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Stability for Sums of i.i.d. Random Variables when Extreme Terms are Excluded

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Let $\{X_n\}$ be a sequence of i.i.d. random variables and let $X_n^{(r)}=X_j$ if $|X_j|$ is the r-th maximum of $|X_1|, \dots, |X_n|$. Let $S_n=X_1+\dots+X_n$ and ${}^{(r)}S_n=S_n-(X_n^{(1)}+\dots+X_n^{(r)})$. Suppose a sequence $\{a_n\}$ of normalizing constants satisfies (i) $a_n/n^{1/\alpha}$ is nondecreasing for some α, $0<\alpha<2$, and (ii) $\sup(a_{2n}/a_n)<\infty$. An integral criterion for the stability of ${}^{(r)}S_n/a_n$ is obtained. This extends a previous result [4] on the stability of ${}^{(r)}S_n/n$.

1. Introduction and Results

Let $\{X_n\}_{n\geqq 1}$ be a sequence of i.i.d. random variables with common d.f. F and put $\mathscr{F}(x)=P\{|X_1|>x\}$. For $r\geqq 1$ and $n\geqq r$ let $X_n^{(r)}=X_j$ if $|X_j|$ is the r-th maximum of $|X_1|, \dots, |X_n|$. More precisely let $M_n(j)$, $n\geqq 1$, $1\leqq j\leqq n$, be the number of X_i's satisfying either $|X_i|>|X_j|$, $1\leqq i\leqq n$, or $|X_i|=|X_j|$, $1\leqq i\leqq j$, and let $X_n^{(r)}=X_j$ if $M_n(j)=r$. Let $S_n=\sum_{i=1}^{n} X_i$, ${}^{(0)}S_n=S_n$ and ${}^{(r)}S_n=S_n-\sum_{k=1}^{r} X_n^{(k)}$ for $r\geqq 1$, $n\geqq r$.

In [4] an integral criterion for the stability of ${}^{(r)}S_n/n$ was obtained. In this paper we consider the stability of ${}^{(r)}S_n/a_n$. Throughout this paper we suppose the sequence $\{a_n\}$ of normalizing constants $a_n>0$ satisfies the following two conditions:

(A1) $\{a_n/n^{1/\alpha}\}$ is nondecreasing for some α, $0<\alpha<2$,

(A2) $\sup_{n\geqq 1}(a_{2n}/a_n)<\infty$.

If $\{a_n\}$ satisfies (A1) and (A2) then we can define an absolutely continuous increasing function A on $[0,\infty)$ with $A(0)=0$, $A(n)=a_n$ for $n=1,2,\dots$ and satisfying

(A1′) $A(x)/x^{1/\alpha}$ is nondecreasing,

(A2′) $\sup_{x>0}(A(2x)/A(x))<\infty$.

Since $A(\infty)=\infty$ the inverse function B of A is absolutely continuous increasing on $[0, \infty)$ with $B(0)=0$ and $B(\infty)=\infty$. Let us denote by J_r, $r>0$, the integral $\int_0^\infty \mathscr{F}^r(x)\, dB^r(x)$, where $\mathscr{F}^r(x)=\{\mathscr{F}(x)\}^r$ etc. If $0<r<s$ then

$$J_s=\frac{s}{r}\int_0^\infty (\mathscr{F}(x)\, B(x))^{s-r}\, \mathscr{F}^r(x)\, dB^r(x)$$

and therefore $J_r<\infty$ implies $J_s<\infty$ (see Lemma 3 below).

The purpose of this paper is to prove the following two theorems which extend a result of [4]. Theorem 1 should be compared with a classical result due to Feller ([1] or [6] p. 132). When $r=0$ Theorem 2 reduces to Marcinkiewicz strong law of large numbers ([6] p. 126).

Theorem 1. *Suppose $r\geqq 0$ is a fixed integer and $\{a_n\}$ is a sequence satisfying* (A1) *and* (A2). If $J_{r+1}<\infty$ *then*

$$\lim X_n^{(r+1)}/a_n=0 \quad \text{a.s.} \tag{1}$$

and there exists a sequence $\{c_n\}$ of constants satisfying

$$\lim({}^{(r)}S_n/a_n-c_n)=0 \quad \text{a.s.} \tag{2}$$

In this case c_n may be chosen according to the formula

$$c_n=\frac{n}{a_n}\int_{|x|\leqq \tau a_n} x\, dF(x) \tag{3}$$

where $\tau>0$ is an arbitrary constant. If $J_{r+1}=\infty$ then

$$\limsup |X_n^{(r+1)}|/a_n=\infty \quad \text{a.s.} \tag{4}$$

and

$$\limsup |{}^{(r)}S_n/a_n-c_n|=\infty \quad \text{a.s.} \tag{5}$$

for every sequence $\{c_n\}$.

