{10}) \Rightarrow \left(Z_{n1}>m_1+\frac{a_{n101}}{k_1(n)}, Z_{n2}<m_2+\frac{a_{n102}}{k_2(n)}\right),$$

(12d) $$M=K_n(X(n)|\alpha_{11}) \Rightarrow \left(Z_{n1}>m_1+\frac{a_{n111}}{k_1(n)}, Z_{n2}>m_2+\frac{a_{n112}}{k_2(n)}\right).$$

(13) **Theorem** (Weiss and Wolfowitz). *Let* $\{Z_{n1}(X(n), r), Z_{n2}(X(n), r)\}$ *be a sequence of* GMLE's. *Let* $\{T_n\}$ *be any sequence of estimators of* θ *such that, for fixed*

$r=(r_1, r_2)>0$ and all integers h_1, h_2

$$\lim P_{\theta_1,\theta_2}\left[-\frac{r_1}{2}<k_1(n)(T_{n1}-\theta_1)\leqq\frac{r_1}{2},\ -\frac{r_2}{2}<k_2(n)(T_{n2}-\theta_2)\leqq\frac{r_2}{2}\right]$$

$$=\lim P_{\theta_1+\frac{h_1r_1}{k_1(n)},\,\theta_2+\frac{h_2r_2}{k_2(n)}}\left[-\frac{r_1}{2}<k_1(n)\left(T_{n1}-\theta_1-\frac{h_1r_1}{k_1(n)}\right)\leqq\frac{r_1}{2},\right.$$
$$\left.-\frac{r_2}{2}<k_2(\mathrm{n})\left(T_{n2}-\theta_2-\frac{h_2r_2}{k_2(n)}\right)\leqq\frac{r_2}{2}\right]$$

for any $\theta\in\Theta$. Then

$$\lim P_\theta\left[-\frac{r_1}{2}<k_1(n)(Z_{n1}-\theta_1)<\frac{r_1}{2},\ -\frac{r_2}{2}<k_2(n)(Z_{n2}-\theta_2)<\frac{r_2}{2}\right]$$
$$\geqq\lim\sup P_\theta\left[-\frac{r_1}{2}<k_1(n)(T_{n1}-\theta_1)\leqq\frac{r_1}{2},\ -\frac{r_2}{2}<k_2(n)(T_{n2}-\theta_2)\leqq\frac{r_2}{2}\right].$$

Note that on page 78 of Weiss and Wolfowitz [6], condition (B') is mis-stated; therein, in (3.13) through (3.16) (corresponding to our (12a) through (12d) above)

$$\{a_{n001}, a_{n011}, a_{n101}, a_{n111};\ a_{n002}, a_{n012}, a_{n102}, a_{n112}\}$$

should be

$$\left\{\frac{a_{n001}}{k_1(n)}, \frac{a_{n011}}{k_1(n)}, \frac{a_{n101}}{k_1(n)}, \frac{a_{n111}}{k_1(n)};\ \frac{a_{n002}}{k_2(n)}, \frac{a_{n012}}{k_2(n)}, \frac{a_{n102}}{k_2(n)}, \frac{a_{n112}}{k_2(n)}\right\}.$$

Examination of the modification of the proof of pages 73–74 of Weiss and Wolfowitz [6] used for the proof of their Theorem 3.2 (Theorem (13) above) shows that without this change the quantities a_{nijl} multiplied by the normalizing factors $k_1(n)$ and $k_2(n)$ would occur, and would not necessarily converge stochastically to zero (under the appropriate parameters). In their multi-parameter examples VI, VII, and VIII Weiss and Wolfowitz [6] seem to satisfy the corrected (B'). (In example VIII this is not as clear as in examples VI and VII.)

We now investigate the application of these results to the estimation of $\mu_{[1]}, \ldots, \mu_{[k]}$. For $k\geqq 2$ we now choose

(14) $X(n)=(\bar{X}_{[1]}, \ldots, \bar{X}_{[k]})$
$K_n(x|\theta)=K_n(x|\mu)=f^{(\mu)}_{\bar{X}_{[1]},\ldots,\bar{X}_{[k]}}(x_1, \ldots, x_k)\equiv f_{\bar{X}_{[1]},\ldots,\bar{X}_{[k]}}(x_1, \ldots, x_k)$
$\mu_n=$ Lebesgue measure on $\mathscr{R}^k$.

Define $\Omega_0=\{(\mu_1, \ldots, \mu_k)\colon \mu_i\in\mathscr{R}\ (i=1, \ldots, k)\}$. We would also like to choose $\Theta=\{\mu\colon \mu\in\Omega_0, \mu_1=\mu_{[1]}, \ldots, \mu_k=\mu_{[k]}\}$, $\bar{\Theta}=\mathscr{R}^k$ (which would satisfy (1)), but by Theorem (A.10) (see Appendix) this would not allow satisfaction of condition (A') (essentially because $\mu\in\Theta\cap[\Omega(\neq)]^c$ would not uniquely specify the limiting distribution). Thus, we fix $\eta^*>0$ and choose

(15) $\Theta(\eta^*)=\{\mu\colon \mu\in\Theta, \mu_k-\mu_{k-1}\geqq\eta^*, \mu_{k-1}-\mu_{k-2}\geqq\eta^*, \ldots, \mu_2-\mu_1\geqq\eta^*\}$
$\bar{\Theta}=\Theta(\eta^*/2)$.

(Although our results below would hold if we simply excluded the boundaries of our desired Θ, that set would not be closed.) Since our results lack real dependence on η^*, we have essentially only eliminated the boundary (where equalities exist).

For $k \geqq 2$, consider the sequence

(16) $\{Z_{n1}(X(n), r), \ldots, Z_{nk}(X(n), r)\} = \{\bar{X}_{[1]}, \ldots, \bar{X}_{[k]}\}$

with $r = (r_1, \ldots, r_k)$ fixed and positive.

(17) **Theorem.** *For $k \geqq 2$, condition (A') (or, more properly, its generalization to $k \geqq 2$) holds for the sequence (16) for arbitrary $r > 0$, with $k_1(n) = k_2(n) = \sqrt{n}/\sigma$.*

Proof. This follows from Theorem (A.8).

(18) **Lemma.** *Let h_1 and h_2 be any integers. Choose*

$$S_n(\mu, h_1, h_2) = \mathscr{R}^k \cap \{\mu_{[1]} - \varepsilon_n \leqq \bar{X}_{[1]} \leqq \mu_{[1]} + \varepsilon_n, \mu_{[2]} - \varepsilon_n \leqq \bar{X}_{[2]} \leqq \mu_{[2]} + \varepsilon_n\},$$

where $\varepsilon_n = \sigma/n^\delta$ ($0 < \delta < \frac{1}{2}$ fixed). Then (for $i, j = 0, 1$)

$$\lim P_{\alpha_{ij}}[X(n) \in S_n(\mu, h_1, h_2)] = 1.$$

Proof. By (8), here $\alpha_{ij} = (\mu_{[1]} + (h_1 + i) r_1 \sigma/\sqrt{n}, \mu_{[2]} + (h_2 + j) r_2 \sigma/\sqrt{n})$, and (setting $a_1 = (h_1 + i) r_1$, $a_2 = (h_2 + j) r_2$)

$$(19) \quad \begin{aligned} P_{\alpha_{ij}}[X(n) \in S_n(\mu, h_1, h_2)] &= P_{\mu + a\sigma/\sqrt{n}}[\mu_{[i]} - \sigma/n^\delta \leqq \bar{X}_{[i]} \leqq \mu_{[i]} + \sigma/n^\delta \ (i = 1, 2)] \\ &= P_{\mu + a\sigma/\sqrt{n}}\Bigg[-n^{\frac{1}{2}-\delta} - a_1 \leqq \frac{\bar{X}_{[1]} - \mu_{[1]} - a_1 \sigma/\sqrt{n}}{\sigma/\sqrt{n}} \leqq n^{\frac{1}{2}-\delta} - a_1, \\ &\qquad -n^{\frac{1}{2}-\delta} - a_2 \leqq \frac{\bar{X}_{[2]} - \mu_{[2]} - a_2 \sigma/\sqrt{n}}{\sigma/\sqrt{n}} \leqq n^{\frac{1}{2}-\delta} - a_2\Bigg]. \end{aligned}$$

