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Uniform Dimension Results for Processes with Independent Increments

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Let $X_t(\omega)$ be a process in $\mathbb{R}^d$ with stationary independent increments and let $X(E,\omega)$ denote the image under $X_t(\omega)$ of a time set E. It is shown that $\dim X(E,\omega) \leqq \beta \dim E$, with probability one, simultaneously for all time sets E where β is the upper index of the process. This result combines with previous work to show that for a strictly stable process of index α with $\alpha \leqq d$, $\dim X(E,\omega) = \alpha \dim E$, while for a subordinator $T_t(\omega)$, $\sigma \dim E \leqq \dim T(E,\omega) \leqq \beta \dim E$ where σ is the lower subordinator index. Both these results hold simultaneously for all time sets E with probability one; they were known previously for fixed time sets. An example is given which shows that the subordinator result cannot be improved. It is shown, however, that $\dim T(E,\omega) = \sigma \dim E$ for a restricted class of regular time sets. As an application, the dimension of the collision set of a subordinator and a stable process of index $\alpha > 1$ in $\mathbb{R}^1$ is found to be $\sigma(1-1/\alpha)$.

1. Introduction

Let $X_t(\omega)$ be a process in $\mathbb{R}^d$ with stationary independent increments and let $X(E,\omega)$ denote the image of a time set E under $X_t(\omega)$. Let β be the upper index of the process and when $X_t(\omega)$ is a subordinator let σ be its lower (subordinator) index. These indices were introduced by Blumenthal and Getoor [2]; we will define them in the next section. In [2], Blumenthal and Getoor established some inequalities relating the Hausdorff dimensions of $X(E,\omega)$ and E. Their upper bound stated that if $\beta < 1$, then for all Borel $E \subset [0,1]$,

$$P\{\dim X(E,\omega) \leqq \beta \dim E\} = 1. \tag{1.1}$$

The restriction $\beta < 1$ was later removed by Millar [15].

In many applications we have a time set $E(\omega)$, depending on ω, and wish to know the dimension of $X(E(\omega),\omega)$. The above result gives no information in this case. In Theorem 3.1 we will prove a uniform version of (1.1) which can be used in this situation:

$$P\{\dim X(E,\omega) \leqq \beta \dim E \text{ for all } E\} = 1. \tag{1.2}$$

The key to the proof is a Covering Principle (Lemma 3.1) which has been stated separately because of its wide applicability.

Uniform lower bounds of this type have been obtained previously for special classes of processes. In [8], Hawkes showed that if X_t is a strictly stable process of index α with $\alpha \leqq d$, then

$$P\{\dim X(E,\omega) \geqq \alpha \dim E \text{ for all } E\} = 1. \tag{1.3}$$

Since $\beta = \alpha$ for a stable process of index α, this result combines with (1.2) to give equality in (1.3). This establishes a conjecture of Hawkes [8] and gives a generali-

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zation (for $\alpha \leqq d$) of the result of Blumenthal and Getoor [1] that for all Borel $E \subset [0,1]$,

$$P\{\dim X(E,\omega) = \min(\alpha \dim E, d)\} = 1. \tag{1.4}$$

A simple example shows that there is no uniform generalization of (1.4) when $\alpha > d$. However, we do obtain a (smaller) uniform lower bound for $\dim X(E,\omega)$ in this case in Theorem 4.2.

Uniform lower bounds for a general subordinator T_t have also been obtained by Hawkes [10]. These combine with our upper bounds to give

$$P\{\sigma \dim E \leqq \dim T(E,\omega) \leqq \beta \dim E \text{ for all } E\} = 1. \tag{1.5}$$

This result had also been obtained previously for fixed time sets by Blumenthal and Getoor [2]. We will give an example to show that (1.5) cannot be improved in general even for a fixed time set. Specifically, we will construct a subordinator T_t with $\sigma < \beta$ such that for every $\theta \in (\sigma, \beta)$ there is a set E with $\dim X(E,\omega) = \theta \dim E$ a.s. The result (1.5) can be improved, however, if the class of time sets is somewhat restricted. We will define a class of regular time sets $\mathscr{D}_\sigma$ and prove that

$$P\{\dim T(E,\omega) = \sigma \dim E \text{ for all } E \in \mathscr{D}_\sigma\} = 1.$$

As an application of this result, we show that the dimension of the collision set of an arbitrary subordinator of lower index σ and an independent stable process of index $\alpha > 1$ in $\mathbb{R}^1$ is $\sigma(1 - 1/\alpha)$.

A natural question that arises is whether general uniform lower bounds can be obtained, perhaps in terms of one of the lower indices β' or β'' introduced by Blumenthal and Getoor [2]. The example already mentioned for stable processes shows that there can be no bound without some assumption. Blumenthal and Getoor [2] show that for a fixed Borel $E \subset [0,1]$, if $\beta' \leqq d$, then

$$P\{\dim X(E,\omega) \geqq \beta' \dim E\} = 1. \tag{1.6}$$

We will give an example to show that there is no uniform generalization of this result. It would be interesting to know if there is a smaller uniform bound, perhaps $\beta'' \dim E$, under the assumption that $\beta \leqq d$.