Theorem 2. (i) *If*

$$\int_0^\infty x^{\alpha(r+1)-1}\mathscr{F}^{r+1}(x)\, dx<\infty \tag{6}$$

for some α, $0<\alpha<1$, and $r\geqq 0$ then

$${}^{(r)}S_n/n^{1/\alpha}\to 0 \quad \text{a.s.}$$

(ii) *If* (6) *holds with $\alpha=1$ and $r\geqq 0$ then for every $\tau>0$*

$${}^{(r)}S_n/n-\int_{|x|\leqq n\tau} x\, dF(x)\to 0 \quad \text{a.s.}$$

(iii) *If* (6) *holds for some* α, $1<\alpha<2$, *and* $r\geqq 0$ *then* $E|X_1|<\infty$ *and*

$({}^{(r)}S_n - nEX_1)/n^{1/\alpha}\to 0$ a.s.

(iv) *Conversely if* ${}^{(r)}S_n/n^{1/\alpha}-c_n\to 0$ a.s. *for some* α, $0<\alpha<2$, *and for some* $\{c_n\}$ *then* (6) *holds.*

2. Proofs

Lemma 1. *If* $0<b_n\uparrow\infty$ *then*

$$P\{|X_n^{(r+1)}|>b_n \text{ i.o.}\}=0 \text{ or } 1$$

according as $\sum_{n=1}^{\infty} n^r \mathscr{F}^{r+1}(b_n)$ *converges or diverges.*

Proof. This is Lemma 3 of [4].

Lemma 2. *For every* $\varepsilon>0$

$$P\{|X_n^{(r+1)}|>\varepsilon a_n \text{ i.o.}\}=0 \text{ or } 1$$

according as $J_{r+1}<\infty$ *or* $J_{r+1}=\infty$.

Proof. It is easy to see that

$$\sum_{n=1}^{\infty} n^r \mathscr{F}^{r+1}(\varepsilon a_n)<\infty$$

iff

$$\int_0^{\infty} x^r \mathscr{F}^{r+1}(\varepsilon A(x))\,dx<\infty. \tag{7}$$

By (A1′) we have $A(\varepsilon^\alpha x)\leqq\varepsilon A(x)\leqq A(x)$ if $0<\varepsilon<1$ and $A(x)\leqq\varepsilon A(x)\leqq A(\varepsilon^\alpha x)$ if $\varepsilon>1$. Therefore for every $\varepsilon>0$ (7) holds iff

$$\int_0^{\infty} x^r \mathscr{F}^{r+1}(A(x))\,dx=(r+1)^{-1} J_{r+1}<\infty.$$

Hence the lemma follows from Lemma 1.

Lemma 3. *If* $J_{r+1}<\infty$ *for some* $r\geqq 0$ *then*

$$\lim_{x\to\infty}\mathscr{F}(x)\,B(x)=0.$$

Proof. Write $\mathscr{F}^{r+1}(x)\,B^{r+1}(x)$ as

$$\int_0^{x} (\mathscr{F}(x)/\mathscr{F}(y))^{r+1}\mathscr{F}^{r+1}(y)\,dB^{r+1}(y)$$

and apply the dominated convergence theorem.

Lemma 4. *If $J_{r+1}<\infty$ for some $r\geqq 0$ then*

$$\int_{|x|\leqq y} x^2\,dF(x)=o(y^2/B(y)) \quad \text{as } y\to\infty.$$

Proof. Integrating by parts we have

$$\int_{|x|\leqq y} x^2\,dF(x)=-y^2\mathscr{F}(y)+2\int_0^y x\mathscr{F}(x)\,dx, \qquad y>0.$$

It is immediate from Lemma 3 that $y^2\mathscr{F}(y)=o(y^2/B(y))$. It follows from (A 1′) that $x/B^{1/\alpha}(x)\leqq y/B^{1/\alpha}(y)$ for $x\leqq y$ and therefore

$$y^{-2}B(y)\int_0^y x\mathscr{F}(x)\,dx\leqq y^{-2}\int_0^y (y/x)^\alpha x B(x)\mathscr{F}(x)\,dx$$

$$=y^{-2+\alpha}\int_0^y x^{1-\alpha}B(x)\mathscr{F}(x)\,dx.$$

Applying Lemma 3 again we obtain $\int_0^y x\mathscr{F}(x)\,dx=o(y^2/B(y))$.