However, by Theorem (A.8) the random quantities of (19) approach a joint limiting distribution, while the respective upper and lower limits on those quantities tend to $\pm\infty$. (In fact, the result is proven for any fixed $a = (a_1, a_2)$ and not just for $((h_1 + i) r_1, (h_2 + j) r_2)$.)

As noted in the proof of Lemma (18), for our case we have (for $i, j = 0, 1$)

(20) $\alpha_{ij} = (\mu_{[1]} + (h_1 + i) r_1 \sigma/\sqrt{n}, \mu_{[2]} + (h_2 + j) r_2 \sigma/\sqrt{n})$.

(21) **Lemma.** *If $k = 2$, then (for $i, j = 0, 1$)*

$$K_n(x|\alpha_{ij})\, 2\pi \frac{\sigma^2}{n} e^{\frac{r_1^2 h_1^2}{2} + \frac{r_2^2 h_2^2}{2}} = a' e^{r_1 i \frac{x_1 - \mu_{[1]}}{\sigma/\sqrt{n}} - i(h_1 + \frac{1}{2} i) r_1^2 + r_2 j \frac{x_2 - \mu_{[2]}}{\sigma/\sqrt{n}} - j(h_2 + \frac{1}{2} j) r_2^2}$$
$$+ b' e^{r_2 j \frac{x_1 - \mu_{[2]}}{\sigma/\sqrt{n}} - j(h_2 + \frac{1}{2} j) r_2^2 + r_1 i \frac{x_2 - \mu_{[1]}}{\sigma/\sqrt{n}} - i(h_1 + \frac{1}{2} i) r_1^2},$$

where

$$a' = e^{-\frac{1}{2}\left(\frac{x_1 - \mu_{[1]}}{\sigma/\sqrt{n}}\right)^2 - \frac{1}{2}\left(\frac{x_2 - \mu_{[2]}}{\sigma/\sqrt{n}}\right)^2} e^{r_1 h_1 \frac{x_1 - \mu_{[1]}}{\sigma/\sqrt{n}}} e^{r_2 h_2 \frac{x_2 - \mu_{[2]}}{\sigma/\sqrt{n}}},$$
$$b' = e^{-\frac{1}{2}\left(\frac{x_1 - \mu_{[2]}}{\sigma/\sqrt{n}}\right)^2 - \frac{1}{2}\left(\frac{x_2 - \mu_{[1]}}{\sigma/\sqrt{n}}\right)^2} e^{r_2 h_2 \frac{x_1 - \mu_{[2]}}{\sigma/\sqrt{n}}} e^{r_1 h_1 \frac{x_2 - \mu_{[1]}}{\sigma/n}}.$$

Proof. (Note that $a'>0$ and $b'>0$ involve only $\sigma, n, x_1, x_2, \mu_{[1]}, \mu_{[2]}, r_1, r_2, h_1$, and h_2, and not i and j.) From (14), (20), and the joint density of $\bar{X}_{[1]}, \ldots, \bar{X}_{[k]}$ (see [3]), one finds $K_n(x|\alpha_{ij})2\pi\sigma^2/n$ and the result follows.

(22) **Lemma.** *There exist a_{nij1} and a_{nij2} (which may depend on $X(n), \mu, h_1$, and h_2) which converge stochastically to zero when α_{ij} is the parameter of the density of $X(n)$ $(i,j=0,1)$ such that, if $X(n)\in S_n(\mu, h_1, h_2)$ and $M=K_n(X(n)|\alpha_{ij})$, then*

(i) *for* $i,j=0,0$

$$\frac{\bar{X}_{[1]}-\mu_{[1]}}{\sigma/\sqrt{n}}<(h_1+\tfrac{1}{2})r_1+a_{n001}$$

(23) *and*

$$\frac{\bar{X}_{[2]}-\mu_{[2]}}{\sigma/\sqrt{n}}<(h_2+\tfrac{1}{2})r_2+a_{n002}$$

(ii) *for* $i,j=0,1$

$$\frac{\bar{X}_{[1]}-\mu_{[1]}}{\sigma/\sqrt{n}}<(h_1+\tfrac{1}{2})r_1+a_{n011}$$

(24) *and*

$$\frac{\bar{X}_{[2]}-\mu_{[2]}}{\sigma/\sqrt{n}}>(h_2+\tfrac{1}{2})r_2+a_{n012}$$

(iii) *for* $i,j=1,0$

$$\frac{\bar{X}_{[1]}-\mu_{[1]}}{\sigma/\sqrt{n}}>(h_1+\tfrac{1}{2})r_1+a_{n101}$$

(25) *and*

$$\frac{\bar{X}_{[2]}-\mu_{[2]}}{\sigma/\sqrt{n}}<(h_2+\tfrac{1}{2})r_2+a_{n102}$$

(iv) *for* $i,j=1,1$

$$\frac{\bar{X}_{[1]}-\mu_{[1]}}{\sigma/\sqrt{n}}>(h_1+\tfrac{1}{2})r_1+a_{n111}$$

(26) *and*

$$\frac{\bar{X}_{[2]}-\mu_{[2]}}{\sigma/\sqrt{n}}>(h_2+\tfrac{1}{2})r_2+a_{n112}.$$

Proof. (i) Case $i,j=0,0$. For simplicity, write x for $X(n)$, x_1 for $\bar{X}_{[1]}$, x_2 for $\bar{X}_{[2]}$, μ_1 for $\mu_{[1]}$, and μ_2 for $\mu_{[2]}$. Since $K_n(x|\alpha_{00})\geqq K_n(x|\alpha_{10})$, by Lemma (21)

$$(27)\quad a'+b'\geqq a'e^{r_1\frac{x_1-\mu_1}{\sigma/\sqrt{n}}-(h_1+\frac{1}{2})r_1^2}+b'e^{r_1\frac{x_1-\mu_1}{\sigma/\sqrt{n}}-(h_1+\frac{1}{2})r_1^2}$$

since $x_1 \leqq x_2$. Thus

$$(28)\quad \frac{x_1-\mu_1}{\sigma/\sqrt{n}} \leqq (h_1+\tfrac{1}{2})r_1.$$

We may take $a_{n001}=\dfrac{1}{n}$.