Some definitions and notation are given in Section 2. The uniform upper bound (1.2) is obtained in Section 3. Stable processes are discussed in Section 4, subordinators in Section 5, and the examples are in Section 6.

2. Preliminaries

The d-dimensional characteristic function of X_t has the form $\exp[-t\psi(z)]$ where

$$\psi(z) = i(a,z) + \tfrac{1}{2} z S z' + \int \left[1 - e^{i(x,z)} + \frac{i(x,z)}{1+|x|^2}\right] \nu(dx), \tag{2.1}$$

with $a \in \mathbb{R}^d$, S a non-negative definite symmetric d by d matrix, and ν a Borel measure on $\mathbb{R}^d$ satisfying

$$\int \frac{|x|^2}{1+|x|^2} \nu(dx) < \infty.$$

The function ψ is called the *exponent* of the process and ν is called the *Lévy measure.* We will assume that $X_0=0$ although this is clearly not important to the conclusions. It is a standard fact that we may assume that the sample functions are right continuous and have left limits and that the process X_t enjoys the strong Markov property.

The *upper index* β is defined by

$$\beta=\inf\{\alpha\geqq 0\colon\ |z|^{-\alpha}|\psi(z)|\to 0 \text{ as } |z|\to\infty\}. \tag{2.2}$$

This index is always in the interval $[0,2]$. By Theorem 3.2 of [2], this definition agrees with that of Blumenthal and Getoor when $S=0$ and a is chosen appropriately. By using the definition (2.2) we can include processes with a Gaussian component and/or a linear drift. However, for these processes which were excluded in [2] the index β does not have all the properties discussed there; for example, it cannot be determined from the Lévy measure alone.

A *subordinator* T_t is a non-decreasing process with stationary independent increments in $\mathbb{R}^1$. This corresponds to the assumption that ν is supported on $[0,\infty)$ and $\int_0^1 x\,\nu(dx)<\infty$, that $S=0$, and that a is chosen so that $\psi(z)=\int(1-e^{ixz})\,\nu(dx)$. For subordinators it is useful to introduce the Laplace transform

$$E\{\exp(-u\,T_t)\}=\exp[-t\,g(u)]$$

where

$$g(u)=\int_0^\infty(1-e^{-ux})\,\nu(dx). \tag{2.3}$$

Then g is called the *subordinator exponent* of T_t. The *lower subordinator index* σ is defined by

$$\sigma=\sup\{\alpha\colon\ u^{-\alpha}g(u)\to\infty \ \textit{as}\ u\to\infty\}. \tag{2.4}$$

By Theorem 6.1 of [2], in the case of a subordinator we have

$$\beta=\inf\{\alpha\geqq 0\colon\ u^{-\alpha}g(u)\to 0\ \textit{as}\ u\to\infty\}, \tag{2.5}$$

and so $0\leqq\sigma\leqq\beta\leqq 1$.

The process X_t is called *strictly stable* if $0<\alpha<1$ or $1<\alpha<2$ and

$$\psi(z)=\lambda|z|^\alpha\int_{S^d}|(z/|z|,\theta)|^\alpha[1-i\,\mathrm{sgn}(z,\theta)\tan\pi\alpha/2]\,m(d\theta),$$

or, if $\alpha=1$,

$$\psi(z)=i(a,z)+\lambda|z|\int_{S^d}|(z/|z|,\theta)|\,m(d\theta),$$

where $a\in\mathbb{R}^d$, $\lambda>0$, and m is a probability measure on S^d, the unit sphere in $\mathbb{R}^d$. We assume that m is not supported on a subspace of lower dimension so that X_t is truly d-dimensional. There are also strictly stable processes of index $\alpha=2$; these are characterized by $a=0$, $\nu=0$, and arbitrary S in (2.1). (S should be positive definite so that X_t will be d-dimensional.) The upper index β is equal to α for a stable process of index α. There are strictly stable subordinators of index $0<\alpha<1$. These are characterized by the subordinator exponent $g(u)=\lambda u^\alpha$ for $\lambda>0$. It is clear that $\sigma=\alpha$ for these processes.

Now we recall, briefly, the definition of Hausdorff dimension. For any subset B of $\mathbb{R}^d$ and each pair γ, $\delta>0$ we define

$$\mu_\delta^\gamma(B)=\inf\sum_i [d(S_i)]^\gamma,$$

where the infimum is taken over all covers of B by a collection of sets $\{S_i\}$ of diameters $d(S_i)<\delta$. Now let $\mu^\gamma(B)=\lim_{\delta\to 0}\mu_\delta^\gamma(B)$. For any set B there exists a number b called the dimension of B defined by

$$b=\sup\{\gamma\geqq 0\colon\ \mu^\gamma(B)=\infty\}=\inf\{\gamma>0\colon\ \mu^\gamma(B)=0\}.$$

We conclude the preliminaries by stating as a lemma a result of Kingman [14].