In the next three lemmas we impose the following condition:

(F) $\mathscr{F}$ is positive and differentiable on $(0,\infty)$.

Let us define ψ by $\psi(x)=(B(x)/\mathscr{F}(x))^{1/2}$, $x\geqq 0$. Under the assumption (F) ψ is absolutely continuous strictly increasing with $\psi(0)=0$ and $\psi(\infty)=\infty$. Hence the inverse function φ of ψ is also absolutely continuous increasing with $\varphi(0)=0$ and $\varphi(\infty)=\infty$. By Lemma 3 $J_{r+1}<\infty$ implies $\lim_{x\to\infty}\psi(x)/B(x)=\infty$ and therefore $\lim_{x\to\infty} x/B(\varphi(x))=\infty$. Since (A 1′) implies

$$A(y)/\varphi(y)=A(y)/A(B(\varphi(y)))\geqq (y/B(\varphi(y)))^{1/\alpha}$$

for large y we have $\lim_{y\to\infty} A(y)/\varphi(y)=\infty$ if $J_{r+1}<\infty$.

Lemma 5. *If $J_{r+1}<\infty$ for some $r\geqq 0$ and if $k\geqq 2r+2$ then*

$$\int_0^\infty x^{k-1}\mathscr{F}^k(\varphi(x))\,dx=\int_0^\infty x^{-k-1}B^k(\varphi(x))\,dx=(2/k)\,J_{k/2}<\infty.$$

Proof. The first equality is obvious from $B(\varphi(x))=x^2\mathscr{F}(\varphi(x))$. The second equality is shown by a routine calculation using Lemma 3.

Let I_m, $m\geqq 0$, denote the set $\{2^m, 2^m+1,\ldots,2^{m+1}-1\}$ and let $\theta_n=\varphi(2^m)$ if $n\in I_m$. Define $X_n'=X_n\cdot I(|X_n|<\theta_n)$. The following lemma plays the central role in this paper. The proof is obtained by modifying the method used in Nagaev [5].

Lemma 6. *Assume* (F). *If $J_{r+1}<\infty$ for some $r\geqq 0$ then there exists a sequence $\{c_n\}$ of constants satisfying*

$$X_k'/a_n-c_n\to 0 \quad \text{a.s.}$$

Proof. Let $\{X_n''\}$ be a sequence of independent random variables independent of $\{X_n'\}$ and having the same distribution as $\{X_n'\}$. Then $\{Y_n\}$, $Y_n = X_n' - X_n''$, is a sequence of independent symmetrically distributed random variables with $|Y_n| \leqq 2\theta_n$, and every Y_n, $n \in I_m$, has the same d.f. F_m.

For a proof of the lemma it suffices to show that

$$\sum_{k=1}^{n} Y_k/a_n \to 0 \quad \text{a.s.}$$

(see [3] p. 247 or [6] p. 147). It is known ([6] p. 158) that under the assumptions (A1) and (A2) this is equivalent to

$$\sum_{m=1}^{\infty} P\{\sum_{n\in I_m} Y_n > \varepsilon A(2^m)\} < \infty \tag{8}$$

for every $\varepsilon > 0$.

Let G_m and Q_m denote the d.f. and the m.g.f. of $\sum_{n\in I_m} Y_n$ resp. Denoting by f_m the m.g.f. of Y_n, $n \in I_m$, we have $Q_m(h) = \{f_m(h)\}^{2^m}$. It is shown that

$$\frac{d}{dh}\log Q_m(h) = 2^m \frac{d}{dh}\log f_m(h)$$

is an increasing function of h and vanishes at $h=0$ (see [5]). For an arbitrarily fixed $\varepsilon > 0$ let h_m be the (minimum) solution of the equation

$$\frac{d}{dh}\log Q_m(h) = \varepsilon A(2^m)/2.$$

If this equation does not have solution then define $h_m = 1/\gamma_m$ where $\gamma_m = \varphi(2^m)$. By this definition we have

$$\log Q_m(h_m) = \int_0^{h_m} \frac{d}{dh}\log Q_m(h)\, dh \leqq \varepsilon\, h_m A(2^m)/2.$$