Since $K_n(x|\alpha_{00}) \geqq K_n(x|\alpha_{01})$,

$$(29)\quad a'+b' \geqq a' e^{r_2 \frac{x_2-\mu_2}{\sigma/\sqrt{n}} - (h_2+\frac{1}{2}) r_2^2}$$

or

$$(30)\quad 1+\frac{b'}{a'} \geqq e^{r_2 \left[\frac{x_2-\mu_2}{\sigma/\sqrt{n}} - (h_2+\frac{1}{2}) r_2\right]}.$$

Now,

$$(31)\quad 0<\frac{b'}{a'}=\begin{cases} e^{-\frac{n}{\sigma^2}(x_2-x_1)\left\{(\mu_2-\mu_1)-\frac{\sigma}{\sqrt{n}}(r_1h_1-r_2h_2)\right\}} & \text{if } \mu_2>\mu_1 \\ e^{\frac{x_2-x_1}{\sigma/\sqrt{n}}(r_1h_1-r_2h_2)} & \text{if } \mu_2=\mu_1. \end{cases}$$

Since $\mu_2>\mu_1$, from (30) and (31) we find that

$$(32)\quad \frac{x_2-\mu_2}{\sigma/\sqrt{n}} \leqq (h_2+\tfrac{1}{2})r_2+\frac{1}{r_2}\ln\left(1+\frac{b'}{a'}\right).$$

The choice $a_{n002}=\dfrac{1}{r_2}\ln\left(1+\dfrac{b'}{a'}\right)$ is effective (recall that a_{n002} may depend on μ, as well as on $X(n)$, h_1, and h_2). For,

$$(33)\quad P_{\alpha_{00}}[|(\bar{X}_{[2]}-\bar{X}_{[1]})-(\mu_{[2]}-\mu_{[1]})|<\varepsilon] \geqq P_{\alpha_{00}}[|\bar{X}_{[2]}-\mu_{[2]}|<\varepsilon/2, |\bar{X}_{[1]}-\mu_{[1]}|<\varepsilon/2].$$

By Theorem (A.8), as $n\to\infty$

$$(34)\quad P_{\alpha_{00}}[|\bar{X}_{[1]}-\mu_{[1]}|<\varepsilon/2] = P_{\alpha_{00}}\left[-h_1r_1-\frac{\varepsilon}{2}\frac{\sqrt{n}}{\sigma}<\frac{\sqrt{n}}{\sigma}(\bar{X}_{[1]}-\mu_{[1]}-h_1r_1\sigma/\sqrt{n})<\frac{\varepsilon}{2}\frac{\sqrt{n}}{\sigma}-h_1r_1\right]\to 1,$$

a similar result holding for $\bar{X}_{[2]}$. By Lemma (A.1), the r.h.s. of (33) converges to 1 as $n\to\infty$, so that the l.h.s. must also $\to 1$ as $n\to\infty$. Taking $\varepsilon=\varepsilon'(\mu_{[2]}-\mu_{[1]})$ with $0<\varepsilon'<1$, this means that (as $n\to\infty$)

$$(35)\quad P_{\alpha_{00}}[(1-\varepsilon')(\mu_{[2]}-\mu_{[1]})<\bar{X}_{[2]}-\bar{X}_{[1]}<(1+\varepsilon')(\mu_{[2]}-\mu_{[1]})]\to 1.$$

Using (35), noting that $x_2-x_1>0$, and taking $n \geqq (r_1h_1-r_2h_2)^2\sigma^2\cdot\delta/(\mu_2-\mu_1)^2$, it follows that the exponent a_n (say) of $b'/a'=e^{-a_n}$ in (31) is such that for all x we have $P_{\alpha_{00}}[a_n\leqq x]\to 0$ as $n\to\infty$. Then it can be shown (successively) that

$$(36)\quad P_{\alpha_{00}}[e^{-a_n}\leqq x]\to\begin{cases}1, x>0\\ 0, x\leqq 0\end{cases};$$

$$(37)\quad P_{\alpha_{00}}[\ln(1+e^{-a_n})\leqq x]\to\begin{cases}1, x>0\\ 0, x\leqq 0.\end{cases}$$

From (37) it follows that a_{n002} converges stochastically to zero under α_{00}.

(ii), (iii), (iv). By methods similar to those used for the case $i, j = 0, 0$ we find we may take $a_{n012} = -1/n$,

$$a_{n011} = \frac{1}{r_1} \ln \left\{1 + e^{-\frac{n}{\sigma^2}(x_2 - x_1)\left\{(\mu_2 - \mu_1) - \frac{\sigma}{\sqrt{n}}(r_1 h_1 - r_2(h_2 + 1))\right\}}\left[1 - e^{r_1 \frac{x_2 - \mu_1}{\sigma/\sqrt{n}} - (h_1 + \frac{1}{2}) r_1^2}\right]\right\},$$

$$a_{n102} = \frac{1}{r_2} \ln \left\{1 + \frac{b'}{a'} e^{r_1 \frac{x_2 - x_1}{\sigma/\sqrt{n}}}\right\},$$

$$a_{n101} = \frac{1}{r_1} \ln \left(1 + \frac{b'}{a'}\right) - \frac{1}{r_1} \ln \left[1 + \frac{b'}{a'} e^{r_1 \frac{x_2 - x_1}{\sigma/\sqrt{n}}}\right],$$

and

$$a_{n111} = -\frac{1}{r_1} \ln \left[1 + \frac{b'}{a'} e^{r_1 \frac{x_2 - x_1}{\sigma/\sqrt{n}}}\right].$$

(38) **Theorem.** *For $k \geqq 2$, condition (B') (or, more properly, its generalization to $k \geqq 2$) holds for the sequence* (16) *for arbitrary $r > 0$.*

Proof. Condition (B') is given at (6). Its first requirement, (7), is satisfied by (the generalization to $k \geqq 2$ of) Lemma (18). The remainder of its requirements are satisfied (for the case $k = 2$) by Lemma (22). We will now show that these remaining requirements are satisfied when $k > 2$. Let S_k be the symmetric group on k elements.

As at (21) and (20), for $i_1, \dots, i_k = 0, 1$

$$(39) \quad K_n(x \mid \mu) = f^{(\mu)}_{\bar{X}_{[1]}, \dots, \bar{X}_{[k]}}(x_1, \dots, x_k) = \sum_{\beta \in S_k} (\sqrt{n}/\sigma)^k \phi\left(\frac{x_{\beta(1)} - \mu_{[1]}}{\sigma/\sqrt{n}}\right) \dots \phi\left(\frac{x_{\beta(k)} - \mu_{[k]}}{\sigma/\sqrt{n}}\right);$$

$$(40) \quad \alpha_{i_1 \dots i_k} = (\mu_{[1]} + (h_1 + i_1) r_1 \sigma/\sqrt{n}, \dots, \mu_{[k]} + (h_k + i_k) r_k \sigma/\sqrt{n}).$$