Lemma 2.1. *Let $\{c_n\}$ be a sequence of real numbers with $c_n\nearrow\infty$, $c_{n+1}-c_n\to 0$. Let G be an open subset of $\mathbb{R}^1$ unbounded above and I any open interval in $\mathbb{R}^1$. Then there is an $x\in I$ such that $x+c_n\in G$ for infinitely many integers n.*

3. Uniform Upper Bound

In this section we obtain uniform upper bounds for the dimension of $X(E,\omega)$, where X_t is an arbitrary process with stationary independent increments. The following lemma plays a key role in our considerations.

First define

$$M_t(\omega)=\sup_{0\leqq s\leqq t}|X_s(\omega)|.$$

Lemma 3.1 (Covering Principle). *Let $\{t_n\}$ be a sequence of positive real numbers with $\sum_n t_n^p$ finite for some $p>0$, and let $\mathscr{C}_n$ be a class of N_n intervals of length t_n with $\log N_n=O(1)|\log t_n|$. If $\{\theta_n\}$ is a sequence of positive real numbers such that for some $\delta>0$ we have*

$$P\{M_{t_n}\geqq\theta_n\}=O(1)\,t_n^\delta \tag{3.1}$$

then there exists a positive integer k such that, with probability one, for sufficiently large n, $X(I,\omega)$ can be covered by k spheres of radius θ_n whenever I is in $\mathscr{C}_n$.

Proof. Choose any $I\in\mathscr{C}_n$ and suppose that $I=[a,a+t_n]$. We define a sequence $\{\tau_j\}$ of stopping times by $\tau_0\equiv a$ and

$$\tau_j=\inf\{s\geqq\tau_{j-1}\colon\ |X_s-X_{\tau_{j-1}}|>\theta_n\},\qquad j\geqq 1.$$

By the strong Markov and independent increment properties we see that $\{\tau_j-\tau_{j-1}\}$ is a sequence of independent identically distributed random variables. The event

$$\{X(I,\omega)\text{ cannot be covered by }k\text{ spheres of radius }\theta_n\}\subset\{\tau_k-\tau_0\leqq t_n\},$$

and so, by the above remark, has probability less than or equal to $(P\{\tau_1-\tau_0\leqq t_n\})^k$. Condition (3.1) tells us that this is $O(1)\,t_n^{k\delta}$. Thus the event that there exists some I in $\mathscr{C}_n$ such that $X(I)$ cannot be covered by k spheres of radius θ_n has probability $O(1)N_n t_n^{k\delta}$. If k is large enough this is summable so that the lemma follows from the Borel Cantelli lemma.

Before we apply the covering principle we shall need the following estimate for the tail of the distribution of M_t.

Lemma 3.2. *If β is the upper index for X_t, $\alpha>\beta$, $0<\eta<1-\beta/\alpha$, and c is a positive constant, then there is a constant A such that*

$$P\{M_t \geqq c\, t^{1/\alpha}\} \leqq A\, t^{\eta}$$

for all t.

Remark. This estimate is a consequence of Theorem 2.2 of [15] in the case where the Lévy measure decays rapidly enough at infinity.

Proof. Since the upper indices corresponding to the coordinate processes are no larger than β, a trivial argument shows that it suffices to prove the lemma for one dimensional processes. Blumenthal and Getoor [2, p. 497] prove that if $\beta<2$ and η is in the prescribed range, there is a constant B such that

$$P\{|X_t| \geqq c\, t^{1/\alpha}\} \leqq B\, t^{\eta}. \tag{3.2}$$

Their proof is also valid for $\beta=2$ since one may take $\beta=\alpha=2$ in their Lemma 3.2 due to the exponential tail of the normal density. The proof on p. 497 then applies directly; indeed it is even possible to take $\eta=1-2/\alpha$ for $\alpha>2$ in this case. If $\mu(t)$ denotes a median of X_t, then by (3.2) we have $\mu(t)=o(1)\, t^{1/\alpha}$ as $t\to 0$. Thus we can discretize time and use Lévy's inequality to complete the proof.

Theorem 3.1. *Let X_t be any process with stationary independent increments and upper index β. Then*

$$P\{\dim X(E,\omega) \leqq \beta \dim E \textit{ for all } E\}=1.$$

Proof. By a standard limiting argument, it suffices to consider $E\subset[0,s]$. We take $\alpha>\beta$ and apply the Covering Principle (Lemma 3.1) with $\theta_n=2^{-n/\alpha}$, $t_n=s2^{-n}$, and

$$\mathscr{C}_n=\{[(j-1)s2^{-n}, js2^{-n}]: j=1,2,\dots,2^n\}.$$

By Lemma 3.2 the hypotheses are satisfied. Thus for sufficiently large n, $X(I)$ can be covered by k spheres of radius $2^{-n/\alpha}$ for every $I\in\mathscr{C}_n$.