Defining a d.f. $\bar{G}_m$ by

$$\bar{G}_m(x) = \int_{y\leqq x} e^{h_m y}\, dG_m(y)/Q_m(h_m)$$

we find that

$$\begin{aligned} 1 - G_m(\varepsilon A(2^m)) &= Q_m(h_m) \int_{x>\varepsilon A(2^m)} e^{-h_m x}\, d\bar{G}_m(x) \\ &= Q_m(h_m)\exp[-h_m\, \varepsilon A(2^m)] \\ &\leqq \exp[-h_m\, \varepsilon A(2^m)/2]. \end{aligned} \tag{9}$$

Suppose $h_m < 1/\gamma_m$. Since $f_m(h) \geqq 1$ and $|Y_n| \leqq 2\gamma_m$ for $n \in I_m$, by using an inequality $e^x - e^{-x} \leqq 2x\, e^x$ we have

$$A(2^m)\,\varepsilon/2 \leqq 2^m f_m'(h_m)/f(h_m) \leqq 2^m \int_{-\infty}^{\infty} x\, e^{h_m x}\, dF_m(x)$$

$$= 2^m \int_0^{\infty} x(e^{h_m x} - e^{-h_m x})\, dF_m(x) \leqq 2^{m+1} h_m \int_0^{\infty} x^2 e^{h_m x}\, dF_m(x)$$

$$\leqq 2^{m+1} h_m e^{2h_m \gamma_m} \int_0^{\infty} x^2\, dF_m(x) \leqq 2^{m+1} e^2 h_m \int_{|x| \leqq 2\gamma_m} x^2\, dF(x).$$

Thus by Lemma 4 if $h_m < 1/\gamma_m$ and if m is sufficiently large then

$$A(2^m) \leqq 2^m h_m \gamma_m^2 / B(\gamma_m).$$

Consequently we find that for large m

$$\text{either} \quad h_m \geqq 1/\gamma_m \quad \text{or} \quad h_m \geqq A(2^m)\, B(\gamma_m)/(2^m \gamma_m^2). \tag{10}$$

Let $s \geqq \max(2\alpha(r+1), 2\alpha(r+1)/(2-\alpha))$. Since

$$\varphi(x)/B^{1/\alpha}(\varphi(x)) \leqq A(x)/x^{1/\alpha}$$

it follows from Lemma 5 that

$$\int_0^{\infty} \frac{1}{x} \left[\frac{\varphi(x)}{A(x)}\right]^s dx \leqq \int_0^{\infty} x^{-s/\alpha - 1} B^{s/\alpha}(\varphi(x))\, dx < \infty.$$

Similarly we obtain

$$\int_0^{\infty} \frac{1}{x} \left[\frac{x\,\varphi^2(x)}{A^2(x)\, B(\varphi(x))}\right]^s dx < \infty.$$

In view of (A 2′) these inequalities imply

$$\sum_{m=1}^{\infty} \left[\frac{\gamma_m}{A(2^m)}\right]^s < \infty \quad \text{and} \quad \sum_{m=1}^{\infty} \left[\frac{2^m \gamma_m^2}{A^2(2^m)\, B(\gamma_m)}\right]^s < \infty.$$

Thus it follows from (10) that

$$\sum_{m=1}^{\infty} [h_m A(2^m)]^{-s} < \infty.$$

This implies

$$\sum_{m=1}^{\infty} \exp[-h_m \varepsilon A(2^m)/2] < \infty.$$

By (9) this proves (8) and therefore the lemma.

Lemma 7. *Assume* (F). *Let* N_m *denote the number of* j's *such that* $|X_j| > \varphi(2^m)$, $j \leqq 2^{m+1} - 1$. *If* $J_{r+1} < \infty$ *then* $P\{N_m \geqq 2r+2 \text{ i.o.}\} = 0$.