Thus,

$$(41) \quad \begin{aligned} &K_n(x \mid \alpha_{i_1 i_2 \dots i_k})(\sqrt{2\pi}\sigma/\sqrt{n})^k e^{\frac{r_1^2 h_1^2}{2} + \dots + \frac{r_k^2 h_k^2}{2}} \\ &= (\sqrt{2\pi})^k e^{\frac{r_1^2 h_1^2}{2} + \dots + \frac{r_k^2 h_k^2}{2}} (1/\sqrt{2\pi})^k \cdot \sum_{\beta \in S_k} e^{-\frac{1}{2}\sum_{j=1}^{k}\left(\frac{x_{\beta(j)} - \mu_{[j]} - (h_j + i_j) r_j \sigma/\sqrt{n}}{\sigma/\sqrt{n}}\right)^2} \\ &= e^{\frac{r_1^2 h_1^2}{2} + \dots + \frac{r_k^2 h_k^2}{2}} \sum_{\beta \in S_k} e^{-\frac{1}{2}\sum_{j=1}^{k}\left\{\left(\frac{x_{\beta(j)} - \mu_{[j]}}{\sigma/\sqrt{n}}\right)^2 - 2(x_{\beta(j)} - \mu_{[j]})(h_j + i_j) r_j \frac{\sqrt{n}}{\sigma} + (h_j + i_j)^2 r_j^2\right\}} \\ &= e^{\frac{r_1^2 h_1^2}{2} + \dots + \frac{r_k^2 h_k^2}{2}} \sum_{\beta \in S_k} e^{\sum_{j=1}^{k}\left\{-\frac{1}{2}\left(\frac{x_{\beta(j)} - \mu_{[j]}}{\sigma/\sqrt{n}}\right)^2 + h_j r_j \frac{(x_{\beta(j)} - \mu_{[j]})}{\sigma/\sqrt{n}}\right\}} \\ &\qquad \cdot e^{\sum_{j=1}^{k}\left\{r_j i_j \frac{(x_{\beta(j)} - \mu_{[j]})}{\sigma/\sqrt{n}} - i_j(h_j + \frac{1}{2} i_j) r_j^2 - \frac{r_j^2 h_j^2}{2}\right\}} \\ &= \sum_{\beta \in S_k} a'(\beta) e^{\sum_{j=1}^{k}\left\{r_j i_j \frac{(x_{\beta(j)} - \mu_{[j]})}{\sigma/\sqrt{n}} - i_j(h_j + \frac{1}{2} i_j) r_j^2\right\}}, \end{aligned}$$

where

$$(42) \quad a'(\beta) = e^{\sum_{j=1}^{k}\left\{-\frac{1}{2}\left(\frac{x_{\beta(j)} - \mu_{[j]}}{\sigma/\sqrt{n}}\right)^2 + h_j r_j \frac{(x_{\beta(j)} - \mu_{[j]})}{\sigma/\sqrt{n}}\right\}}.$$

While for the case $k=2$ there were $2!=2$ terms in the final summation, there are now $k!$ terms.

As there were $2^2=4$ parts to Lemma (22), there are 2^k parts here. We will give the proof for the part corresponding to (23), since it is indicative. I.e., in the case $i_1, \ldots, i_k = 0, \ldots, 0$,

$$\begin{aligned} &\frac{\bar{X}_{[1]}-\mu_{[1]}}{\sigma/\sqrt{n}} < (h_1+\tfrac{1}{2})\, r_1 + a_{n0\ldots01} \\ (43)\quad &\frac{\bar{X}_{[2]}-\mu_{[2]}}{\sigma/\sqrt{n}} < (h_2+\tfrac{1}{2})\, r_2 + a_{n0\ldots02} \\ &\quad\vdots \\ &\frac{\bar{X}_{[k]}-\mu_{[k]}}{\sigma/\sqrt{n}} < (h_k+\tfrac{1}{2})\, r_k + a_{n0\ldots0k} \end{aligned}$$

(where $a_{ni_1\ldots i_k 1}, \ldots, a_{ni_1\ldots i_k k}$ converge stochastically to zero when $\alpha_{i_1\ldots i_k}$ is the parameter of the density of $X(n)$ ($i_1, \ldots, i_k = 0, 1$)) when $X(n) \in S_n(\mu, h_1, \ldots, h_k)$ and $M = K_n(X(n) \mid \alpha_{0\ldots0})$. The $a_{ni_1\ldots i_k j}$ ($j=1, \ldots, k$) may depend on $X(n)$, μ, $h_1, \ldots, h_k$.

For, e.g., the first comparison of (43), $K_n(x \mid \alpha_{00\ldots0}) \geqq K_n(x \mid \alpha_{10\ldots0})$, so by (41) and the fact that $x_1 \leqq x_i$ ($i=2, \ldots, k$),

$$\begin{aligned} \sum_{\beta\in S_k} a'(\beta) &\geqq \sum_{\beta\in S_k} a'(\beta)\, e^{r_1\left(\frac{x_{\beta(1)}-\mu_{[1]}}{\sigma/\sqrt{n}}\right)-(h_1+\frac{1}{2})r_1^2} \\ &\geqq \sum_{\beta\in S_k} a'(\beta)\, e^{r_1\left(\frac{x_1-\mu_{[1]}}{\sigma/\sqrt{n}}\right)-(h_1+\frac{1}{2})r_1^2}; \end{aligned}$$

$$1 \geqq e^{r_1\frac{x_1-\mu_{[1]}}{\sigma/\sqrt{n}}-(h_1+\frac{1}{2})r_1^2}.$$

From here the proof is essentially that which follows (27).

Rule for Making Comparisons. For each of the $k!$ vectors $i_1, \ldots, i_k$, one must prove k relations similar to (43), with appropriate modifications of "$<$" to "$>$". For these, compare the given $\alpha_{i_1\ldots i_k}$ with the k others which have $i_1', \ldots, i_k'$'s which differ from the given $i_1, \ldots, i_k$ in only one place. (This rule, suggested by the $k=2$ results, works when $k>2$.)

To illustrate our method, we will now study, e.g., the second comparison of (43). Since $K_n(x \mid \alpha_{000\ldots0}) \geqq K_n(x \mid \alpha_{010\ldots0})$,

$$\begin{aligned} \sum_{\beta\in S_k} a'(\beta) &\geqq \sum_{\beta\in S_k} a'(\beta)\, e^{r_2\frac{x_{\beta(2)}-\mu_{[2]}}{\sigma/\sqrt{n}}-(h_2+\frac{1}{2})r_2^2} \\ &\geqq \sum_{\substack{\beta\in S_k\\ \beta(2)=2}} a'(\beta)\, e^{r_2\frac{x_2-\mu_{[2]}}{\sigma/\sqrt{n}}-(h_2+\frac{1}{2})r_2^2}; \end{aligned}$$

$$1 + \frac{\sum_{\beta\in S_k} a'(\beta) - \sum_{\substack{\beta\in S_k\\ \beta(2)=2}} a'(\beta)}{\sum_{\substack{\beta\in S_k\\ \beta(2)=2}} a'(\beta)} \geqq e^{r_2\frac{x_2-\mu_{[2]}}{\sigma/\sqrt{n}}-(h_2+\frac{1}{2})r_2^2}.$$

Now the proof proceeds as at (30), and a relation like (31) holds because what is left in $\sum_{\beta \in S_k} a'(\beta)$ after $\sum_{\substack{\beta \in S_k \\ \beta(2)=2}} a'(\beta)$ is removed, makes the "wrong" associations and thus tends to zero, while the denominator does not.

(44) **Theorem.** *For $\Theta(\eta^*)$ and any fixed $r=(r_1, \ldots, r_k)>0$, $(\bar{X}_{[1]}, \ldots, \bar{X}_{[k]})$ is a sequence of* GMLE's *for estimation of $(\mu_{[1]}, \ldots, \mu_{[k]})$ based on $X(n)=(\bar{X}_{[1]}, \ldots, \bar{X}_{[k]})$. It thus possesses, for all $r=(r_1, \ldots, r_k)>0$, the property of Theorem* (13).

Proof. Theorems (17) and (38) establish conditions (A') and (B'), respectively, for all $r>0$. We therefore have a sequence of GMLE's possessing the property of Theorem (13), or more properly its extension to $k \geqq 2$, for all $r>0$.