Now suppose $\dim E=\gamma$. Choose $\varepsilon>0$, $\delta>0$ and cover E by intervals F_i of length d_i less than ε and so that

$$\sum_i d_i^{\gamma+\delta}<\varepsilon.$$

We then choose n_i so that

$$\frac{s}{2^{n_i+1}}<d_i\leqq\frac{s}{2^{n_i}},$$

whence F_i is contained in two intervals of $\mathscr{C}_{n_i}$. Thus, if ε is small enough, $X(F_i)$ can be covered by $2k$ spheres of radius $2^{-n_i/\alpha}\leqq(2s^{-1}d_i)^{1/\alpha}$. This gives a cover of $X(E)$ with a small $\alpha(\gamma+\delta)$ sum. Thus we have

$$\dim X(E)\leqq\alpha(\gamma+\delta).$$

Letting $\delta\to 0$ and $\alpha\to\beta$ through a countable sequence of values gives the final result.

4. Stable Processes

In this section we obtain complete results concerning the upper and lower uniform bounds of $\dim X(E,\omega)$ when X_t is any strictly stable process.

First we observe that Theorem 3.1 combines with Theorem 1 of [8] to give

Theorem 4.1. *Let X_t be any strictly stable process of index α, $\alpha \leqq d$, in $\mathbb{R}^d$. Then*

$$P\{\dim X(E,\omega)=\alpha \dim E \text{ for all } E\}=1.$$

Proof. It is just necessary to remark that the restriction, made in [8], that E be a bounded Borel set can be dropped without changing the validity of the proof. Also the strictly stable processes of index 1 with a drift can be included with no change.

This result is a generalization of (1.4) for processes that are not point recurrent. The generalization breaks down for processes that are point recurrent, as the following example shows. Suppose that $\alpha>1$ and that $Z(\omega)$ is the zero set of $X_t(\omega)$, a stable process of index α in $\mathbb{R}^1$. Then it is known [3] that $\dim Z(\omega)=1-1/\alpha$ almost surely, but $X[Z(\omega),\omega]=\{0\}$ so that $\dim X[Z(\omega),\omega]=0 \neq \alpha(1-1/\alpha)$. In the following theorem we show that this situation is the worst possible in the sense that if $\dim E>1-1/\alpha$ then $\dim X(E)$ is positive, and we have a uniform lower estimate for its dimension.

Theorem 4.2. *Let X_t be a strictly stable process of index α, $\alpha>1$, in $\mathbb{R}^1$ with zero set Z. Then*

$$P\{\alpha[\dim E-\dim Z] \leqq \dim X(E) \leqq \min(\alpha \dim E, 1) \text{ for all } E \in \mathscr{A}\}=1,$$

where $\mathscr{A}$ denotes the class of analytic sets in $\mathbb{R}^1$.

Remark. The proof depends heavily on the results of [7], whose notations we adopt without further reference.

Proof. Since $P\{\dim Z(\omega)=1-1/\alpha\}=1$, it is sufficient to establish the theorem in the case where $\dim E>1-1/\alpha$. Now let $T_t(\omega')$ be a stable subordinator of index β, $\beta=1/\alpha$, defined on $(\Omega', \mathscr{F}', P')$ and taking values in the time set of $X_t(\omega)$. Then $\beta+\dim E>1$ and E is non-polar for $T_t(\omega')$. Let $R(\omega')$ be the range of $T_t(\omega')$ and $S(\omega')$ the set of occupation times of E so that, by Theorem 2 (see also the first paragraph on p. 94) of [7],

$$(\beta+\dim E-1)/\beta=\sup\{\theta\colon\ P'[\dim S(\omega')>\theta]>0\}. \tag{4.1}$$

Let $Z_t(\omega,\omega')=X_{T_t(\omega')}(\omega)$ so that Z_t is a strictly stable process of index $\alpha\beta$, $\alpha\beta=1$, defined on $(\Omega\times\Omega', \mathscr{F}\times\mathscr{F}', P\times P')$. Now let

$$\tilde{\Omega}=\{(\omega,\omega')\colon\ \dim Z(E,\omega,\omega') \geqq \dim E \text{ for all } E\},$$

and $\tilde{\Omega}(\omega)=\{\omega'\colon\ (\omega,\omega')\in\tilde{\Omega}\}$. By Theorem 4.1 and Fubini's theorem we see that if $\Omega^*=\{\omega\colon\ P'[\tilde{\Omega}(\omega)]=1\}$ we have $P(\Omega^*)=1$. We have

$$X(E,\omega)\supset X[E\cap R(\omega'),\omega]=X\{T[S(\omega'),\omega'],\omega\}=Z[S(\omega'),\omega,\omega'].$$

Thus if $\omega\in\Omega^*$

$$\dim X(E,\omega) \geqq \sup_{\omega'\in\tilde{\Omega}(\omega)} \dim S(\omega'),$$

which, by (4.1), is at least $(\beta+\dim E-1)/\beta$. Therefore

$$P\{\dim X(E) \geqq \alpha[\dim E-(1-1/\alpha)] \text{ for all } E\in\mathscr{A}\}=1.$$

The upper inequality follows from Theorem 3.1 and the theorem is proved.