Proof. By Lemma 5 we have

$$\sum_{n=1}^{\infty} n^{2r+1} \mathscr{F}^{2r+1}(\theta_n) \leqq \text{const.} \times \int_0^{\infty} x^{2r+1} \mathscr{F}^{2r+1}(\varphi(x))\, dx < \infty.$$

Hence by Lemma 1

$$P\{N_m \geqq 2r+2 \text{ i.o.}\} \leqq P\{|X_n^{(2r+2)}| \geqq \theta_n \text{ i.o.}\} = 0.$$

Proof of Theorem 1. First suppose $J_{r+1} < \infty$. Then (1) follows from Lemma 2. By the same reasoning as in [4] it suffices to prove (2) assuming (F). For an arbitrary $\varepsilon > 0$ let

$$S_n(\varepsilon) = \sum_{j=1}^{n} X_j \cdot I(|X_j| \leqq a_n \varepsilon)$$

and let $S'_n = \sum_{j=1}^{n} X'_j$. If $n \in I_m$ and n is so large that $\theta_n < \varepsilon a_n$ then

$$\begin{aligned} |S_n(\varepsilon) - S'_n| &\leqq \varepsilon a_n N_m + \sum_{k=1}^{m} \varepsilon A(2^{m-k+1}) N_{m-k} \\ &\leqq \varepsilon a_n \left[N_m + \sum_{k=1}^{m} (\tfrac{3}{4})^{k-1} N_{m-k} \right] \end{aligned}$$

because $a_n/a_{2n} \leqq (n/(2n))^{1/\alpha} = 2^{-1/\alpha} < \frac{3}{4}$. Hence by Lemma 7 we obtain

$$\limsup a_n^{-1} |S_n(\varepsilon) - S'_n| \leqq (2r+1) \left[1 + \sum_{k=1}^{\infty} (\tfrac{3}{4})^{k-1} \right] \varepsilon = 5(2r+1)\varepsilon \qquad \text{a.s.}$$

By Lemma 2 we have almost surely $|S_n(\varepsilon) - {}^{(r)}S_n| \leqq r a_n \varepsilon$ for large n. Thus

$$\limsup a_n^{-1} |{}^{(r)}S_n - S'_n| \leqq (11r+5)\varepsilon \qquad \text{a.s.}$$

Since ε was arbitrary

$$\lim a_n^{-1} |{}^{(r)}S_n - S'_n| = 0 \qquad \text{a.s.}$$

Combined with Lemma 6 this shows (2).

It remains to prove (3). By Lemma 3 and the inequalities used in the proof of Lemma 2 we have $\lim_{x \to \infty} x \mathscr{F}(\varepsilon A(x)) = 0$ for all $\varepsilon > 0$. Hence for every $k \geqq 1$

$$\begin{aligned} P\{|X_n^{(k)}| > \varepsilon a_n\} &= \sum_{j=k}^{n} \binom{n}{j} \mathscr{F}^j(\varepsilon a_n)[1 - \mathscr{F}(\varepsilon a_n)]^{n-j} \\ &\sim [n \mathscr{F}(\varepsilon a_n)]^k / k! \to 0 \end{aligned}$$

as $n \to \infty$. This shows

$$X_n^{(k)}/a_n \to 0 \quad \text{in prob.}$$

for $k \geqq 1$ and therefore (2) implies

$$S_n/a_n - c_n \to 0 \quad \text{in prob.}$$

Hence c_n may be chosen according to the formula (3) (see [2] p. 135).

Next suppose $J_{r+1} = \infty$. Then (4) is immediate from Lemma 2. When $r \geqq 1$ it is found that

$$|{}^{(r)}S_{n+1} - {}^{(r)}S_n| = \min(|X_{n+1}|, |X_n^{(r)}|). \tag{11}$$

Let $r+1 \leqq n_1 \leqq n_2 \leqq \cdots$ be successive indices n with $|X_{n+1}^{(r+1)}| > |X_n^{(r+1)}|$. It is easy to see that

$$|X_{n_j+1}^{(r+1)}| = \min(|X_{n_j+1}|, |X_{n_j}^{(r)}|) = |{}^{(r)}S_{n_j+1} - {}^{(r)}S_{n_j}|.$$