If T and U are estimators of θ, then U is said to be more concentrated (about θ) than T if

(45) $$P_\theta[-r \leqq U-\theta \leqq r] \leqq P_\theta[-r \leqq T-\theta \leqq r]$$

for all $\theta \in \Theta$ and all $r>0$. (This definition appears for perhaps the first time in print in Lawton [5].) If T_n and U_n estimate θ, then U_n is said to be of higher *large sample concentration* (about θ) than T_n if

(46) $$\lim_{n\to\infty} P_\theta[-r \leqq k(n)(U_n-\theta) \leqq r] \geqq \limsup P_\theta[-r \leqq k(n)(T_n-\theta) \leqq r],$$

where $k(n)$ is such that $k(n)(U_n-\theta)$ approaches a limiting distribution, for all $\theta \in \Theta$ and all $r>0$. The GMLE $(\bar{X}_{[1]}, \ldots, \bar{X}_{[k]})$ has, using a k-dimensional generalization of (46), desirable large sample concentration in comparison to the class of estimators of Theorem (13).

We will now show (for $k=2$, the $k>2$ extension being similar) that, by finding one GMLE, we find *a class of* GMLE's.

(47) **Lemma.** *Suppose $\lim_{n\to\infty} P_{\theta_n}[Z_n<y]=L(y)$, with $L(\cdot)$ a continuous d.f. Then, if $\lim_{n\to\infty} c_n=0$,*

$$\lim_{n\to\infty} P_{\theta_n}[Z_n<y+c_n]=L(y).$$

Proof. If all but a finite number of the c_n are positive, then $L(y) \leqq \lim_{n\to\infty} P_{\theta_n}[Z_n<y+c_n]$ and (since eventually all c_n are less than c_m, m fixed)

(48) $$\lim_{n\to\infty} P_{\theta_n}[Z_n<y+c_n] \leqq L(y+c_m).$$

Taking the limit on m in (48) and using the continuity of $L(\cdot)$ the desired result follows. (If all but a finite number of the c_n are negative, the proof is similar.)

If infinitely many c_n are positive and infinitely many c_n are negative, suppose $c_r<0$, $c_s>0$. Then

(49) $$L(y+c_r) \leqq \lim_{n\to\infty} P_{\theta_n}[Z_n<y+c_n] \leqq L(y+c_s)$$

since eventually $c_r \leqq c_n \leqq c_s$. Taking limits in (49) over $\{r: c_r<0\}$ and $\{s: c_s>0\}$ on the l.h.s. and r.h.s. (respectively) the desired result follows. Note that this is a special case of, with an even simpler proof than, Cramér's Theorem (see, e.g., Fisz [4], p. 236).

(50) **Theorem.** *If* $\{Z_{n1}(X(n), r), Z_{n2}(X(n), r)\}$ *is a sequence of* GMLE's *then so is*

(51) $\{Z_{n1}+o_1(1/k_1(n)), Z_{n2}+o_2(1/k_2(n))\}$,

where $o_i(1/k_i(n))$ $(i=1,2)$ *is a quantity such that*

$$\lim_{n\to\infty}\frac{o_i(1/k_i(n))}{1/k_i(n)}=\lim_{n\to\infty}k_i(n)\,o_i(1/k_i(n))=0.$$

An easy generalization of Theorem (50) is that if $\{Z_{n1}(X(n), r), Z_{n2}(X(n), r)\}$ is a sequence of GMLE's then so is $\{Z_{n1}+T'_{n1}, Z_{n2}+T'_{n2}\}$ where (T'_{n1}, T'_{n2}) is such that, uniformly in θ,

(52) $\lim_{n\to\infty} P_\theta[|k_1(n)\,T'_{n1}|<\delta, |k_2(n)\,T'_{n2}|<\delta]=1$

for any given $\delta>0$. In Weiss and Wolfowitz [7] condition (52) has been weakened. These results will now be used to compare *the* MLE (derived in [3]) *and the* GMLE *with regard to asymptotic efficiency* when $k=2$.

(53) **Lemma.** *For any* $a>0$, $P_\mu[\bar{X}_{[2]}-\bar{X}_{[1]}>a\sigma/\sqrt{n}]$ *is minimized (over* $\mu\in\Theta(\eta^*)$ *i.e. over* μ *such that* $\mu_{[2]}=\mu_{[1]}+\eta$ *for some* $\eta\geqq\eta^*>0$) *at* $\mu_{[2]}=\mu_{[1]}+\eta^*$. *Also*

$$P_{\mu_{[2]}=\mu_{[1]}+\eta^*}[\bar{X}_{[2]}-\bar{X}_{[1]}>a\sigma/\sqrt{n}]\to 1 \qquad as\ n\to\infty.$$

Proof. From the joint density of $\bar{X}_{[1]},\ldots,\bar{X}_{[k]}$ (see [3]), we find that

$$\begin{aligned}P_\mu[\bar{X}_{[2]}-\bar{X}_{[1]}>a\sigma/\sqrt{n}]&=\frac{\sqrt{n}}{2\sigma\sqrt{\pi}}\int_{a\sigma/\sqrt{n}}^{\infty}\left\{e^{-\frac{1}{4}\left(\frac{y-\eta}{\sigma/\sqrt{n}}\right)^2}+e^{-\frac{1}{4}\left(\frac{y+\eta}{\sigma/\sqrt{n}}\right)^2}\right\}dy\\
&=\frac{1}{\sqrt{2\pi}}\int_{\frac{a\frac{\sigma}{\sqrt{n}}-\eta}{\sqrt{2}\sigma/\sqrt{n}}}^{\infty}e^{-\frac{1}{2}y^2}dy+\frac{1}{\sqrt{2\pi}}\int_{\frac{a\frac{\sigma}{\sqrt{n}}+\eta}{\sqrt{2}\sigma/\sqrt{n}}}^{\infty}e^{-\frac{1}{2}y^2}dy\\
&=\frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{\frac{-a\frac{\sigma}{\sqrt{n}}-\eta}{\sqrt{2}\sigma/\sqrt{n}}}+\int_{\frac{a\frac{\sigma}{\sqrt{n}}-\eta}{\sqrt{2}\sigma/\sqrt{n}}}^{\infty}\right)e^{-\frac{1}{2}y^2}dy.\end{aligned}$$

By the formula for differentiation with respect to a parameter or by the Chain Rule, since $\left(a\frac{\sigma}{\sqrt{n}}+\eta\right)^2>\left(a\frac{\sigma}{\sqrt{n}}-\eta\right)^2$,

$$\frac{d}{d\eta}P_\mu[\bar{X}_{[2]}-\bar{X}_{[1]}>a\sigma/\sqrt{n}]=\frac{1}{\sqrt{2\pi}}\left[e^{\frac{1}{2}\left(\frac{a\frac{\sigma}{\sqrt{n}}+\eta}{\sqrt{2}\sigma/\sqrt{n}}\right)^2}\frac{1}{\sqrt{2}\sigma/\sqrt{n}}-e^{\frac{1}{2}\left(\frac{a\frac{\sigma}{\sqrt{n}}-\eta}{\sqrt{2}\sigma/\sqrt{n}}\right)^2}\frac{1}{\sqrt{2}\sigma/\sqrt{n}}\right]>0.$$

Hence $P_\mu[\bar{X}_{[2]}-\bar{X}_{[1]}>a\sigma/\sqrt{n}]$ is an increasing function of $\eta\geqq\eta^*>0$, and is therefore minimized when $\eta=\eta^*>0$ (i.e. when $\mu_{[2]}=\mu_{[1]}+\eta^*$). That this minimal probability $\to 1$ as $n\to\infty$ follows from (54).