The above theorem is presumably true without the assumption that E is analytic, but this was needed in [7] in order to establish (4.1).

5. Subordinators

In this section we obtain best possible uniform dimension results for subordinators, and examine possible extensions.

Theorem 5.1. *Let T_t be a subordinator with upper and lower indices β and σ respectively. Then*

$$P\{\sigma \dim E \leqq \dim T(E) \leqq \beta \dim E \text{ for all } E\} = 1. \tag{5.1}$$

Proof. Theorem 3.1 of [10] gives the lower bound whilst Theorem 3.1 of this paper gives the upper bound.

We thus have a uniform version of the result, obtained by Blumenthal and Getoor, [2], that

$$P\{\sigma \dim E \leqq \dim T(E) \leqq \beta \dim E\} = 1 \qquad \text{for every } E. \tag{5.2}$$

During the course of their work they asked whether one could obtain a result like

$$P\{\dim T(E) = \dim T([0,1]) \dim E\} = 1 \qquad \text{for every } E. \tag{5.3}$$

It is known, [13], that $\dim T([0,1]) = \sigma$ and so the question arises as to whether the lower inequality in (5.1) is in fact equality. We show in Section 6 that this is not the case and that the bounds in (5.2) are best possible, so the Blumenthal and Getoor question is answered in the negative. (Hendricks [12] has shown that (5.3) is not true for general processes with stationary independent increments.) If, however, we restrict our attention to sets that are in some way regular we can obtain a class of sets for which the Blumenthal and Getoor question has a positive answer.

Let A be a bounded subset of the real line and let n be a positive integer. We define

$$N_n(A) = \#\{i\colon\ [i2^{-n}, (i+1)2^{-n}) \cap A \neq \phi,\ i \text{ an integer}\}$$

and, following [9], we call A a $\mathscr{D}$ set if

$$\frac{\log_2 N_n(A)}{n} \to \dim A$$

as $n \to \infty$. If A is an increasing union of bounded sets A_i, each of which has

$$\limsup_{n\to\infty} \frac{\log_2 N_n(A_i)}{n} \leqq \dim A$$

we call A a $\mathscr{D}_\sigma$ set. In [9] and [11] we investigated such sets. For $\mathscr{D}_\sigma$ sets the Blumenthal and Getoor question is answered positively.

Theorem 5.2. *Let T_t be a subordinator with lower index σ. Then*

$$P\{\dim T(E) = \sigma \dim E \ \textit{ for all } E \in \mathscr{D}_\sigma\} = 1.$$

Proof. Suppose $\varepsilon>0$, $\alpha=\sigma+\varepsilon$, $0<\delta<\varepsilon/\alpha$, and $\alpha'=\alpha(1-\delta)$, so that $\sigma<\alpha'<\alpha$. Let g be the subordinator exponent of T_t and

$$G_\alpha=\{\lambda: g(\lambda^{1/\alpha})<\lambda^{\alpha'/\alpha}\},$$

so that G_α is an unbounded open set. Kingman's result (Lemma 2.1) shows that there exists φ, $2^{1/(1+\varepsilon)}<\varphi<2$, and an increasing sequence $\{m(n)\}$ of integers such that $\varphi^{m(n)}\in G_\alpha$ for each n (to see this let $f(x)=\exp\exp x$, $G=f^{-1}(G_\alpha)$ and apply Kingman's result with $c_n=\log n$).

Now $P[T_t\geqq a]\leqq 2t\,g(1/a)$, [5, p. 169], so that

$$P[T_{2^{-m(n)}}\geqq \varphi^{-m(n)/\alpha}]\leqq 2\cdot 2^{-m(n)}\,g(\varphi^{m(n)/\alpha})\leqq 2\cdot 2^{-m(n)}\,\varphi^{m(n)\alpha'/\alpha}\leqq 2\cdot 2^{-m(n)\delta}.$$

Let $t_n=2^{-m(n)}$, $\theta_n=\varphi^{-m(n)/\alpha}$ and

$$\mathscr{C}_n=\{[j\,t_n,(j+1)\,t_n): j=0,1,\dots,m(n)\,2^{m(n)}\}.$$

The Covering Principle shows that with probability one, for sufficiently large n, $T(I)$ can be covered by k intervals of length $2\theta_n$, where I is any interval in $\mathscr{C}_n$.

Now let A be a bounded set with

$$\limsup_{n\to\infty}\frac{\log_2 N_n(A)}{n}<\gamma.$$

Then for sufficiently large n, $A\subset\bigcup\{I: I\in\mathscr{C}_n\}$ and A can be covered by $2^{m(n)\gamma}$ intervals in $\mathscr{C}_n$. Thus $T(A)$ can be covered by $O(1)\,2^{m(n)\gamma}$ intervals of length $2\theta_n$, if n is large enough. Letting $\eta=2\theta_n$ we see that

$$\mu_\eta^{\alpha\gamma(1+\varepsilon)}[T(A)]=O(1)\,2^{m(n)\gamma}[2\,\theta_n]^{\alpha\gamma(1+\varepsilon)}=O(1).$$

Thus

$$\dim T(A)\leqq\alpha\gamma(1+\varepsilon).$$

Letting ε tend to zero through a countable sequence, γ decrease to $\dim E$, and A increase to E we obtain

$$P\{\dim T(E)\leqq\sigma\dim E \text{ for all } E\in\mathscr{D}_\sigma\}=1.$$

The opposite inequality follows from Theorem 5.1 and the theorem is proved.