Further $|X_n^{(r+1)}| > a_n M$ for infinitely many n iff $|X_{n_j+1}^{(r+1)}| > a_{n_j+1} M$ for infinitely many j. Thus (4) implies

$$P\{|{}^{(r)}S_{n+1} - {}^{(r)}S_n| > a_n M \text{ i.o.}\} = 1 \tag{12}$$

for every $M > 0$. When $r = 0$ (12) is immediate from (4). On the other hand by the zero-one law $\limsup |{}^{(r)}S_n/a_n - c_n|$ is either $= \infty$ a.s. or $< \infty$ a.s. Suppose

$$\limsup |{}^{(r)}S_n/a_n - c_n| < \infty \quad \text{a.s.} \tag{13}$$

for some $\{c_n\}$. It follows from (11) that

$$a_n^{-1} |{}^{(r)}S_{n+1} - {}^{(r)}S_n| \leqq a_n^{-1} |X_{n+1}| \to 0 \quad \text{in prob.}$$

Therefore (13) implies

$$\sup |c_n - (a_{n+1}/a_n) c_{n+1}| < \infty.$$

Together with (13) this shows

$$\limsup a_n^{-1} |{}^{(r)}S_{n+1} - {}^{(r)}S_n| < \infty \quad \text{a.s.}$$

This contradicts with (12) and therefore (4) must imply (5).

Proof of Theorem 2. Put $a_n = n^{1/\alpha}$, $A(x) = x^{1/\alpha}$ in Theorem 1. Then the theorem is immediate from Theorem 1 except for assertions on the centering constants c_n in (i) and (iii). Let (6) be satisfied and let c_n be chosen according to (3).

If $0 < \alpha < 1$ then by Lemma 3

$$\begin{aligned}
\Big| \int_{|x| \leqq n^{1/\alpha}\tau} x\, dF(x) \Big| &\leqq \int_{|x| \leqq n^{1/\alpha}\tau} |x|\, dF(x) \\
&= -n^{1/\alpha} \mathscr{F}(n^{1/\alpha}\tau) + \int_0^{n^{1/\alpha}\tau} \mathscr{F}(x)\, dx \\
&= -n^{1/\alpha - 1} n \mathscr{F}(n^{1/\alpha}\tau) + \int_0^{n^{1/\alpha}\tau} x^{-\alpha}(x^\alpha \mathscr{F}(x))\, dx \\
&= o(n^{1/\alpha - 1}).
\end{aligned}$$

Hence

$$c_n = n^{1-1/\alpha} \int_{|x| \leqq n^{1/\alpha}\tau} x \, dF(x) \to 0. \tag{14}$$

Suppose $1 < \alpha < 2$. Hölder inequality shows

$$\int_c^\infty \mathscr{F}(x)\, dx \leqq \left[\int_c^\infty (x^{\alpha - 1/p} \mathscr{F}(x))^p \, dx\right]^{1/p} \cdot \left[\int_c^\infty (x^{-\alpha + 1/p})^q \, dx\right]^{1/q}$$

where $p > 1$ and $1/p + 1/q = 1$. If $r \geqq 1$ then by putting $p = r+1$ and $q = (r+1)/r$ we obtain

$$\int_c^\infty \mathscr{F}(x)\, dx \leqq K c^{-\alpha+1} \left[\int_c^\infty x^{\alpha(r+1)-1} \mathscr{F}^{r+1}(x)\, dx\right]^{1/(r+1)} \tag{15}$$

for $c > 0$ where $K = [(\alpha-1)(r+1)/r]^{-r/(r+1)}$. When $r = 0$ it is easy to see that (15) holds with $K = 1$. Thus (6) implies $E|X_1| = \int_0^\infty \mathscr{F}(x)\, dx < \infty$. It follows from (15), (6) and Lemma 3 that

$$\begin{aligned} &|n^{1-1/\alpha} EX_1 - c_n| \\ &= n^{1-1/\alpha} \left| \int_{|x| > n^{1/\alpha}\tau} x\, dF(x)\right| \leqq n^{1-1/\alpha} \int_{|x| > n^{1/\alpha}\tau} |x| \, dF(x) \\ &= n^{1-1/\alpha} [n^{1/\alpha} \tau \mathscr{F}(n^{1/\alpha}\tau) + \int_{n^{1/\alpha}\tau}^\infty \mathscr{F}(x)\, dx] \\ &\leqq n\tau \mathscr{F}(n^{1/\alpha}\tau) + K\tau^{-\alpha+1} [\int_{n^{1/\alpha}\tau}^\infty x^{\alpha(r+1)-1} \mathscr{F}^{r+1}(x)\, dx]^{1/(r+1)} \to 0. \end{aligned} \tag{16}$$

The relations (14) and (16) complete the proof of (i) and (iii) resp.

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