(55) **Lemma.** $\left|\frac{d^2 n}{\sigma^2}-\varepsilon_0\right| \leqq 2$, *where* ε_0 *is the positive solution of*

$d^2 n/\sigma^2 = \varepsilon \coth(\varepsilon/2)$.

Proof. From the fact that $\coth(x)>1$ for $x>0$,

$$\left|\frac{d^2 n}{\sigma^2}-\varepsilon_0\right| = |\varepsilon_0-\varepsilon_0 \coth(\varepsilon_0/2)| = \varepsilon_0(\coth(\varepsilon_0/2)-1).$$

Using an expression for $\coth(\varepsilon_0/2)$, this becomes

$$\left|\frac{d^2 n}{\sigma^2}-\varepsilon_0\right| = \varepsilon_0\left(\frac{e^{\varepsilon_0/2}+e^{-\varepsilon_0/2}}{e^{\varepsilon_0/2}-e^{-\varepsilon_0/2}}-1\right) = \varepsilon_0 \frac{2e^{-\varepsilon_0/2}}{e^{\varepsilon_0/2}-e^{-\varepsilon_0/2}} = 2\frac{\varepsilon_0}{e^{\varepsilon_0}-1} \leqq 2,$$

since (for $x \geqq 0$) $x/(e^x-1) \leqq 1$, or $x \leqq e^x-1$, because $x+1 \leqq e^x = 1+x+\frac{x^2}{2!}+\cdots$.

In the notation at (52), we wish to show that the MLE, which is (say) $\{\bar{X}_{[1]}+T'_{n1}, \bar{X}_{[2]}+T'_{n2}\}$, is such that (52) holds, with $k_1(n)=k_2(n)=\sqrt{n}/\sigma$. By Theorem (37) of [3], the MLE has

$$(56)\quad |T'_{n1}| = |\hat{\mu}_{[1]}-\bar{X}_{[1]}|$$

$$= \begin{cases} \left|\frac{\bar{X}_{[1]}+\bar{X}_{[2]}}{2}-\bar{X}_{[1]}\right| & \text{if } 0 \leqq \bar{X}_{[2]}-\bar{X}_{[1]} \leqq \sqrt{2}\,\sigma/\sqrt{n} \\ \left|\frac{\bar{X}_{[1]}+\bar{X}_{[2]}}{2}-\frac{\bar{X}_{[2]}-\bar{X}_{[1]}}{2\coth(\varepsilon_0/2)}-\bar{X}_{[1]}\right| & \text{if } \bar{X}_{[2]}-\bar{X}_{[1]} > \sqrt{2}\,\sigma/\sqrt{n} \end{cases}$$

$$= \begin{cases} \frac{\bar{X}_{[2]}-\bar{X}_{[1]}}{2} & \text{if } 0 \leqq \bar{X}_{[2]}-\bar{X}_{[1]} \leqq \sqrt{2}\,\sigma/\sqrt{n} \\ \frac{\bar{X}_{[2]}-\bar{X}_{[1]}}{2}\left|1-\frac{1}{\coth(\varepsilon_0/2)}\right| & \text{if } \bar{X}_{[2]}-\bar{X}_{[1]} > \sqrt{2}\,\sigma/\sqrt{n} \end{cases}$$

and $|T'_{n2}| = |\hat{\mu}_{[2]}-\bar{X}_{[2]}|$ turns out to be the same. Thus, using the definition $d=\bar{X}_{[2]}-\bar{X}_{[1]}$ and the fact that $\varepsilon_0 \coth(\varepsilon_0/2) = d^2 n/\sigma^2$, for any $\delta>0$

$$\begin{aligned} &P_\theta[|k_1(n)T'_{n1}|<\delta, |k_2(n)T'_{n2}|<\delta] = P_\mu[|T'_{n1}|<\delta\sigma/\sqrt{n}] \\ &\quad = P_\mu[\bar{X}_{[2]}-\bar{X}_{[1]}<2\delta\sigma/\sqrt{n},\ 0 \leqq \bar{X}_{[2]}-\bar{X}_{[1]} \leqq \sqrt{2}\,\sigma/\sqrt{n}] \\ &\qquad + P_\mu\left[\frac{\bar{X}_{[2]}-\bar{X}_{[1]}}{2}\left|1-\frac{1}{\coth(\varepsilon_0/2)}\right|<\delta\sigma/\sqrt{n},\ \bar{X}_{[2]}-\bar{X}_{[1]}>\sqrt{2}\,\sigma/\sqrt{n}\right] \end{aligned}$$

$$(57)\quad \begin{aligned} &\geqq P_\mu\left[\frac{\bar{X}_{[2]}-\bar{X}_{[1]}}{2}\left|1-\frac{1}{\coth(\varepsilon_0/2)}\right|<\delta\sigma/\sqrt{n},\ \bar{X}_{[2]}-\bar{X}_{[1]}>\sqrt{2}\,\sigma/\sqrt{n}\right] \\ &= P_\mu\left[\frac{\bar{X}_{[2]}-\bar{X}_{[1]}}{2}\left|1-\frac{\varepsilon_0\sigma^2}{d^2 n}\right|<\delta\frac{\sigma}{\sqrt{n}},\ \bar{X}_{[2]}-\bar{X}_{[1]}>\sqrt{2}\,\sigma/\sqrt{n}\right] \\ &= P_\mu\left[\frac{\bar{X}_{[2]}-\bar{X}_{[1]}}{2}\frac{|d^2 n-\varepsilon_0\sigma^2|}{d^2 n}<\delta\sigma/\sqrt{n},\ \bar{X}_{[2]}-\bar{X}_{[1]}>\sqrt{2}\,\sigma/\sqrt{n}\right] \\ &= P_\mu\left[\tfrac{1}{2}\frac{|d^2 n-\varepsilon_0\sigma^2|}{n}<(\bar{X}_{[2]}-\bar{X}_{[1]})\delta\sigma/\sqrt{n},\ \bar{X}_{[2]}-\bar{X}_{[1]}>\sqrt{2}\,\sigma/\sqrt{n}\right] \\ &= P_\mu\left[\bar{X}_{[2]}-\bar{X}_{[1]}>\frac{1}{2\delta}\frac{\sigma}{\sqrt{n}}\left|\frac{d^2 n}{\sigma^2}-\varepsilon_0\right|,\ \bar{X}_{[2]}-\bar{X}_{[1]}>\sqrt{2}\,\sigma/\sqrt{n}\right]. \end{aligned}$$

(58) **Theorem.** *For the* MLE *when* $k=2$, *uniformly in* μ, *for any given* $\delta>0$,

$$\lim_{n\to\infty} P_\mu[|k_1(n)T'_{n1}|<\delta, |k_2(n)T'_{n2}|<\delta]=1.$$

Proof. By Lemma (55) and equation (57),

$$(59)\quad P_\mu[|k_1(n)T'_{n1}|<\delta, |k_2(n)T'_{n2}|<\delta]$$
$$\geqq P_\mu\left[\bar{X}_{[2]}-\bar{X}_{[1]}>\frac{1}{2\delta}\frac{\sigma}{\sqrt{n}}2, \bar{X}_{[2]}-X_{[1]}>\sqrt{2}\sigma/\sqrt{n}\right]$$
$$=P_\mu\left[\bar{X}_{[2]}-\bar{X}_{[1]}>\max\left(\sqrt{2},\frac{1}{\delta}\right)\sigma/\sqrt{n}\right].$$

By Lemma (53), the last member of (59) can be bounded below for $\mu\in\Theta(\eta^*)$, in such a way that the bound $\to 1$ as $n\to\infty$.