In [8] we showed how uniform dimension results can be applied to obtain other dimension results. Here we content ourselves with showing how the above result can be used to obtain the dimension of the collision set of a subordinator and an arbitrary stable process.

Let T_t be a subordinator with lower index σ and X_t a strictly stable process of index α, $\alpha>1$, in $\mathbb{R}^1$ defined on the same space as T_t, independent of T_t, and taking values in the same space as T_t. Then

$$E(\omega)=\{t: T_t(\omega)=X_t(\omega)\}$$

and

$$C(\omega)=\{x: x=T_t(\omega)=X_t(\omega) \text{ for some } t\}$$

are respectively called the collision times and collision set of X_t and T_t. We have

Theorem 5.3. *Let T_t and X_t be as above. Then if $C(\omega)$ is the collision set of T_t and X_t we have*

$$P\{\dim C(\omega)=\sigma(1-1/\alpha)\}=1.$$

Proof. Let $Z_t(\omega)$ be the infinitely divisible process T_t-X_t. Then Z_t has exponent $\psi(z)+|z|^\alpha(c_1+i\,c_2 \operatorname{sgn} z)$ where $\psi(z)$ is the exponent of T_t and c_1, c_2 are constants, $c_1>0$. Now $E(\omega)=\{t: Z_t(\omega)=0\}$. The results of Blumenthal and Getoor [4] apply to show that $E(\omega)$ is stochastically equivalent to the range of a (perhaps exponentially killed) subordinator S_t, whose subordinator exponent, $g(\lambda)$, satisfies

$$\frac{1}{g(\lambda)+\gamma}=\int \frac{dz}{\lambda+\psi(z)+|z|^\alpha(c_1+i\,c_2 \operatorname{sgn} z)},$$

where γ is the killing parameter. It follows that this subordinator has $\sigma(S)=\beta(S)=1-1/\alpha$. Theorem 4.4 of [11] shows that the range of S (and hence $E(\omega)$) is a $\mathscr{D}_\sigma$ set of dimension $1-1/\alpha$. Since $C(\omega)=T[E(\omega),\omega]$ we see, from Theorem 5.2, that

$$P\{\dim C(\omega)=\sigma(1-1/\alpha)\}=1,$$

and the theorem is proved.

6. Examples

Example 1. The first example is to show that the bounds (5.2) obtained by Blumenthal and Getoor cannot, in general, be improved even in the context of a fixed time set. Thus we shall construct a subordinator with $\sigma<\beta$ such that for any $\theta\in(\sigma,\beta)$, there is a set E of positive dimension with $\dim T(E)=\theta \dim E$ a.s.

Hendricks [12] has constructed an example which shows that the analogue of (5.3) is not true for general processes with stationary independent increments. The main purpose of our example is to show that (5.3) is not true even for subordinators. Another interesting feature is that in Hendrick's example $\dim X(E)$ is determined by $\dim E$ even though it is not of the form $\dim X[0,1]\cdot\dim E$, while in our example $\dim T(E)$ depends on other characteristics of the set E. Thus we can find two time sets E_1 and E_2 of the same dimension but with $\dim T(E_1)\neq \dim T(E_2)$.

Let the Lévy measure ν be atomic with atoms $p_n=2^{2^n}$ at $x_n=p_n^{-2}$, $n=1,2,\dots$. First we need bounds for the subordinator exponent (2.3). For any n,

$$p_n(1-e^{-u x_n})\leqq g(u)\leqq \sum_{k=1}^{n-1} p_k+u\sum_{k=n}^{\infty} p_k\, x_k.$$

This leads to the bounds

$$c_1(p_{n-1}+u\,x_{n-1})\leqq g(u)\leqq c_2(p_{n-1}+u\,x_{n-1}),\qquad p_n\leqq u\leqq p_{n+1}. \tag{6.1}$$

Thus for u in this range,

$$g(u)\leqq c_2(p_{n-1}+u^{\frac12}\,u^{\frac12}\,x_{n-1})\leqq 2\,c_2\,u^{\frac12},$$

while $g(p_n)\geqq c_1\,p_{n-1}=c_1\,p_n^{\frac12}$. Therefore the upper index $\beta=\frac12$ by (2.5). Similarly, if $p_n\leqq u\leqq p_n^{\frac32}$, $g(u)\geqq c_1\,p_n^{\frac12}\geqq c_1\,u^{\frac13}$, and if $p_n^{\frac32}\leqq u\leqq p_{n+1}$, $g(u)\geqq c_1\,u^{\frac23}\,u^{\frac13}\,x_{n-1}\geqq c_1\,u^{\frac13}$, while $g(p_n^{\frac32})\leqq 2\,c_2\,(p_n^{\frac32})^{\frac13}$. Thus the lower subordinator index $\sigma=\frac13$ by (2.4).

Now we need some information about the growth of the subordinator. Uniform lower functions for subordinators were obtained by Fristedt and Pruitt in [6]. Let η denote the inverse function of the subordinator exponent g. Then by Lemma 5 of [6], we have with probability one for t sufficiently small

$$T(s+t)-T(s)\geqq\frac{\log t^{-1}}{\eta(3\,t^{-1}\log t^{-1})}\qquad \text{for all } s\in[0,1].$$

Let $\alpha\in(\frac{3}{2},2)$. If $p_n^\alpha\leqq u\leqq p_{n+1}$, then $g(u)\geqq c_1\,x_{n-1}\,u\geqq c_1\,u^{(\alpha-1)/\alpha}$ and if $p_{n+1}\leqq u\leqq p_n^{\alpha/(\alpha-1)}$, $g(u)\geqq c_1\,p_n\geqq c_1\,u^{(\alpha-1)/\alpha}$. Thus

$$g(u)\geqq c_1\,u^{(\alpha-1)/\alpha}\qquad \text{for } p_n^\alpha\leqq u\leqq p_n^{\alpha/(\alpha-1)}.$$

Converting this to a bound for η gives

$$\eta(x)\leqq\left(\frac{x}{c_1}\right)^{\alpha/(\alpha-1)}\qquad \text{for } c_1\,p_n^{\alpha-1}\leqq x\leqq c_1\,p_n.$$

Therefore, we have if $\gamma>1$ and n is sufficiently large,

$$T(s+t)-T(s)\geqq t^{\alpha\gamma/(\alpha-1)} \tag{6.2}$$

for $c\,p_n^{\alpha-1}\leqq t^{-1}\log t^{-1}\leqq c\,p_n$ and all $s\in[0,1]$.

Next we construct the time set. Let $\theta\in(\sigma,\beta)=(\frac{1}{3},\frac{1}{2})$. We will actually construct a family of time sets E depending on a parameter ξ such that $0<\dim E<1$ and $\dim T(E)=\theta\dim E$. It will be clear that the parameter ξ will allow us to have two time sets of the same dimension corresponding to different values of θ. First let $\alpha=(1-\theta)^{-1}$ and let $\xi\in(\alpha-1,1)$. Note that $\alpha\in(\frac{3}{2},2)$. Define $s_n=x_{n-2}^\xi$ and $t_n=x_{n-1}^{\alpha-1}$, $n\geqq 3$. Observe that $t_n<s_n<t_{n-1}$. The set E is to be a Cantor type set constructed in the following way. Let $E_2=[0,1]$. Then E_n is to be a finite union of closed intervals of length t_n formed by placing as many of these intervals as possible in each interval of length t_{n-1} of E_{n-1} while keeping the intervals of E_n separated by intervals of length s_n which are not in E_n. Also E_n is to be a subset of E_{n-1}. Then $E=\bigcap_n E_n$. If N_n is the number of intervals of length t_n in E_n, then it is straightforward to see that for any $\varepsilon>0$, if n is sufficiently large,

$$p_n^{\xi-\alpha+1-\varepsilon}\leqq N_n\leqq p_n^{\xi-\alpha+1+\varepsilon}. \tag{6.3}$$

Standard arguments now show that

$$\dim E=\frac{\xi-\alpha+1}{\alpha-1}. \tag{6.4}$$

(Or this can be obtained from Ohtsuka's theorem [16] together with the fact that the capacitary dimension equals the Hausdorff dimension.)

Finally we will show that $\dim T(E)=\theta\dim E$ a.s. For the lower bound, we give an argument which applies to all paths which satisfy (6.2) for a sequence of γ's approaching one. For given $\delta>0$, $\varepsilon>0$, $\rho>\dim T(E)$, and $\gamma>1$, cover $T(E)$ by a sequence of intervals $\{S_i\}$ with the length of S_i being d_i and $d_i<\varepsilon^{\alpha\gamma/(\alpha-1)}$, $\sum_i d_i^\rho<\delta$. Let

$$a_i=\inf\{t\colon T_t\in S_i\},\qquad b_i=\sup\{t\colon T_t\in S_i\}.$$

We want to obtain a good covering for E. For each i, choose n (depending on i) such that $t_{n+1} < b_i - a_i \leqq t_n$. Now if $b_i - a_i \leqq s_{n+1}$, the interval (a_i, b_i) can meet at most one interval of E_{n+1}. Thus we can replace it with an interval of length t_{n+1} with no loss so far as covering E is concerned. In this way we can assume that for each i there is an n with $s_{n+1} < b_i - a_i \leqq t_n$. Then we can find $e_i > 0$ with $s_{n+1} \leqq b_i - a_i - e_i \leqq t_n$. Since $T(b_i - e_i) \in S_i$ and $T(a_i) \in \bar{S}_i$, we have $T(b_i - e_i) - T(a_i) \leqq d_i$. Also, for n sufficiently large,