By Theorem (58) it follows, as noted above (52), that the MLE and the GMLE have (for $k=2$) the same asymptotic efficiency, and that the MLE is a GMLE. This proves asymptotic efficiency properties for the MLE which do not follow directly from the standard theory, which assumes i.i.d. observations.

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Appendix A. Limit Distribution of $\bar{X}_{[1]}, \ldots, \bar{X}_{[k]}$

The limiting distribution of $\bar{X}_{[1]}, \ldots, \bar{X}_{[k]}$ (under certain parameter configurations) is of interest to us. Let $\{A_n, n\geqq 1\}$ and $\{B_n, n\geqq 1\}$ be sequences of events on some probability space (which may depend on n). Let $a=(a_1, \ldots, a_k)\in\mathscr{R}^k$ be fixed, and denote the vector $(\mu_1+a_1\sigma/\sqrt{n}, \ldots, \mu_k+a_k\sigma/\sqrt{n})$ by $\mu+a\sigma/\sqrt{n}$.

(A.1) **Lemma.** *If* $\lim_{n\to\infty} P_n(B_n)=1$, *then (if either of the following limits exists)* $\lim_{n\to\infty} P_n(A_n B_n)=\lim_{n\to\infty} P_n(A_n)$.

(A.2) *Definition.* For $\mu\in\Omega_0$, let $p(n|\mu)=P_\mu[\bar{X}_{(1)}<\cdots<\bar{X}_{(k)}]$; where $\Omega(\neq)=\{\mu\colon \mu_{[1]}\neq\mu_{[2]}\neq\cdots\neq\mu_{[k]}\}$, if $\mu\in\Omega(\neq)$, $\bar{X}_{(i)}$ denotes the sample mean produced by the population associated with $\mu_{[i]}$ $(i=1, \ldots, k)$, and if there is at least one break in the string of inequalities $\mu_{[1]}\neq\cdots\neq\mu_{[k]}$, then the situation is that we have $l(1\leqq l<k)$ groups of equal parameters

$$\mu_{[1]}=\cdots=\mu_{[i_1]}\neq\mu_{[i_1+1]}=\cdots=\mu_{[i_2]}\neq\cdots\neq\mu_{[i_{l-1}+1]}=\cdots=\mu_{[k]}$$

with $i_1, \ldots, i_{l-1}$ integers

$$(0\equiv i_0<1\leqq i_1<i_2<\cdots<i_{l-1}\leqq k-1<i_l\equiv k),$$

and we let

$$\bar{X}_{(i_j+1)}\leqq\bar{X}_{(i_j+2)}\leqq\cdots\leqq\bar{X}_{(i_{j+1}-1)}\leqq\bar{X}_{(i_{j+1})}$$

be the ranked values of the sample means from the population(s) associated with parameter $\mu_{[i_j+1]}$ $(j=0, \ldots, l-1)$.

(A.3) **Lemma.** *Let* $\Theta=\{\mu\colon \mu\in\Omega_0, \mu_1=\mu_{[1]}, \ldots, \mu_k=\mu_{[k]}\}$. *For all* $\mu\in\Omega(\neq)\cap\Theta$,

$$\lim_{n\to\infty} p(n|\mu+a\sigma/\sqrt{n})=\lim_{n\to\infty} P_{\mu+a\sigma/\sqrt{n}}[\bar{X}_{(1)}<\cdots<\bar{X}_{(k)}]=1.$$

Proof. **1.** Suppose that $\mu\in\Omega(\neq)\cap\Theta$. Then for all n large enough,

$$\mu+a\sigma/\sqrt{n}\in\Omega(\neq)\cap\Theta.$$

Then the $\bar{X}_{(j)}$ are independent and $\bar{X}_{(j)}$ is the sample mean of n i.i.d.

$$N(\mu_{[j]}+a_j\sigma/\sqrt{n}, \sigma^2) \qquad \text{r.v.'s.}$$

The characteristic function of a $N(m,\sigma^2)$ r.v. is $\varphi(t)=\exp\{itm-\frac{1}{2}t^2\sigma^2\}$, so that $\lim_{n\to\infty}\varphi_{\bar{X}_{(j)}}(t)=e^{it\mu_{[j]}}$ and $\bar{X}_{(j)}$ converges in probability to $\mu_{[j]}$ $(j=1,\ldots,k)$. Thus, since the $\bar{X}_{(j)}$ are independent, it is clear that the probability that $\{\bar{X}_{(j)}$ converges to $\mu_{[j]}$ $(j=1,\ldots,k)\}$ approaches 1 as $n\to\infty$. However, by Lemma (A.1)

(A.4) $$\lim P_{\mu+a\sigma/\sqrt{n}}[\bar{X}_{(1)}<\cdots<\bar{X}_{(k)}]$$
$$=\lim_{n\to\infty} P_{\mu+a\sigma/\sqrt{n}}[\bar{X}_{(1)}<\cdots<\bar{X}_{(k)}, |\bar{X}_{(1)}-\mu_{[1]}|<\varepsilon, \ldots, |\bar{X}_{(k)}-\mu_{[k]}|<\varepsilon]$$

for any $\varepsilon>0$. If we choose $2\varepsilon\leqq\min_{1\leqq i<j\leqq k}(\mu_{[j]}-\mu_{[i]})$, then the r.h.s. of (A.4) equals 1 since $P[\bar{X}_{(j)}$ converges to $\mu_{[j]}$ $(j=1,\ldots,k)]$ approaches 1 as $n\to\infty$.

2. Suppose that $\mu\in[\Omega(\neq)]^c\cap\Theta$. (Eventually $\mu+a\sigma/\sqrt{n}\in\Theta\cap\Omega(\neq)$, or $\Theta\cap[\Omega(\neq)]^c$.) Then there are l distinct values in $\{\mu_{[1]}+a_1\sigma/\sqrt{n}, \ldots, \mu_{[k]}+a_k\sigma/\sqrt{n}\}$ $(1\leqq l\leqq k-1)$ and (see (A.2))

$$P_{\mu+a\sigma/\sqrt{n}}[\bar{X}_{(1)}<\cdots<\bar{X}_{(k)}]$$
$$=P_{\mu+a\sigma/\sqrt{n}}[\bar{X}_{(i_1)}<\bar{X}_{(i_1+1)}, \bar{X}_{(i_2)}<\bar{X}_{(i_2+1)}, \ldots, \bar{X}_{(i_{l-1})}<\bar{X}_{(i_{l-1}+1)}].$$

However, the result will *not* follow as before since $\min_{1\leqq i<j\leqq k}(\mu_{[j]}-\mu_{[i]})=0$ here. It can be seen (e.g., consider the case $k=2$) that the limit $\not\to 1$ as $n\to\infty$. (In fact, it depends on a.)

(A.5) **Lemma.** *For* $\mu\in\Theta\cap\Omega(\neq)$, *as* $n\to\infty$

$$F^{(\mu+a\sigma/\sqrt{n})}_{\bar{X}_{[1]},\ldots,\bar{X}_{[k]}}(x_1,\ldots,x_k)-P_{\mu+a\sigma/\sqrt{n}}[\bar{X}_{(i)}\leqq x_i\ (i=1,\ldots,k)]\to 0.$$

Proof.

$$\begin{aligned}
&\lim_{n\to\infty} F^{(\mu+a\sigma/\sqrt{n})}_{\bar{X}_{[1]},\ldots,\bar{X}_{[k]}}(x_1,\ldots,x_k)\\
&\quad=\lim_{n\to\infty}\{p(n|\mu+a\sigma/\sqrt{n})\\
&\qquad\quad\cdot P_{\mu+a\sigma/\sqrt{n}}[\bar{X}_{[1]}\leqq x_1,\ldots,\bar{X}_{[k]}\leqq x_k|\bar{X}_{(1)}<\cdots<\bar{X}_{(k)}]\\
&\qquad\quad+(1-p(n|\mu+a\sigma/\sqrt{n}))\\
&\qquad\quad\cdot P_{\mu+a\sigma/\sqrt{n}}[\bar{X}_{[1]}\leqq x_1,\ldots,\bar{X}_{[k]}\leqq x_k|\text{not}\,(\bar{X}_{(1)}<\cdots<\bar{X}_{(k)})]\}\\
&\quad=\lim_{n\to\infty} P_{\mu+a\sigma/\sqrt{n}}[\bar{X}_{[1]}\leqq x_1,\ldots,\bar{X}_{[k]}\leqq x_k;\ \bar{X}_{(1)}<\cdots<\bar{X}_{(k)}]\\
&\quad=\lim_{n\to\infty} P_{\mu+a\sigma/\sqrt{n}}[\bar{X}_{(1)}\leqq x_1,\ldots,\bar{X}_{(k)}\leqq x_k;\ \bar{X}_{(1)}<\cdots<\bar{X}_{(k)}]\\
&\quad=\lim_{n\to\infty} P_{\mu+a\sigma/\sqrt{n}}[\bar{X}_{(1)}\leqq x_1,\ldots,\bar{X}_{(k)}\leqq x_k].
\end{aligned}$$

Here the second equality follows from Lemma (A.3), while the last equality follows from Lemmas (A.3) and (A.1).

(A.6) **Lemma.** *As $n\to\infty$, if $\mu+a\sigma/\sqrt{n}\in\Theta\cap\Omega(\neq)$ then*

$$P_{\mu+a\sigma/\sqrt{n}}[\bar{X}_{(1)}\leqq x_1,\ldots,\bar{X}_{(k)}\leqq x_k]\to P_\mu[\bar{X}_{(1)}\leqq x_1,\ldots,\bar{X}_{(k)}\leqq x_k].$$

Proof. As $n\to\infty$,

$$\begin{aligned}&P_{\mu+a\sigma/\sqrt{n}}[\bar{X}_{(1)}\leqq x_1,\ldots,\bar{X}_{(k)}\leqq x_k]\\&\quad=P_{\mu+a\sigma/\sqrt{n}}[\bar{X}_{(1)}-a_1\sigma/\sqrt{n}\leqq x_1-a_1\sigma/\sqrt{n},\ldots,\bar{X}_{(k)}-a_k\sigma/\sqrt{n}\leqq x_k-a_k\sigma/\sqrt{n}]\\&\quad=P_\mu[\bar{X}_{(1)}\leqq x_1-a_1\sigma/\sqrt{n},\ldots,\bar{X}_{(k)}\leqq x_k-a_k\sigma/\sqrt{n}]\\&\to P_\mu[\bar{X}_{(1)}\leqq x_1,\ldots,\bar{X}_{(k)}\leqq x_k].\end{aligned}$$

The second equality follows because, when $\mu+a\sigma/\sqrt{n}\in\Theta\cap\Omega(\neq)$, $\bar{X}_{(i)}$ is $N(\mu_{[i]}+a_i\sigma/\sqrt{n},\sigma^2/n)$ iff $\bar{X}_{(i)}-a_i\sigma/\sqrt{n}$ is $N(\mu_{[i]},\sigma^2/n)$ $(i=1,\ldots,k)$.

(A.7) *Definition.* Let $\Phi(z_1,\ldots,z_s)$ denote the d.f. of the $1,\ldots,s$ order statistics in a sample of size s from a $N(0,1)$ population.

(A.8) **Theorem.** *As $n\to\infty$, if $\mu\in\Theta\cap\Omega(\neq)$ then*

$$F^{(\mu+a\sigma/\sqrt{n})}_{\frac{\sqrt{n}}{\sigma}(\bar{X}_{[1]}-\mu_{[1]}-a_1\sigma/\sqrt{n}),\ldots,\frac{\sqrt{n}}{\sigma}(\bar{X}_{[k]}-\mu_{[k]}-a_k\sigma/\sqrt{n})}(x_1,\ldots,x_k)\to\prod_{i=1}^{k}\Phi(x_i).$$

(A.9) **Corollary.** *As $n\to\infty$, if $\mu\in\Theta\cap\Omega(\neq)$ then*

$$F_{\frac{\sqrt{n}}{\sigma}(\bar{X}_{[1]}-\mu_{[1]}),\ldots,\frac{\sqrt{n}}{\sigma}(\bar{X}_{[k[}-\mu_{[k]})}(x_1,\ldots,x_k)\to\prod_{i=1}^{k}\Phi(x_i).$$

(A.10) **Theorem.** *If $\mu\in\Theta\cap[\Omega(\neq)]^c$ then*

$$\lim_{n\to\infty}F^{(\mu+a\sigma/\sqrt{n})}_{\frac{\sqrt{n}}{\sigma}(\bar{X}_{[1]}-\mu_{[1]}-a_1\sigma/\sqrt{n}),\ldots,\frac{\sqrt{n}}{\sigma}(\bar{X}_{[k]}-\mu_{[k]}-a_k\sigma/\sqrt{n})}(x_1,\ldots,x_k)$$

depends on a.

Proof. (A hint of this dependence was given in part 2 of the proof of Lemma (A.3).) Suppose $k=2$, $a=(a_1,a_2)$ with $a_1\leqq a_2$, and let Y_1, Y_2 denote i.i.d. $N(0,1)$ r.v.'s. Then $\mu_{[1]}=\mu_{[2]}$ and

$$\begin{aligned}&F^{(\mu+a\sigma/\sqrt{n})}_{\frac{\sqrt{n}}{\sigma}(\bar{X}_{[1]}-\mu_{[1]}-a_1\sigma/\sqrt{n}),\frac{\sqrt{n}}{\sigma}(\bar{X}_{[2]}-\mu_{[2]}-a_2\sigma/\sqrt{n})}(x_1,x_2)\\&\quad=P_{\mu+a\sigma/\sqrt{n}}\Big[\frac{\sqrt{n}}{\sigma}(\min(\bar{X}_1,\bar{X}_2)-\mu_{[1]}-a_1\sigma/\sqrt{n})\leqq x_1,\\&\qquad\qquad\frac{\sqrt{n}}{\sigma}(\max(\bar{X}_1,\bar{X}_2)-\mu_{[1]}-a_2\sigma/\sqrt{n})\leqq x_2\Big]\\&\quad=P[\min(Y_1,Y_2+(a_2-a_1))\leqq x_1,\max(Y_1-(a_2-a_1),Y_2)\leqq x_2].\end{aligned}$$

For $a_2-a_1=0$, this is $\Phi(x_1,x_2)$. However, for $a_2\gg a_1$ it is approximately $\Phi(x_1)\Phi(x_2)$, and therefore depends on a.

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