$$c\, p_n^{\alpha-1} \leqq \frac{1}{t_n} \log \frac{1}{t_n} \leqq \frac{1}{b_i - a_i - e_i} \log \frac{1}{b_i - a_i - e_i} \leqq \frac{1}{s_{n+1}} \log \frac{1}{s_{n+1}} \leqq c\, p_n$$

so that by (6.2), if ε is small enough,

$$d_i \geqq T(b_i - e_i) - T(a_i) \geqq (b_i - e_i - a_i)^{\alpha\gamma/(\alpha-1)}.$$

Letting $e_i \to 0$, we have

$$b_i - a_i \leqq d_i^{(\alpha-1)/\alpha\gamma} < \varepsilon, \qquad \sum_i (b_i - a_i)^{\alpha\gamma\rho/(\alpha-1)} \leqq \sum_i d_i^\rho < \delta.$$

Since the intervals $[a_i, b_i]$ cover E, $\dim E \leqq \alpha\gamma\rho/(\alpha-1)$. Letting $\rho \searrow \dim T(E)$ and $\gamma \searrow 1$ gives the lower bound for $\dim T(E)$ since $(\alpha-1)/\alpha = \theta$.

For the upper bound we use the Covering Principle to get a cover for $T(E)$. Let $\gamma \in (3/2(\alpha-1), \alpha/(\alpha-1))$. By Lemma 1 of [5] and (6.1),

$$P[T_{t_n} \geqq t_n^\gamma] \leqq 2\, t_n\, g(t_n^{-\gamma}) \leqq 2\, c_2\, t_n (p_{n-1} + t_n^{-\gamma} x_{n-1}).$$

The second term is dominant due to the lower bound for γ. Thus

$$P[T_{t_n} \geqq t_n^\gamma] \leqq c\, t_n^{1-\gamma+1/(\alpha-1)}$$

and the exponent is positive. Since $\sum t_n$ converges and $\log N_n = O(\log t_n^{-1})$ by (6.3), the Covering Principle applies with $\mathscr{C}_n$ being the intervals in E_n. Thus for n large enough, $T(I)$ can be covered by k intervals of length $2\, t_n^\gamma$ for every I in E_n. Thus if $\eta = 2\, t_n^\gamma$

$$\mu_\eta^\rho[T(E)] \leqq k\, N_n (2\, t_n^\gamma)^\rho$$

and this tends to zero if $\rho > (\xi - \alpha + 1 + \varepsilon)/(\alpha-1)\gamma$ so that

$$\dim T(E) \leqq \frac{\xi - \alpha + 1 + \varepsilon}{(\alpha-1)\gamma}.$$

Letting $\varepsilon \searrow 0$, $\gamma \nearrow \alpha/(\alpha-1)$, and recalling (6.4) completes the proof.

The next two examples concern the possible candidates for uniform lower bounds for general processes.

Example 2. This shows that $\beta' \dim E$ is not a uniform lower bound when $\beta < d$. Let $X_t = (U_t, V_t)$, where U_t, V_t are independent symmetric stable processes in $\mathbb{R}^1$ of indices α_1 and α_2 with $0 < \alpha_2 < 1 < \alpha_1 < 2$. Then the indices for X_t are $\beta = \alpha_1$, $\beta' = 1 + \alpha_2(1 - 1/\alpha_1)$, $\beta'' = \alpha_2$; they satisfy $\beta'' < \beta' < \beta < 2$. Now let $E = \{t\colon U_t = 0\}$. Then $X(E) = \{0\} \times V(E)$ and by Theorem 4.1,

$$\dim X(E) = \dim V(E) = \alpha_2 \dim E \quad \text{a.s.},$$

while $\dim E = 1 - 1/\alpha_1 > 0$ as observed above. Since $\alpha_2 = \beta'' < \beta'$, this completes the example.

Example 3. This example shows that there can be no uniform version of Blumenthal and Getoor's lower bound (1.6) under the assumption $\beta' \leqq d$. In fact, for this example, there is no uniform lower bound for $\dim X(E, \omega)$ of the form $c \dim E$ with $c>0$. In [10], a parameter b is defined with the property that for processes X_t in $\mathbb{R}^1$ with zero set $Z=\{t: X_t=0\}$, $\dim Z=1-1/b$ a.s. An example is given there of a process X_t in $\mathbb{R}^1$ with $0<\beta''<\beta'<1<b<\beta<2$ so that for this process, for any $c>0$,

$$\dim X[Z(\omega), \omega]=0<c(1-1/b)=c \dim Z(\omega), \text{ a.s.},$$

and the example is complete.

The question remains as to whether $\beta'' \dim E$ is a uniform lower bound for $\dim X(E)$ when $\beta \leqq d$.

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